Field description of rotational motion

Marlen E. Podolsky
ttt88@mail.ru

Abstract

Two ways of description of stable rotational motion and the examples of their simple applications are considered. The first way is based on investigation of nonstationary case. According to the second one Euler's field approach is used. In the last case the procedure of differentiation with respect to vector elaborated by author was used.

1 Introduction

The typical feature of rotational motion is transition from stationary state in immovable space to nonstationary one in rotating body. For example let us consider toothed gear rotation. If the tooth system is involute and the load on the gear is constant, the force in engagement is constant too and, consequently, it does not depend on time. In spite of this the load on the individual tooth varies, because every tooth is loaded only at that moment when it engages with the tooth of another gear. From here it follows that as the result of rotation the load on the elements of moving body depends on time. In other words the problem appears to be nonstationary.

One more example is stress state of rotating shaft, which is bent by the moment, determined as a constant vector in immovable space. It is obvious that under such conditions the stresses in all fibres are not stationary again. All these facts are well-known, but they suggest that the way of problems solution, similar to the ones mentioned above, may be based on formal definition of field, according to which field is not considered as a part of moving body. In such interpretation, if a body appears to be in the field, it receives those characteristics which are determined in the corresponding part of space according to field description.

2 The acceleration of point

Let the point be rigidly coupled with the shaft, and let us suppose that the shaft rotates around its axis with constant angular velocity. Then the magnitude u of the velocity of a point is constant. But because of the fact that angular position of vector $\underline{u}$ varies together with tangent to the circle, $\underline{u}$ is the function of t, and acceleration $\underline{a} \neq \underline{0}$, even though $\underline{\omega} = \underline{const}$.

In order to find $\underline{a}$ two ways can be used. The first one is essentially nonstationary. Bearing in mind that position of a point is determined by the radius-vector $\underline{r}$, which is rigidly coupled with the shaft, we obtain ($\underline{\omega}$ is supposed to be constant)

$$\underline{u} = \frac{d\underline{r}}{dt} = \underline{\omega} \times \underline{r}, \qquad \underline{a} = \frac{d\underline{u}}{dt} = \underline{\omega} \times \underline{u}. \tag{1}$$

The second way uses Euler's approach and is based on consideration of immovable space and on determination of the vector field $\underline{f}$ in it. During body rotation, points of the body coincide with corresponding points of immovable space, and vectors $\underline{f}$ in these points will be equal to those ones, which are defined by equations of field. If the field is stationary, all vectors of this field are functions of radius-vector $\underline{r}$ only. It means that differential $d\underline{f}$ is equal to scalar product of $d\underline{r}$ and $\frac{d\underline{f}}{d\underline{r}}$:

$$d\underline{f} = d\underline{r} \bullet \frac{d\underline{f}}{d\underline{r}}. \tag{2}$$

During the body motion the point of the body which has velocity $\underline{u}$ appears to be in the point with radius-vector $\underline{r}(t + dt) = \underline{r}(t) + \underline{u}dt$.

Therefore

$$d\underline{r} = \underline{u}dt \tag{3}$$

and formulae (2) and (3) yield

$$\frac{d\underline{f}}{dt} = \underline{u} \bullet \frac{d\underline{f}}{d\underline{r}}. \tag{4}$$

Let $\underline{f} = \underline{u}$, $\underline{u} = \underline{\omega} \times \underline{r}$, $\underline{\omega} \neq \underline{\omega}(t)$. Then from (1) and (4) taking into account that $\underline{\omega} \neq \underline{\omega}(\underline{r})$ it follows that

$$\underline{a} = \underline{u} \bullet \frac{d}{d\underline{r}}(\underline{\omega} \times \underline{r}) = \underline{u} \bullet (-\underline{\underline{E}} \times \underline{\omega}) = -\underline{u} \times \underline{\omega} = \underline{\omega} \times \underline{u}. \tag{5}$$

Here and further formulae from Appendix are used.

Formula (5) is the same, as (1_2).

3 The moment of inertia forces

The moment of inertia forces of rigid body, rotating around the fixed center, is equal to

$$\underline{M} = -\int_m \underline{r} \times \underline{a} dm. \tag{6}$$

Here $\underline{r}$ is radius-vector of the point with mass dm.

