On Simulation of the Flow Around an Airfoil Using Different Numerical Schemes of Vortex Element Method

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Abstract

The problem of numerical flow simulation around an airfoil is considered. Two approaches are considered for boundary conditions satisfaction: the normal component of the velocity is equal to zero (classical method) and the tangent component of the velocity is equal to zero (modified method).

In terms of these approaches several numerical schemes for vortex element method are developed. For the model problems results obtained with these numerical schemes are compared with exact solutions, which can be found using conformal mappings technique.

Analytical expressions for the coefficients of linear equations approximating corresponding integral equation are obtained for all developed numerical schemes. The results of flow simulation for some simple cases are presented.

1 Introduction

Flow simulation around an airfoil is a very important problem for number of engineering applications. Different numerical methods have been developed for its solving, most of them presuppose mesh generation in flow region. But there are also the so-called meshfree lagrangian numerical methods which don't need mesh in flow region at all. Vortex element method [1, 2, 3, 4] is one of these methods and it is especially effective for flow simulation in coupled aeroelastical problems when the airfoil can be not rigid or it can be elastically fixed. When using vortex element method the airfoil is simulated with a vortex layer on the airfoil surface. Its intensity depends on time, so it should be computed every time step. The accuracy of the vortex layer intensity computation defines the accuracy of the boundary condition satisfaction on the airfoil surface and consequently the accuracy of vortex wake simulation near the airfoil. However, the existing well-known numerical schemes, normally being used in vortex element method, sometimes lead to significant errors, especially when simulating flow around airfoils with angle points or sharp edges (wing airfoils). The aim of this paper is to develop some numerical schemes for vortex element method based on different approaches to boundary condition satisfaction and to compare their accuracies when flow simulating around smooth airfoils and airfoils with sharp edge.

2 Governing equations

Viscous incompressible media movement is described by continuity equation

$$\nabla \cdot \underline{V} = 0$$

and Navier — Stokes equations

$$\frac{\partial \underline{V}}{\partial t} + (\underline{V} \cdot \nabla)\underline{V} = \nu \Delta \underline{V} - \nabla \left(\frac{p}{\rho}\right)$$

where $\underline{V}(\underline{r}, t)$ is flow velocity, $p(\underline{r}, t)$ — pressure, $\rho = \text{const}$ — density of the media, ν — kinematic viscosity coefficient. No-slip boundary condition on the airfoil surface

$$\underline{V}(\underline{r}, t) = 0, \quad \underline{r} \in K$$

and boundary conditions of perturbation decay on infinity

$$\underline{V}(\underline{r}, t) \to \underline{V}_\infty, \quad p(\underline{r}, t) \to p_\infty, \quad |\underline{r}| \to \infty$$

should be satisfied.

Navier — Stokes equations can be written down in Helmholtz form using vorticity vector $\underline{\Omega}(\underline{r}, t) = \nabla \times \underline{V}(\underline{r}, t)$:

$$\frac{\partial \underline{\Omega}}{\partial t} + \nabla \times (\underline{\Omega} \times \underline{U}) = 0. \tag{1}$$

Here $\underline{U}(\underline{r}, t) = \underline{V}(\underline{r}, t) + \underline{W}(\underline{r}, t)$, $\underline{W}(\underline{r}, t)$ is the so-called ‘diffusive velocity’, which is proportional to viscosity coefficient:

$$\underline{W}(\underline{r}, t) = \nu \frac{(\nabla \times \underline{\Omega}) \times \underline{\Omega}}{|\underline{\Omega}|^2}.$$

If the vorticity distribution is known, flow velocity can be computed using Biot — Savart law:

$$\underline{V}(\underline{r}, t) = \underline{V}_\infty + \frac{1}{2\pi} \int_S \frac{\underline{\Omega}(\underline{\xi}, t) \times (\underline{r} - \underline{\xi})}{|\underline{r} - \underline{\xi}|^2} dS.$$

Equation (1) means that vorticity which exists in the flow moves and its velocity is $\underline{U}$. ‘New’ vorticity is being generated only on airfoil surface. This vortex layer influence on the flow is equivalent to streamlined airfoil influence, so the vortex layer intensity can be found from boundary condition on airfoil surface. We assume that there is no vorticity in the flow and we need to compute the vortex layer intensity on airfoil surface. From mathematical point of view this problem is equivalent to ideal incompressible steady flow simulation around the airfoil. In real unsteady viscous flow similar problem should be solved every time step.

