On modeling heterogeneous residual stress

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Abstract

In the present research three groups of the most common models of residual stress (RS) distinguishing by a form of constitutive relation are described. The comparison of models in frames of beam theory is made; the effect of RS on dynamic characteristics of plates is estimated. The inverse problem (IP) on the identification of nonhomogeneous RS in a plate on the basis of acoustical probing is investigated. The technique of solving the IP using the projection method and Airy function approximation by biharmonic polynomials is proposed. The computational experiments on RS reconstruction are conducted and reveal enough efficiency of the method proposed.

1 Introduction

The analysis of nonhomogeneous RS fields in bodies is significant problem of solid mechanics. The interest in the study of this problem has been shown by scientists from different countries from the beginning of XX century. The presence of RS in solids is typical for all real objects. Generally, such stress state arises either during technological processing (such as welding, hardening, heat treatment, rolling-and-pressing) or as a result of loading under elastic or viscoelastic deforming, and it may gain large magnitudes.

At present, methods of RS state diagnostics in different objects are divided into three classes by a way of obtaining experimental data: destructive, semi-destructive and non-destructive [15, 10]. The acoustical probing method is one of the most effective (cost-saving and informative) ways to get additional data about the object of study. Mathematical treatment of acoustical probing data allows to define RS levels and to identify essentially inhomogeneous RS fields.

2 Residual stress models

For today, there are almost no reviews of RS models in the world literature with their comparison analysis and revealing the most appropriate one; at the same time, it is vital to understand which models adequately describe the presence of RS in bodies.

The most common mathematical models of RS were proposed in the second half of XX century by V.V. Novozhilov [8], A.I. Lurie [4], C. Truesdell [11], K. Vasidzu [12], A.N. Guz [2], A. Hoger [3], L. Robertson [9].

2.1 General boundary problem of steady-state vibration of a prestressed elastic body

Consider elastic body in the following three configurations: in the first one κ_1 the body is unstressed; in the second one κ_2 the self-balanced stress field $\underline{\underline{\sigma}}^0 \quad (\nabla \cdot \underline{\underline{\sigma}}^0 = 0)$ is created as a result of some finite strains. Let us term $\underline{\underline{\sigma}}^0$ as RS tensor. Next, by applying small strains in the form of steady-state vibration to the configuration κ_2, we get κ_3. We assume that the body in the third configuration κ_3 occupies the volume V, bounded by the surface $S = S_u \cup S_\sigma$; the body is clamped at the boundary part S_u, and it is loaded by the load changing periodically in time with amplitude P at the boundary part S_σ.

On the basis of the constitutive relations obtained in papers [2, 11, 3] one can easily show that in the case described above the linearized boundary problem of steady-state vibration of a prestressed body takes the following form:

$$\nabla \cdot \underline{\underline{T}} + \rho\omega^2 \underline{u} = 0, \tag{1}$$

$$\underline{\underline{T}} = \underline{\underline{\Gamma}}(\underline{\underline{\sigma}}^0, \nabla\underline{u}) + \underline{\underline{L}}(\underline{\underline{\epsilon}}) \tag{2}$$

$$\underline{u}|_{S_u} = 0, \tag{3}$$

$$\underline{\underline{T}} \cdot n\Big|_{S_\sigma} = \underline{P}, \tag{4}$$

where

$\underline{u}$ – displacement vector with respect to the configuration κ_2,

$\underline{\underline{T}}$ – nonsymmetric Piola stress tensor,

$\underline{\underline{\Gamma}}$ – tensor depending generally on RS tensor components $\underline{\underline{\sigma}}^0$ and strain gradient $\nabla\underline{u}$,

$\underline{\underline{\epsilon}} = \frac{1}{2}\left(\nabla\underline{u} + \nabla\underline{u}^{\mathrm{T}}\right)$ – strain tensor,

$\underline{\underline{L}}$ – tensor defined by material constants and strain tensor [11]. In case of linear elastic body this tensor is set by the classical Hook's law $\underline{\underline{\sigma}} = \underline{\underline{C}} : \underline{\underline{\epsilon}} = \underline{\underline{L}}$, where $\underline{\underline{C}}$ – fourth-rank tensor of elastic modules.

2.2 Three groups of models

In the boundary problem (1)-(4) the Eq. (2) is the constitutive relation binding together the generall Piola stress tensor with RS tensor and tensor of linear elasticity. Note that only one tensor $\underline{\underline{\Gamma}}$ depend on RS components; in fact, this tensor is bilinear operator acting on $\underline{\underline{\sigma}}^0$ and on $\nabla\underline{u}$. The various RS models used at present are distinguished by the structure of the tensor $\underline{\underline{\Gamma}}$.