Bearing in mind (1_1) and (1_2) and taking into account, that $\frac{d\underline{r}}{dt} \times \underline{u} = \underline{0}$, we obtain from (6)

$$\underline{M} = -\int_m \frac{d}{dt}(\underline{r} \times \underline{u}) dm. \tag{7}$$

For every vector $\underline{A}$, rigidly coupled with rotating body,

$$\frac{d\underline{A}}{dt} = \underline{\omega} \times \underline{A}, \tag{8}$$

where as above $\underline{\omega}$ is angular velocity.

Using formula (1_1) and formulae from Appendix, we obtain

$$\underline{r} \times \underline{u} = \underline{\omega} r^2 - \underline{r}(\underline{\omega} \bullet \underline{r}) = \underline{\omega} \bullet (r^2 \underline{\underline{E}} - \underline{r} \otimes \underline{r}), \tag{9}$$

$$\underline{\omega} \times (\underline{r} \times \underline{u}) = -(\underline{r} \times \underline{u}) \times \underline{\omega} = -\underline{\omega} \bullet (r^2 \underline{\underline{E}} - \underline{r} \otimes \underline{r}) \times \underline{\omega}. \tag{10}$$

From (7), (8) and (10) it follows that

$$\underline{M} = \underline{\omega} \bullet \underline{\underline{J}} \times \underline{\omega}. \tag{11}$$

Here

$$\underline{\underline{J}} = \int_m (r^2 \underline{\underline{E}} - \underline{r} \otimes \underline{r}) dm \tag{12}$$

is inertia tensor.

The second way of formula (11) receiving is field description. In this case, if we set $\underline{f} = \underline{r} \times \underline{u}$, formula (4) yields

$$\frac{d(\underline{r} \times \underline{u})}{dt} = \underline{u} \bullet \frac{d(\underline{r} \times \underline{u})}{d\underline{r}}.$$

Further taking into account (1_1) and bearing again in mind that $\underline{\omega}$ does not depend on $\underline{r}$, we have

$$\frac{d}{d\underline{r}}(\underline{r} \times \underline{u}) = \underline{\underline{E}} \times \underline{u} - \frac{d\underline{u}}{d\underline{r}} \times \underline{r}, \qquad \frac{d\underline{u}}{d\underline{r}} = -\underline{\underline{E}} \times \underline{\omega},$$

$$\underline{u} \bullet \frac{d}{d\underline{r}}(\underline{r} \times \underline{u}) = \underline{u} \bullet (\underline{\underline{E}} \times \underline{u}) - \underline{u} \bullet (-\underline{\underline{E}} \times \underline{\omega}) \times \underline{r} = \underline{u} \times \underline{u} + (\underline{u} \times \underline{\omega}) \times \underline{r}.$$

But

$$(\underline{u} \times \underline{\omega}) \times \underline{r} = \underline{u} \times (\underline{\omega} \times \underline{r}) - \underline{\omega} \times (\underline{u} \times \underline{r}) = \underline{\omega} \times (\underline{r} \times \underline{u})$$

and (see (10))

$$\frac{d(\underline{r} \times \underline{u})}{dt} = \underline{u} \bullet \frac{d}{d\underline{r}}(\underline{r} \times \underline{u}) = -\underline{\omega} \bullet (r^2 \underline{\underline{E}} - \underline{r} \otimes \underline{r}) \times \underline{\omega}.$$

From here and (7) we obtain formula (11) again.

4 Compound motion of point

Let us consider rotation of rigid body around the fixed center and motion of the point which moves with velocity $\underline{w}$ relative to the body. Absolute velocity of the point is

$$\underline{v} = \underline{u} + \underline{w}, \qquad \underline{u} = \underline{\omega} \times \underline{r}. \tag{13}$$

Using formula (4), replacing in it $\underline{u}$ by $\underline{v}$, $\underline{f}$ by $\underline{v}$ and considering steady motion we receive

$$\underline{a} = \underline{v} \bullet \frac{d\underline{v}}{d\underline{r}} = \underline{u} \bullet \frac{d\underline{u}}{d\underline{r}} + \underline{u} \bullet \frac{d\underline{w}}{d\underline{r}} + \underline{w} \bullet \frac{d\underline{u}}{d\underline{r}} + \underline{w} \bullet \frac{d\underline{w}}{d\underline{r}}. \tag{14}$$