3 Exact solution for simplest airfoils

Using methods of complex analysis, exact solutions for the vortex layer intensity in ideal incompressible steady flow can be found for some simplest airfoils (circular, elliptical, Zhukovsky airfoils). The vortex layer intensity is equal to velocity tangential component on the airfoil surface. Complex value of flow velocity can be found using the following formula [5]:

$$V^*(p) = \frac{R|\underline{V}_\infty| \sin(\phi + \beta - p) + \frac{G}{2\pi}}{\frac{iRe^{i(p-\phi)}}{2} \left(1 - \frac{a^2}{(Re^{i(p-\phi)} + H)^2}\right)}.$$

Here V^* means complex conjugate quantity to velocity V, $p \in [0, 2\pi)$ defines the point on airfoil surface, β — angle of incidence.

For elliptical airfoil

$$a = \sqrt{a_1^2 - b_1^2}, \quad R = a_1 + b_1, \quad \phi = 0, \quad H = 0,$$

a_1 and b_1 are major and minor semiaxes of the ellipse.

For Zhukovsky airfoil

$$R = \sqrt{(a + d\cos\phi)^2 + (h + d\sin\phi)^2}, \quad \phi = \arctan\frac{h}{a}, \quad H = ih - de^{-i\phi},$$

a, d and h are arbitrary parameters, which correspond to length, width and curvature of the airfoil.

The flow velocity circulation G for elliptical airfoil can be chosen arbitrarily (from mathematical point of view); we assume it to be equal to zero independently on angle of incidence, while for Zhukovsky airfoil it is proportional to uniform flow velocity and depends on the airfoil shape and its angle of incidence:

$$G = -2\pi|\underline{V}_\infty|\sin(\beta + \phi)\big(\sqrt{h^2 + a^2} + d\big).$$

Using the previous formulae we can obtain the exact solution for the vortex layer intensity. These exact solutions will be used for numerical schemes comparison and their accuracy estimation.

4 Vortex element method

We consider 2D model problem of ideal incompressible flow simulation around an airfoil. Vorticity is equal to zero everywhere in the flow region and the airfoil is simulated with thin vortex layer with intensity $\gamma(\underline{r}_0) = \gamma(x_0, y_0, 0)$ on the airfoil surface K. In this problem velocity $\underline{V} = (v_x, v_y, 0)^T$ can be determined in every point $\underline{r} = (x, y, 0)^T$ in the flow region using Biot — Savart law (point $\underline{r}_0 = (x_0, y_0, 0)^T$ lies on the airfoil surface K, $\underline{\gamma}(\underline{r}_0) = \gamma(\underline{r}_0)\underline{k}$ is the vortex layer intensity vector, $\underline{k} = (0, 0, 1)^T$):

$$\underline{V}(\underline{r}) = \underline{V}_\infty + \oint_K \frac{\underline{\gamma}(\underline{r}_0) \times (\underline{r} - \underline{r}_0)}{2\pi|\underline{r} - \underline{r}_0|^2} dl_{r_0}.$$

Limit value of flow velocity on the airfoil surface is equal to

$$\underline{V}_-(\underline{r}\,) = \underline{V}_\infty + \oint_K \frac{\underline{\gamma}(\underline{r}_0) \times (\underline{r} - \underline{r}_0)}{2\pi|\underline{r} - \underline{r}_0|^2} dl_{r_0} - \left(\frac{\underline{\gamma}(\underline{r}\,)}{2} \times \underline{n}(\underline{r}\,)\right).$$

Here $\underline{n}(\underline{r})$ is unit normal vector on the airfoil surface in point $\underline{r}$, $\underline{V}_-(\underline{r})$ corresponds to limit value of velocity from the airfoil side.