Let us view three basic groups of RS models:

I. Model proposed by V.V. Novozhilov [8], K. Vasidzu [12], A.N. Guz [2], L. Robertson [9].

$$\underline{\underline{\Gamma}}(\underline{\underline{\sigma}}^0, \nabla\underline{u}) = \nabla\underline{u}^{\mathrm{T}} \cdot \underline{\underline{\sigma}}^0, \tag{5}$$

II. Hoger's model [3];

$$\underline{\underline{\Gamma}}(\underline{\underline{\sigma}}^0, \nabla\underline{u}) = \nabla\underline{u}^{\mathrm{T}} \cdot \underline{\underline{\sigma}}^0 - \frac{1}{2}\left(\underline{\underline{\epsilon}} \cdot \underline{\underline{\sigma}}^0 + \underline{\underline{\sigma}}^0 \cdot \underline{\underline{\epsilon}}\right), \tag{6}$$

III. Truesdell's model [11].

$$\underline{\underline{\Gamma}}(\underline{\underline{\sigma}}^0, \nabla\underline{u}) = \left(\mathrm{tr}\ \underline{\underline{\epsilon}}\right)\underline{\underline{\sigma}}^0 + \underline{\underline{\omega}} \cdot \underline{\underline{\sigma}}^0 - \underline{\underline{\sigma}}^0 \cdot \underline{\underline{\epsilon}}, \tag{7}$$

where $\underline{\underline{\omega}}$ – rotation tensor.

2.3 Comparison analysis of models in frames of uniaxial case

Let us carry out the comparison of the models I, II and III in frames of uniaxial distribuation of RS field in a rod. To do this, we will derive the formulations of boundary problems of longitudinal and bending vibration of pretensioned (or precompressed) cantilever for each model and we will conduct numerical comparison of the models by dynamic characteristics of a plane prestressed rectangelar area.

2.3.1 Boundary problems of longitudinal and bending vibration of prestressed beam for each model

Now let us use the variational principle of Lagrange to derive boundary problems for prestressed beam in frames of each RS model considered. For that view displacement variations satisfying the main boundary conditions: $\delta u_i|_{S_u} = 0$. Then let us use the standard transformation: multiply the Eq. (1) by δu_i, integrate by V and apply the Gauss–Ostrogradski formula, taking into account the boundary conditions (3)-(4). After that let us view longitudinal and bending vibration of a cantilever under axial residual tension or compression (Fig. 1):

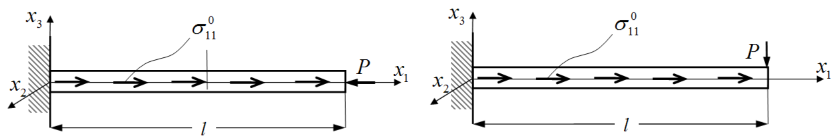

Figure 1

Here we take on hypothesis that the only one nonzero component of the RS tensor is $\sigma_{11}^0(x_1)$. Also let us use hypotheses of longitudinal and bending vibration of a cantilever. In case of longitudinal vibration we have

$$u_1 = u(x_1), \quad u_2 = u_3 = 0, \tag{8}$$

and in bending vibration case

$$u_1 = -x_3 w'(x_1), \quad u_2 = 0, \quad u_3 = w(x_1). \tag{9}$$

By applying the hypotheses(8) and (9) we finally obtain the following boundary problems:

Longitudinal vibration

For the model I we have:

$$\begin{cases} \left[(E + \sigma_{11}^0) F u'\right]' + \rho F \omega^2 u = 0, \\ u(0) = 0, \\ \left[(E + \sigma_{11}^0) F u'\right](l) = P, \end{cases} \tag{10}$$

where F is cross-sectional area of a beam.

In frames of the models II and III the components of the RS tensor are not included in the boundary problems of longitudinal and bending vibration of a cantilever:

$$\begin{cases} \left[E F u'\right]' + \rho F \omega^2 u = 0, \\ u(0) = 0, \\ \left[E F u'\right](l) = P, \end{cases} \tag{11}$$

Bending vibration

For model I:

$$\begin{cases} \left[(E+\sigma_{11}^0)Jw''\right]'' - (F\sigma_{11}^0 w')' + \rho F\omega^2 w = 0, \\ w(0) = w'(0) = 0, \\ \left[(E+\sigma_{11}^0)Jw''\right](l) = 0, \\ \left\{\left[(E+\sigma_{11}^0)Jw''\right]' - F\sigma_{11}^0 w'\right\}(l) = P, \end{cases} \tag{12}$$

where J is a moment of inertia of the beam's cross section.