By analogy with (5)

$$\underline{w} \bullet \frac{d\underline{u}}{d\underline{r}} = \underline{w} \bullet (-\underline{\underline{E}} \times \underline{\omega}) = -\underline{w} \times \underline{\omega} = \underline{\omega} \times \underline{w}. \tag{15}$$

Formulae (5), (14) and (15) yield

$$\underline{a} = \underline{\omega} \times \underline{u} + \underline{\omega} \times \underline{w} + \underline{u} \bullet \frac{d\underline{w}}{d\underline{r}} + \underline{w} \bullet \frac{d\underline{w}}{d\underline{r}}. \tag{16}$$

The sense of the first term of (16) is bulk acceleration $\underline{a}_e$ (see (1_2)), the sense of the last one is relative acceleration $\underline{a}_r$ (according to (16) and (13_2) $\underline{a} = \underline{w} \bullet \frac{d\underline{w}}{d\underline{r}}$, if $\underline{\omega} = \underline{0}$).

Let us consider the third term. It may be rewritten in the form

$$\underline{u} \bullet \frac{d\underline{w}}{d\underline{r}} = \underline{u} \bullet (\underline{\nabla} \otimes \underline{w}) = (\underline{u} \bullet \underline{\nabla})\underline{w}. \tag{17}$$

Using cylindrical coordinates ρ, φ, z, we have

$$\underline{u} = \omega\rho\underline{e}_\varphi, \qquad \underline{w} = w_\rho\underline{e}_\rho + w_\varphi\underline{e}_\varphi + w_z\underline{e}_z, \qquad \underline{\omega} = \omega\underline{e}_z, \tag{18}$$

$$\underline{\nabla} = \underline{e}_\rho\frac{\partial}{\partial\rho} + \underline{e}_\varphi\frac{\partial}{\rho\partial\varphi} + \underline{e}_z\frac{\partial}{\partial z}, \tag{19}$$

$$\underline{u} \bullet \underline{\nabla} = \omega\rho\underline{e}_\varphi \bullet \underline{\nabla} = \omega\frac{\partial}{\partial\varphi}, \qquad \underline{u} \bullet \frac{d\underline{w}}{d\underline{r}} = \omega\frac{\partial\underline{w}}{\partial\varphi}. \tag{20}$$

Here $\underline{e}_\rho$, $\underline{e}_\varphi$, $\underline{e}_z$ are unit vectors, besides

$$\underline{e}_\rho \times \underline{e}_\varphi = \underline{e}_z, \qquad \underline{e}_\varphi \times \underline{e}_z = \underline{e}_\rho, \qquad \underline{e}_z \times \underline{e}_\rho = \underline{e}_\varphi. \tag{21}$$

Components w_ρ, w_φ, w_z in formula (18_2) do not depend on φ. This assumption means that vector $\underline{w}$, being function of $\underline{r}$, turns with $\underline{r}$ at the same angle.

Taking this circumstance into account we can write

$$\frac{\partial\underline{w}}{\partial\varphi} = w_\rho\frac{\partial\underline{e}_\rho}{\partial\varphi} + w_\varphi\frac{\partial\underline{e}_\varphi}{\partial\varphi} + w_z\frac{\partial\underline{e}_z}{\partial\varphi}. \tag{22}$$

Bearing in mind that $\underline{e}_z$ is unit vector for the axis of turn at the angle φ, one can receive (see formula (A52) in Appendix)

$$\frac{\partial\underline{e}}{\partial\varphi} = \underline{e}_z \times \underline{e}, \tag{23}$$

where $\underline{e}$ is an arbitrary unit vector.