In order to determine the vortex layer intensity γ we should solve equation $\underline{V}_-(\underline{r}) = 0$ on the airfoil surface. It can be easily shown that we can solve either scalar equation

$$\underline{V}_-(\underline{r}) \cdot \underline{n}(\underline{r}) = 0$$

or scalar equation

$$\underline{V}_-(\underline{r}) \cdot \underline{\tau}(\underline{r}) = 0$$

instead of vector equation $\underline{V}_-(\underline{r}) = 0$. Here $\underline{\tau}(\underline{r}\,)$ is unit tangent vector on the airfoil surface. From mathematical point of view there is no difference between solutions of these equations, but from computational point of view numerical schemes based on these approaches are very different.

4.1 NVEM approach for vortex element method

In 'classical' approach [1, 2, 4] unknown vortex intensities satisfy equation $\underline{V}_{-} \cdot \underline{n} = 0$, which corresponds to zero normal component of flow velocity on the airfoil surface and leads to singular integral equation

$$\oint_K \frac{\big[\underline{k} \times (\underline{r} - \underline{r}_0)\big] \cdot \underline{n}(\underline{r})}{2\pi|\underline{r} - \underline{r}_0|^2} \gamma(\underline{r}_0)\, dl_{r_0} = -\underline{n}(\underline{r}) \cdot \underline{V}_{\infty}. \tag{2}$$

It should be noted that solution of (2) certainly exists due to form of right side of this equation, but it is not unique. In order to select the unique solution an additional integral condition should be added:

$$\oint_K \gamma(\underline{r}) dl_r = G. \tag{3}$$

The kernel of equation (2) is unbounded and it has nonintegrable singularity when $|\underline{r} - \underline{r}_0| \to 0$, so special numerical schemes are used for Cauchy principal value computation. They allow to obtain the solution of linear system approximating (2) with high accuracy when number of collocating points on the airfoil is large and its surface is smooth curve. It is proved [1] that in this case numerical solution converges to exact one in some integral (Hölder) norm. This approach which lies in the basis of the 'classical' method we will call 'NVEM' (Vortex element method with normal components of velocity on airfoil surface).

At the same time if we simulate flow around the airfoil with angle points or sharp edges using NVEM, the difference between numerical and exact solutions (in uniform norm) becomes significant and it increases proportionally to number of collocating points on the airfoil surface. So it is impossible to determine the vortex layer intensity with high accuracy. So well-known numerical schemes, which are effective in vortex element method for inviscous fluids, can be generalized for viscous case for smooth airfoils, but they can't be applied for 2D Navier — Stockes equations solution for airfoils with angle points and sharp edges. The main problem is that in viscous case all vortex elements generated on the airfoil surface become part of vortex wake near the airfoil. It also should be noted that linear algebraic system corresponding to (2) becomes ill-conditioned for airfoils with angle points or sharp edges.

In this paper some types of numerical schemes for vortex element method are developed, which are based on NVEM approach but differ from the 'classical' scheme.

4.2 TVEM approach for vortex element method

The vortex layer intensity also can be determined from equation $\underline{V}_{-} \cdot \underline{\tau} = 0$ [3] which corresponds to zero limit value of flow velocity tangential component. It leads to Fredholm integral equation with bounded (for smooth airfoils) kernel:

$$\oint_K \frac{\big[\underline{k} \times (\underline{r} - \underline{r}_0)\big] \cdot \underline{\tau}(\underline{r}\,)}{2\pi|\underline{r} - \underline{r}_0|^2} \gamma(\underline{r}_0) dl_{r_0} - \frac{\gamma(\underline{r})}{2} = -\underline{\tau}(\underline{r}) \cdot \underline{V}_{\infty}. \tag{4}$$

Solution of equation (4) is also non-unique, so the same additional condition (3) as in the previous case is used. This equation kernel is bounded by value $\varkappa/4\pi$, where $\varkappa$ is the airfoil curvature. This approach we will call 'TVEM' (Vortex element method with tangential components of velocity on the airfoil surface).