For model II:

$$\begin{cases} \left[EJw''\right]'' - (F\sigma_{11}^0 w')' + \rho F\omega^2 w = 0, \\ w(0) = w'(0) = 0, \\ \left[EJw''\right](l) = 0, \\ \left\{\left[EJw''\right]' - F\sigma_{11}^0 w'\right\}(l) = P, \end{cases} \tag{13}$$

For model III:

$$\begin{cases} \left[(E+2\sigma_{11}^0)Jw''\right]'' - (F\sigma_{11}^0 w')' + \rho F\omega^2 w = 0, \\ w(0) = w'(0) = 0, \\ \left[(E+2\sigma_{11}^0)Jw''\right](l) = 0, \\ \left\{\left[(E+2\sigma_{11}^0)Jw''\right]' - F\sigma_{11}^0 w'\right\}(l) = P, \end{cases} \tag{14}$$

Remark In real mechanical constructions, RS magnitudes usually change within the range $\frac{\max|\sigma_{11}^0|}{E} = 10^{-5} \div 10^{-3}$. If we neglect the value σ_{11}^0 as compared with the Young modulus E, then we have the formulation of boundary problem on longitudinal vibration (11) and on bending vibration (13) for all the models considered.

2.3.2 Numerical comparison of calculation results for different models

Consider a thin rectangular in in-plane vibration regime as a finite-element model (Fig. 2). The plate parameters are taken as follows: $l = 1\text{m}, b = 0.5\text{m}, h = 0.1\text{m}$ (thickness), $E = 1.96 \cdot 10^{11}\text{Pa}, \nu = 0.28, \rho = 7.8 \cdot 10^3\text{kg/m}^3$ (steel). The plate contains homogeneous RS field defined by RS component $\sigma_{11}^0 = \text{const}$ (the rest components of the RS tensor equal zero).

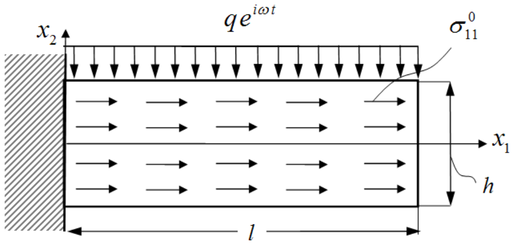

Figure 2

Below the comparison of frequency response functions (FRF) of the plate in the point $\{x = l, y = 0\}$ obtained using finite element method are depicted. In Fig. 3 the branches of FRF are shown for vertical displacement component u_2 below the first resonant frequency

f_1 (values on the y-axis are given in cm for convenience). The parameter σ^0_{11}/E is taken $0.5{\cdot}10^{-2}$. One can see that even such a large value of tensioning RS gives good agreement of FRF branches for all the RS models considered. Note that the lowest resonant frequency for unstressed state is $f_1 = 10.65$Hz, whereas in uniaxial RS case viewed here it is $f_1 = 11.21$Hz (almost the same for all models).

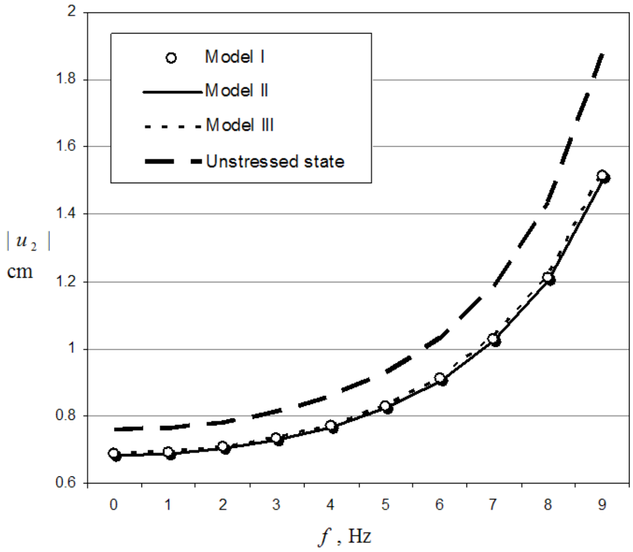

Figure 3

Hence, in the uniaxial case considered all three RS models are close to each other, from the point of view of the effect of RS on FRF.