Therefore

$$\frac{\partial\underline{e}_\rho}{\partial\varphi} = \underline{e}_z \times \underline{e}_\rho, \qquad \frac{\partial\underline{e}_\varphi}{\partial\varphi} = \underline{e}_z \times \underline{e}_\varphi, \qquad \frac{\partial\underline{e}_z}{\partial\varphi} = \underline{e}_z \times \underline{e}_z, \tag{24}$$

and from (20_2), (22) and (18) it follows that

$$\underline{u} \bullet \frac{d\underline{w}}{d\underline{r}} = \underline{\omega} \times \underline{w}. \tag{25}$$

Formulae (16) and (25) yield

$$\underline{a} = \underline{\omega} \times \underline{u} + \underline{w} \bullet \frac{d\underline{w}}{d\underline{r}} + 2\underline{\omega} \times \underline{w}. \tag{26}$$

Together with the results and interpretations received above formula (26) is in complete agreement with well-known formula, which takes Coriolis acceleration into account.

One more way of receiving of formula (25) is based on finding the derivative $\frac{d\underline{w}}{d\underline{r}}$.

According to (18_2)

$$\frac{d\underline{w}}{d\underline{r}} = \frac{dw_\rho}{d\underline{r}} \otimes \underline{e}_\rho + \frac{dw_\varphi}{d\underline{r}} \otimes \underline{e}_\varphi + \frac{dw_z}{d\underline{r}} \otimes \underline{e}_z + w_\rho\frac{d\underline{e}_\rho}{d\underline{r}} + w_\varphi\frac{d\underline{e}_\varphi}{d\underline{r}} + w_z\frac{d\underline{e}_z}{d\underline{r}}. \tag{27}$$

Below following notation will be used

$$\underline{e}_\rho = \underline{\rho}^o, \quad \underline{\rho} = \rho\underline{\rho}^o, \quad \underline{e}_z = \underline{e}_3 = \underline{k}, \quad z = x_3, \quad \underline{r} = \underline{\rho} + x_3\underline{e}_3, \quad \underline{e}_\varphi = \underline{k} \times \underline{\rho}^o. \tag{28}$$

Here $\underline{r}$ is radius-vector .

Because $\underline{k}$ does not depend on $\underline{r}$, from (28_6) it follows that

$$\frac{d\underline{e}_\varphi}{d\underline{r}} = -\frac{d\underline{\rho}^o}{d\underline{r}} \times \underline{k}. \tag{29}$$

Derivative $\dfrac{d\underline{\rho}^o}{d\underline{r}}$ is equal to

$$\frac{d}{d\underline{r}}\left(\frac{\underline{\rho}}{\rho}\right) = \frac{d}{d\underline{r}}\left(\frac{1}{\rho}\right) \otimes \underline{\rho} + \frac{1}{\rho}\frac{d\underline{\rho}}{d\underline{r}}. \tag{30}$$

Let us find $\dfrac{d\underline{\rho}}{d\underline{r}}$. According to ($28_5$)

$$\frac{d\underline{\rho}}{d\underline{r}} = \underline{\underline{E}} - \frac{dx_3}{d\underline{r}} \otimes \underline{e}_3. \tag{31}$$

But

$$\frac{dx_3}{d\underline{r}} = \underline{\nabla} x_3 = \left(\underline{e}_i \frac{\partial}{\partial x_i}\right) x_3 = \underline{e}_i \delta_{3i} = \underline{e}_3 \tag{32}$$

and formula (31) yields

$$\frac{d\underline{\rho}}{d\underline{r}} = \underline{\underline{E}} - \underline{e}_3 \otimes \underline{e}_3. \tag{33}$$

Further we have

$$\frac{d}{d\underline{r}}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}\frac{d\rho}{d\underline{r}} \tag{34}$$

and

$$\frac{d\rho^2}{d\underline{r}} = \frac{d(\underline{\rho} \bullet \underline{\rho})}{d\underline{r}} \Rightarrow \rho\frac{d\rho}{d\underline{r}} = \underline{\rho} \bullet \frac{d\underline{\rho}}{d\underline{r}}. \tag{35}$$

From here taking (33) into account and bearing in mind that $\underline{\rho} \bullet \underline{e}_3 = 0$, we obtain

$$\frac{d\rho}{d\underline{r}} = \frac{\underline{\rho}}{\rho} = \underline{\rho}^o \tag{36}$$

and (see (34))

$$\frac{d}{d\underline{r}}\left(\frac{1}{\rho}\right) \otimes \underline{\rho} = -\frac{1}{\rho^2}\underline{\rho}^o \otimes \underline{\rho}. \tag{37}$$