Equation (4) also can be approximated with linear algebraic system which is well-conditioned both for smooth and non-smooth airfoils. Due to equation kernel boundness an arbitrary quadrature formula can be used for integral approximation in (4).

In this paper some types of numerical schemes for vortex element method based on TVEM approach are also developed.

4.3 Numerical schemes based on NVEM and TVEM approaches

Let's consider different numerical schemes for vortex element method which can be used for computation of the vortex layer intensity on the airfoil surface. Eight numerical schemes are developed for NVEM and TVEM approaches, which follow from two different methods for the boundary condition satisfaction and two methods for vortex layer discretization:

- boundary condition (BC) can be satisfied either in collocation points on the airfoil surface or on an average on the airfoil surface parts (panels);
- vorticity from every panel can be either concentrated in point vortex element or it can be uniformly distributed on every panel.

Designations of suggested schemes are shown in Table 18.

NVEM/TVEM approach	BC in **coll**ocation points	BC on **aver**age at panels
Concentrated **vort**ices	$\mathcal{N}^{coll}_{vort}$ / $\mathcal{T}^{coll}_{vort}$	$\mathcal{N}^{aver}_{vort}$ / $\mathcal{T}^{aver}_{vort}$
Vortex **layer**	$\mathcal{N}^{coll}_{layer}$ / $\mathcal{T}^{coll}_{layer}$	$\mathcal{N}^{aver}_{layer}$ / $\mathcal{T}^{aver}_{layer}$

Table 18: Designations of numerical schemes for NVEM and TVEM approaches

For each of the numerical methods boundary condition (2) or (4) correspondingly leads to linear algebraic system:

$$\sum_{j=1}^{n}\left(A_{ij} - D_i\left(\frac{1}{2}\delta_{ij}\right)\right)\gamma_j = -B_i. \tag{5}$$

Here A_{ij} is matrix coefficient: for NVEM approach all A_{ij} are calculated while for TVEM approach A_{ij} are calculated only if $i \neq j$ and $A_{ii} = 0$; coefficient $D_i = 0$ for all numerical schemes of NVEM approach ($\mathcal{N}$-schemes), $D_i = 1$ for $\mathcal{T}^{coll}_{layer}$ and $\mathcal{T}^{aver}_{layer}$ numerical schemes and $D_i = \dfrac{1}{L_i}$ for $\mathcal{T}^{coll}_{vort}$ and $\mathcal{T}^{aver}_{vort}$ schemes; L_i is length of i-th panel; B_i is flow influence on i-th panel; unknown variable γ_j is the vortex layer intensity for $\mathcal{N}/\mathcal{T}^{coll}_{layer}$ schemes and $\mathcal{N}/\mathcal{T}^{aver}_{layer}$ and γ_j is the vortex element intensity for $\mathcal{N}/\mathcal{T}^{coll}_{vort}$ and $\mathcal{N}/\mathcal{T}^{aver}_{vort}$ schemes, n is number of panels.

Vector $\underline{V}_\infty$ is constant for all cases, so $B_i = \underline{V}_\infty \cdot \underline{\xi}_i$, where $\underline{\xi}_i = \underline{n}_i$ and $\underline{\xi}_i = \underline{\tau}_i$ for $\mathcal{N}$ and $\mathcal{T}$-schemes correspondingly. On fig. 1 the design model on the airfoil is shown. Note that $C_{n+1} \equiv C_1$ and $K_0 \equiv K_n$.

The formulae for A_{ij} coefficients for all developed numerical schemes are presented below. We also assume that all the panels are rectilinear, so $\underline{\xi}_i$ is constant vector on i-th panel.