3 Identification of plane residual stress state in a plate

3.1 Inverse problem statement

Let us take on the problem of the identification of nonhomogeneous RS field $\sigma^0_{xx}(x, y)$, $\sigma^0_{yy}(x, y)$, $\sigma^0_{xy}(x, y)$ contained in a thin plate clamped by one side (Fig. 4)[3]. We shall use the idea of the acoustical probing method of non-destructive inspection: apply periodically changing probing tangent load at the upper plate side and implement its in-plane vibration (in paper[6] various types of probing loads were described; it was revealed that this type gave the most precise results of uniaxial RS identification).

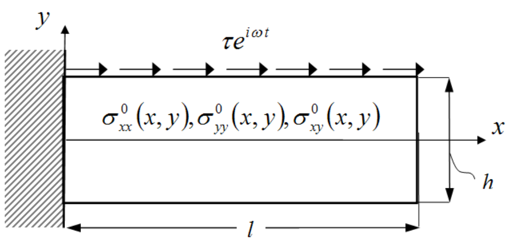

Figure 4

[3]Here and after we shall use the following notation for the axes: x, y

Material parameters (Lamé coefficients and a density), plate dimensions and a load $\underline{\tau}|_{l_\sigma}$ are given. Also assume that we possess an additional information about the displacement field $\underline{u}|_{l_\sigma}$ under the loading in a finite set of vibration frequencies $\omega_k \in [\omega_-, \omega_+]$ simulating the probing data [13, 14].

3.2 Iterative process

Note that the problem on searching the RS field formulated above is scantily explored coefficient IP which is nonlinear and ill-posed problem. In paper [5] in frames of the model I the relation is obtained which allows to carry out an iterative process (for the plate considered above) of determining corrections of RS components with respect to some chosen initial approximation:

$$\int\limits_{\Omega} \left[\sigma_{xx}^{0(n)} K_{xx}^{(n-1)} + \sigma_{yy}^{0(n)} K_{yy}^{(n-1)} + \tau_{xy}^{0(n)} K_{xy}^{(n-1)}\right] d\Omega = F^{(n-1)}, \tag{15}$$

where $\sigma_{xx}^{0(n)}$, $\sigma_{yy}^{0(n)}$, $\sigma_{xy}^{0(n)}$ – corrections to the corresponding RS components at a current iteration (at that $\sigma_{ij}^{0(0)}$ – preselected initial approximation),

$$K_{xx}^{(n-1)} = \left(u_{x,x}^{(n-1)}\right)^2 + \left(u_{y,x}^{(n-1)}\right)^2,$$
$$K_{yy}^{(n-1)} = \left(u_{x,y}^{(n-1)}\right)^2 + \left(u_{y,y}^{(n-1)}\right)^2,$$
$$K_{xy}^{(n-1)} = 2\left(u_{x,x}^{(n-1)} u_{x,y}^{(n-1)} + u_{y,x}^{(n-1)} u_{y,y}^{(n-1)}\right),$$
$$F^{(n-1)} = \int\limits_{l_\sigma} \tau(u_x^{(n-1)} - f_x) dl_\sigma .$$

f_x – component of given displacement field under the probing load (additional information in the IP statement),

$l_\sigma = \{x \in [0, l], y = h/2\}$ – upper plate side under the probing load.

This relation may be treated as the integral Fredholm equation of the first kind with respect to RS corrections, if the direct problem of searching displacement and strain fields in a whole plate region is preliminarily solved (the investigations of different direct problems for rods and plates are carried out in papers [7, 1]).

Let us express the corrections $\sigma_{xx}^{0(n)}$, $\sigma_{yy}^{0(n)}$, $\sigma_{xy}^{0(n)}$ by one Airy stress function $\Phi(x, y)$:

$$\sigma_{xx}^{0(n)} = \Phi_{,yy}, \quad \sigma_{yy}^{0(n)} = \Phi_{,xx}, \quad \sigma_{xy}^{0(n)} = -\Phi_{,xy}. \tag{16}$$

Next, by representing the Airy function as $\Phi = \sum_{k=1}^{N} \alpha_k \psi_k$ (α_k – unknown coefficients of an expansion, ψ_k – basis functions), defining coordinate functions ψ_k and fixing several vibration frequencies $\omega_k \in [\omega_-, \omega_+]$, one can reduce solving of the Eq. (15) to an ill-conditioned system of linear algebraic equations with respect to α_k (for its solving we used the Tikhiniv regularization procedure) [5].