Insertion of (37) and (33) into (30) and (29) gives

$$\frac{d\underline{\rho}^o}{d\underline{r}} = -\frac{1}{\rho}(\underline{\rho}^o \otimes \underline{\rho}^o - \underline{\underline{E}} + \underline{e}_3 \otimes \underline{e}_3), \tag{38}$$

$$\frac{d\underline{e}_\varphi}{d\underline{r}} = \frac{1}{\rho}(\underline{\rho}^o \otimes \underline{\rho}^o - \underline{\underline{E}}) \times \underline{k} \tag{39}$$

(product $(\underline{e}_3 \otimes \underline{e}_3) \times \underline{k} = \underline{\underline{0}}$).

Because of $\underline{e}_\varphi \bullet \underline{\rho}^o = 0$, $\underline{e}_\varphi \bullet \underline{e}_3 = 0$, $\underline{e}_\varphi \bullet \underline{\underline{E}} = \underline{e}_\varphi$, $\underline{e}_\varphi = \underline{k} \times \underline{\rho}$, $\underline{e}_\rho = \underline{\rho}^o$, we find

$$\omega\rho\underline{e}_\varphi \bullet \left(w_\rho \frac{d\underline{e}_\rho}{d\underline{r}} + w_\varphi \frac{d\underline{e}_\varphi}{d\underline{r}} \right) = \omega \left(w_\rho \underline{e}_\varphi - w_\varphi \underline{e}_\varphi \times \underline{k} \right) = \omega \underline{k} \times \left(w_\rho \underline{\rho}^o + w_\varphi \underline{e}_\varphi \right). \tag{40}$$

For the product $\underline{u} \bullet \dfrac{d\underline{e}_z}{d\underline{r}}$ it is more convenient to use formula

$$\underline{u} \bullet \frac{d\underline{k}}{d\underline{r}} = \omega \underline{k} \times \underline{k},$$

analogous to general formulae (20_2) and (24).

From here taking into account (18_2) and (40) we obtain

$$\omega\rho\underline{e}_\varphi \bullet \left(w_\rho \frac{d\underline{e}_\rho}{d\underline{r}} + w_\varphi \frac{d\underline{e}_\varphi}{d\underline{r}} + w_z \frac{d\underline{e}_z}{d\underline{r}} \right) = \omega \underline{k} \times \underline{w} = \underline{\omega} \times \underline{w}. \tag{41}$$

In order to find the sum of first three terms in (27) let us use the independence of w_ρ, w_φ and w_z from φ.

Taking this circumstance into account we must consider function $F = F(\rho, z)$ and receive the formula for derivative of $F = F(\rho, z)$ with respect to the $\underline{r}$.

Because

$$dF = \frac{\partial F}{\partial \rho} d\rho + \frac{\partial F}{\partial z} dz$$

and, according to definition of derivative with respect to the vector,

$$d\rho = d\underline{r} \bullet \frac{d\rho}{d\underline{r}}, \qquad dz = d\underline{r} \bullet \frac{dz}{d\underline{r}},$$

one can write

$$dF = d\underline{r} \bullet \left(\frac{\partial F}{\partial \rho} \frac{d\rho}{d\underline{r}} + \frac{\partial F}{\partial z} \frac{dz}{d\underline{r}} \right),$$

so that

$$\frac{dF}{d\underline{r}} = \frac{\partial F}{\partial \rho} \frac{d\rho}{d\underline{r}} + \frac{\partial F}{\partial z} \frac{dz}{d\underline{r}}. \tag{42}$$

From (42), (36), (28_4) and (32) it follows that if F depends on ρ and z only, product

$$\underline{e}_\varphi \bullet \frac{dF}{d\underline{r}} = 0,$$

and according to (27) and (41) we receive formula (26) again.