1. For $\mathcal{N}^{coll}_{vort}$ and $\mathcal{T}^{coll}_{vort}$ schemes vorticity is concentrated in one point per panel and has form of Dirac δ-function. So integrals in equations (2) and (4) transform to sums and their coefficients are the following:

$$A_{ij} = \frac{\underline{k} \times (\underline{K}_i - \underline{Q}_j)}{2\pi|\underline{K}_i - \underline{Q}_j|^2} \cdot \underline{\xi}_i,$$

where $\underline{Q}_j = \underline{C}_j$ for $\mathcal{N}^{coll}_{vort}$ scheme, $\underline{Q}_j = \underline{K}_j$ for $\mathcal{T}^{coll}_{vort}$ scheme.

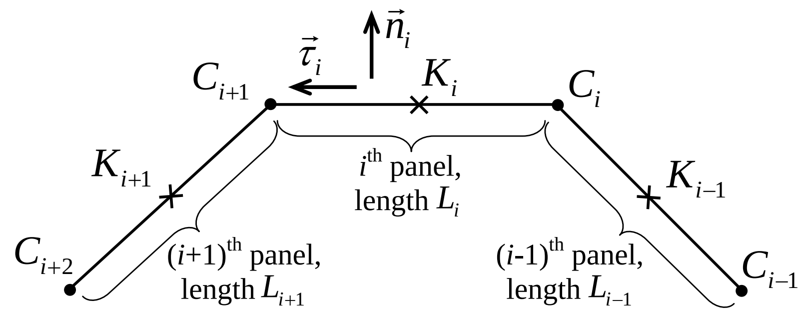

Figure 1: The design model on the airfoil surface

2. For $\mathcal{N}^{coll}_{layer}$ and $\mathcal{T}^{coll}_{layer}$ schemes unknown value is the vortex layer intensity on the airfoil surface, which is assumed to be piecewise constant function, so we obtain the following coefficients of the linear systems:

$$A_{ij} = \left(\int\limits_{Q_{j-1}}^{Q_j} \frac{\underline{k} \times (\underline{K}_i - \underline{r})}{2\pi |\underline{K}_i - \underline{r}|^2} dl_r \right) \cdot \underline{\xi}_i,$$

where $Q_j = K_j$ for $\mathcal{N}^{coll}_{layer}$ scheme, $Q_j = C_{j+1}$ for $\mathcal{T}^{coll}_{layer}$ scheme.

If $|\underline{K}_i - \underline{r}| \to 0$ in $\mathcal{N}^{coll}_{layer}$ case, then integration is carried out not over all the part $[K_{j-1}; K_j]$ of the airfoil surface, but over smaller part distancing from the 'singular' point at small length ϵ.

3. For $\mathcal{N}^{aver}_{vort}$ and $\mathcal{T}^{aver}_{vort}$ schemes boundary conditions are satisfied on an average at panels, thus

$$A_{ij} = \left(\frac{1}{L_i} \int\limits_{C_i}^{C_{i+1}} \frac{\underline{k} \times (\underline{r} - \underline{Q}_j)}{2\pi |\underline{r} - \underline{Q}_j|^2} dl_r \right) \cdot \underline{\xi}_i,$$

where $\underline{Q}_j = \underline{C}_j$ for $\mathcal{N}^{aver}_{vort}$ scheme, $\underline{Q}_j = \underline{K}_j$ for $\mathcal{T}^{aver}_{vort}$ scheme.

If $|\underline{r} - \underline{C}_j| \to 0$ in $\mathcal{N}^{aver}_{vort}$ case, then integration is carried out not over all the airfoil panel $[C_i; C_{i+1}]$, but over smaller part of this panel distancing from its 'singular' end at small length ϵ.

4. For $\mathcal{N}^{aver}_{layer}$ and $\mathcal{T}^{aver}_{layer}$ schemes we consider boundary conditions also on an average at panels and distributed vorticity (vortex layer) along the panels:

$$A_{ij} = \left(\frac{1}{L_i} \int\limits_{C_i}^{C_{i+1}} dl_r \int\limits_{Q_{j-1}}^{Q_j} \frac{\underline{k} \times (\underline{r} - \underline{\rho})}{2\pi |\underline{r} - \underline{\rho}|^2} dl_\rho \right) \cdot \underline{\xi}_i,$$

where $\underline{Q}_j = \underline{K}_j$ for $\mathcal{N}^{aver}_{layer}$ scheme, $\underline{Q}_j = \underline{C}_{j+1}$ for $\mathcal{T}^{aver}_{layer}$ scheme.