3.3 Computational experiment

In paper [5] the computational experiments on the RS identification are represented in case when the reconstruction was occurred within the same function class in which the exact RS were defined. In the present research more general IP statement is set: it is required to reconstruct RS which are defined in arbitrary class of smooth enough functions in the class of biharmonic polynomials. Results of computational experiments showed that in this case the reconstruction procedure succeeds if the polynomials degree in the Airy function representation does not exceed three:

$$\Phi = \frac{a_2}{2}x^2 + b_2xy + \frac{c_2}{2}y^2 + \frac{a_3}{6}x^3 + \frac{b_3}{2}x^2y + \frac{c_3}{2}xy^2 + \frac{d_3}{6}y^3, \tag{17}$$

that corresponds to linear laws of RS variation:

$$\begin{cases} \sigma^0_{xx} = c_2 + c_3x + d_3y, \\ \sigma^0_{yy} = a_2 + a_3x + b_3y, \\ \sigma^0_{xy} = -b_2 - b_3x - c_3y. \end{cases} \tag{18}$$

In Fig. 5 the computational experiment result is shown for steel plate with the following parameters: $l = 1\text{m}, b = 0.5\text{m}, h = 0.1\text{m}, E = 1.96 \cdot 10^{11}\text{Pa} = 2 \cdot 10^6\text{kg/cm}^2, \nu = 0.28, \rho = 7.8 \cdot 10^3\text{kg/m}^3$. The exact RS functions were defined as follows:

$$\begin{cases} \sigma^{0*}_{xx} = 40\sin(0.03x+1)\exp(0.02y) - 360\sin(0.06y+1), \\ \sigma^{0*}_{yy} = -90\sin(0.03x+1)\exp(0.02y), \\ \sigma^{0*}_{xy} = -60\cos(0.03x+1)\exp(0.02y), \end{cases} \tag{19}$$

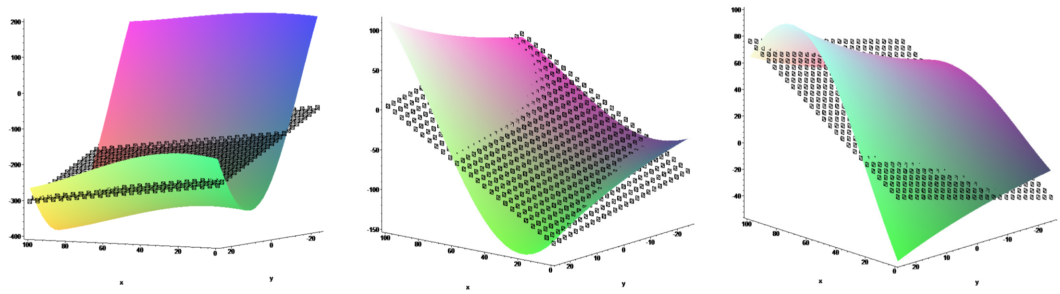

Figure 5: From left to right: σ^0_{xx} (kg/cm^2), σ^0_{yy} (kg/cm^2), σ^0_{xy} (kg/cm^2). Solid surface – exact law, points – reconstruction. Scales x, y are given in cm. Number of iterations: 7

The reconstructed RS function have the form (18) with the following values $a_2 = -106.692, b_2 = 22.4383, c_2 = -148.192, a_3 = 1.31656, b_3 = -0.813834, c_3 = -0.690735, d_3 = -3.42323$.

Computational experiments were conducted for a frequency range lying between the first and the second resonant frequencies (in total 7 frequencies uniformly distributed in the range selected were fixed); it is revealed that range limits should be selected quite close to the resonances to obtain high accuracy of the reconstruction. Also we should note that investigation of the IP in the range lying below the first resonance gives a reconstruction of a worse accuracy.

Thus, the computational experiment results demonstrate a possibility of employing the scheme proposed to identify smooth enough nonhomogeneous RS components in the class of linear functions.

4 Conclusion

The theoretical RS models used at present are described. In frames of each models considered the comparison of boundary problems of longitudinal and bending vibration of a prestressed cantilever is made. Using the finite element method the numerical comparison of FRFs of a plate is done for different models. The IP of an identification of nonhomogeneous RS components in a plate using acoustical probing and registration of FRFs in some frequency range is investigated. The iterative process of solving the IP is built; a way of solving the IP-equation is proposed on the basis of the projection method and Airy function approximation by biharmonic polynomials. The computational experiments on a reconstruction of smooth enough RS from arbitrary function class in the class of linear functions are conducted; the experiment results showed enough efficiency of the algorithm suggested.

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