5 Appendix. Short information about vector and tensor analysis

The main formulae and notation are the following (summation convention is used).

$$\underline{f} = f_i \underline{e}_i, \qquad \underline{\underline{\alpha}} = \alpha_{ij} \underline{e}_i \otimes \underline{e}_j, \qquad \underline{\underline{E}} = \underline{e}_i \otimes \underline{e}_i; \tag{A43}$$

$$\underline{a} \times (\underline{b} \times c) = \underline{b}(\underline{a} \bullet \underline{c}) - \underline{c}(\underline{a} \bullet \underline{b}), \qquad \underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \times \underline{c} + \underline{b} \times (\underline{a} \times \underline{c}); \qquad \text{(A44)}$$

$$\underline{\underline{E}} \bullet \underline{\underline{A}} = \underline{\underline{A}} \bullet \underline{\underline{E}}, \qquad \underline{a} \bullet (\underline{b} \otimes \underline{c}) = (\underline{a} \bullet \underline{b})\underline{c}, \qquad \underline{a} \bullet (\underline{\underline{E}} \times \underline{b}) = (\underline{a} \times \underline{b}); \qquad \text{(A45)}$$

$$\frac{d\underline{e}}{dt} = \underline{\omega} \times \underline{e}, \qquad d\underline{f} = d\underline{r} \bullet \frac{d\underline{f}}{d\underline{r}} \quad (\underline{f} = \underline{f}(\underline{r})), \qquad d\alpha = d\underline{r} \bullet \frac{d\alpha}{d\underline{r}} \quad (\alpha = \alpha(\underline{r})); \qquad \text{(A46)}$$

$$\frac{d}{d\underline{r}} = \underline{\nabla}, \qquad \frac{d\underline{A}}{d\underline{r}} = \underline{\nabla} \otimes \underline{A}, \qquad \frac{d\underline{r}}{d\underline{r}} = \underline{\underline{E}}, \qquad \underline{\nabla} = \underline{e}_i \frac{\partial}{\partial x_i}; \qquad \text{(A47)}$$

$$\frac{d(\alpha \underline{a})}{d\underline{r}} = \frac{d\alpha}{d\underline{r}} \otimes \underline{a} + \alpha \frac{d\underline{a}}{d\underline{r}}, \qquad \frac{d(\underline{a} \times \underline{b})}{d\underline{r}} = \frac{d\underline{a}}{d\underline{r}} \times \underline{b} - \underline{b} \times \frac{d\underline{a}}{d\underline{r}}; \qquad \text{(A48)}$$

$$\underline{\underline{\varphi}} = cos\varphi \underline{\underline{E}} + (1 - \cos\varphi)\underline{\varphi}^o \otimes \underline{\varphi}^o - \sin\varphi \underline{\underline{E}} \times \underline{\varphi}^o. \qquad \text{(A49)}$$

Here α, φ are scalars; $\underline{a}$, $\underline{b}$, ... are vectors; $\underline{\underline{\alpha}}$, $\underline{\underline{\varphi}}$, ... are tensors; $\underline{\underline{E}}$ is unit tensor; $\underline{e}$ is unit vector; $\underline{\omega}$ is angular velocity vector; $\underline{\nabla}$ is Hamilton's operator. By formula (A49) tensor of turn is defined (φ and $\underline{\varphi}^o$ are angle and vector of turn).

If a vector $\underline{a}$ is turning about the fixed point on the immovable axis, new position of vector $\underline{a}$ is given by formula $\underline{a}(\Delta\varphi) = \underline{a} \bullet \underline{\underline{\varphi}}(\Delta\varphi, \underline{\varphi}^o)$. Let $\underline{\varphi}^o = \underline{k}$ and $\Delta\varphi \to 0$. Then according to (A49)

$$\frac{d\underline{a}}{d\varphi} = \underline{k} \times \underline{a}. \qquad \text{(A50)}$$

Every unit vector $\underline{e}$ may be written in the form

$$\underline{e} = \underline{b} - \underline{a}, \qquad \text{(A51)}$$

where the origins of $\underline{b}$ and $\underline{a}$ are the same.

From (A50) and (A51) it follows that

$$\frac{d\underline{e}}{d\varphi} = \underline{k} \times \underline{e}. \qquad \text{(A52)}$$

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Marlen E. Podolsky, 3, Lotsmanskaya Str., Saint Petersburg, 190008, Russia