For integrals in all expressions for coefficients A_{ij} analytical formulae were derived.

5 Numerical results

Now we compare numerical solutions which can be obtained using developed numerical schemes. We consider 2 test problems for which exact analytical solution for the vortex layer intensity is known.

5.1 Flow around an elliptical airfoil

Results of the vortex layer intensity computation for steady flow around the elliptical airfoil with major and minor semiaxes equal to $a_1 = 1.0$ and $b_1 = 0.1$ for angle of incidence $\beta = \dfrac{\pi}{6}$ for $n = 150$ panels on the airfoil surface are shown on fig. 2. We can see that there is significant difference between exact solution and NVEM scheme solution near ends of ellipse major axis while TVEM scheme solution is closer to exact one for the same number of panels on the airfoil.

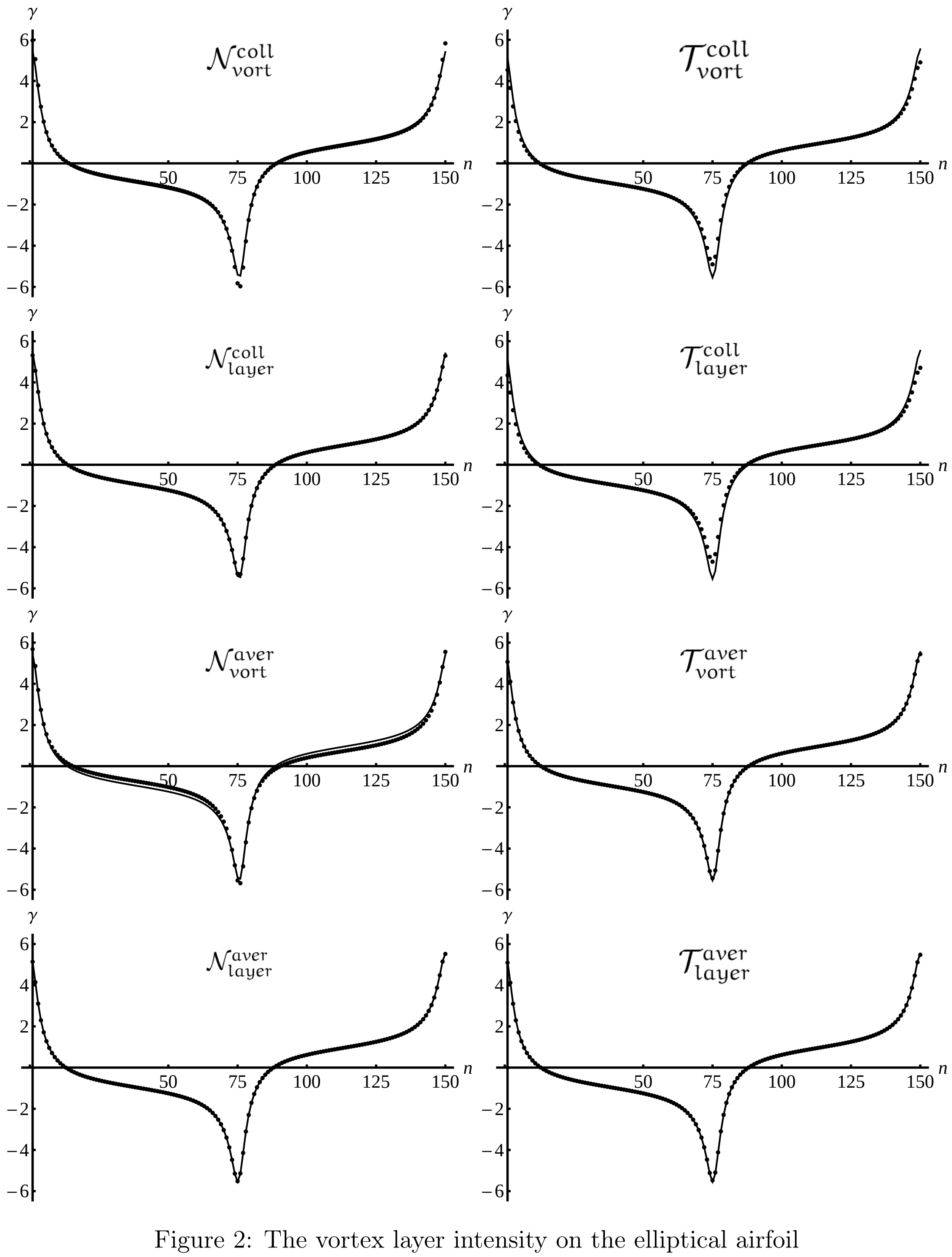

Figure 2: The vortex layer intensity on the elliptical airfoil

5.2 Flow around Zhukovsky airfoil

Results of the vortex layer intensity computation for steady flow around symmetrical Zhukovsky airfoil with thickness ratio 20 % for angle of incidence $\beta = \frac{\pi}{6}$ using the developed numerical schemes are shown on fig. 3. Number of panels on the airfoil is $n = 150$.

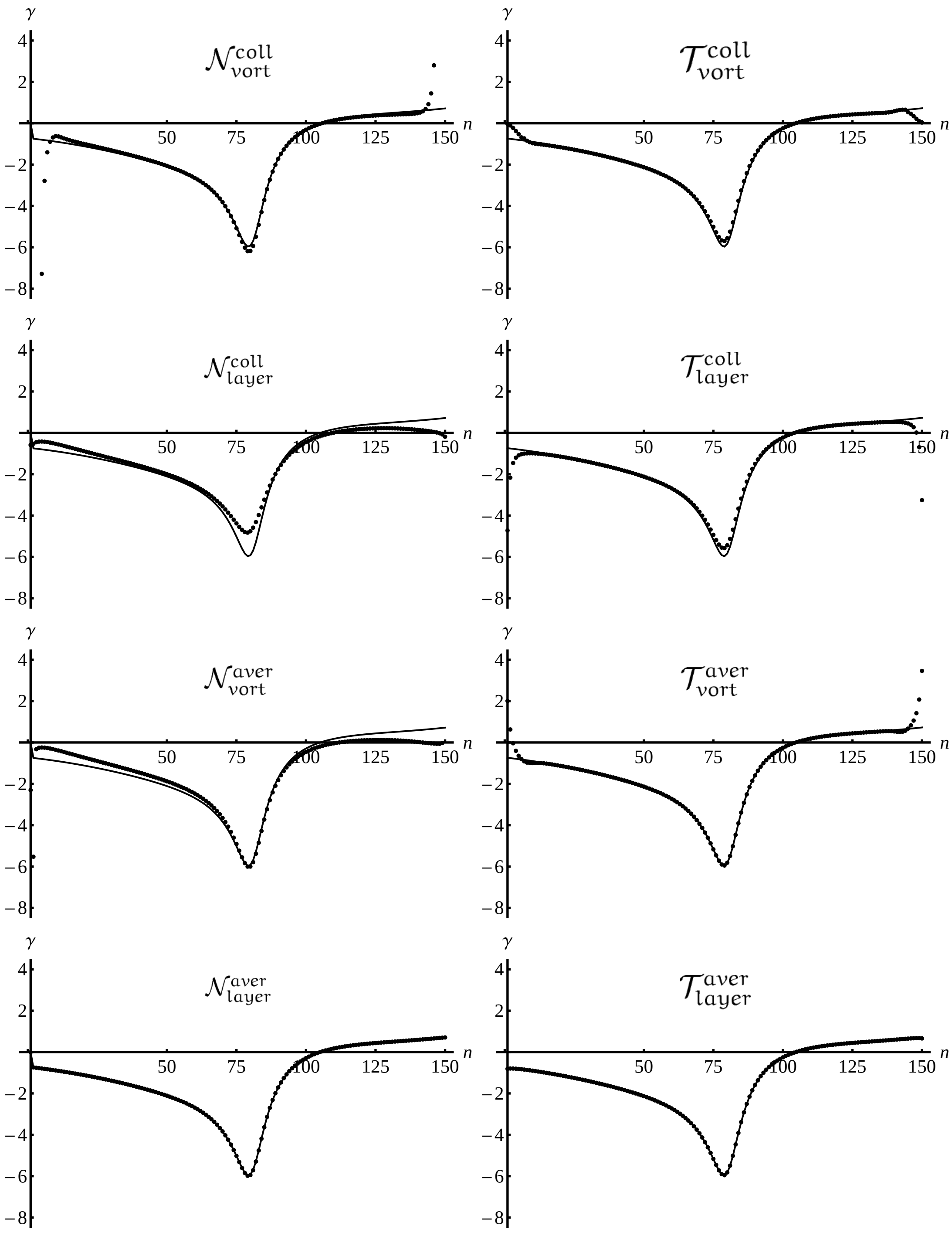

Figure 3: The vortex layer intensity on Zhukovsky airfoil

Results for non-symmetrical Zhukovsky airfoil are nearly the same: $\mathcal{N}$-schemes lead to significant errors near airfoil sharp edge while $\mathcal{T}$-schemes produce solution very close to exact one.

In order to compare accuracies of different schemes we calculate the vortex intensity error for $\mathcal{N}$- and $\mathcal{T}$-schemes, where

$$\|\Delta\|_{\mathcal{N}} = \frac{1}{2}\max_i[(|\gamma_i^0 - \gamma_i|)(L_{i-1} + L_i)], \quad L_0 \equiv L_n,$$

$$\|\Delta\|_{\mathcal{T}} = \max_i[(|\gamma_i^0 - \gamma_i|)L_i].$$

Here γ_i^0 is exact solution (average vortex layer intensity on i-th panel), γ_i is computed value for it, L_i is i-th panel length. Results are shown on Table 19.

$\|\Delta\|_{\mathcal{N}}/\|\Delta\|_{\mathcal{T}}$	Elliptical airfoil	Zhukovsky airfoil
$\mathcal{N}_{vort}^{coll}$ / $\mathcal{T}_{vort}^{coll}$	0.0058/0.0055	4.0815/0.0108
$\mathcal{N}_{layer}^{coll}$ / $\mathcal{T}_{layer}^{coll}$	0.0013/0.0071	0.0398/0.0163
$\mathcal{N}_{vort}^{aver}$ / $\mathcal{T}_{vort}^{aver}$	0.0182/0.0009	0.0544/0.0158
$\mathcal{N}_{layer}^{aver}$ / $\mathcal{T}_{layer}^{aver}$	0.0008/ 0.0006	0.0011/0.0004

Table 19: Errors in test problems for $\mathcal{N}$- and $\mathcal{T}$-schemes

6 Conclusion

The problem of 2D flow numerical simulation around an airfoil is considered. For its solution two approaches are proposed: NVEM (Vortex element method with normal components of velocity on the airfoil surface) and TVEM (Vortex element method with tangential components of velocity). Eight numerical schemes based on these approaches are developed. In order to compare accuracies of these methods the vortex layer intensities were computed for some simplest airfoils with known exact solution (elliptical and Zhukovsky airfoils). In TVEM method piecewise constant function is used to approximate the vortex layer intensity on the airfoil. Results for smooth airfoil (elliptical) are close in both methods. The most accurate results for both airfoils were obtained using $\mathcal{N}_{layer}^{aver}$ and $\mathcal{T}_{layer}^{aver}$ schemes.

All the developed schemes can be used for unsteady flow simulation and also for solving complicated aeroelastic problems.

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