# Coriolis inertia forces in the problem of Euler's turbine equation

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**Abstract**

Motion of the fluid pumped through the canal, which rotates around fixed point, is considered. Formula for the moment of forces about this point is received. It is shown that the well-known Euler's turbine equation is a special case of general formula, obtained in this work. The role of Coriolis inertia forces is discovered and it is shown that in general case these forces influence the moment of forces, acting on the fluid, by two ways, but in the plane problem one of these ways is absent. Physical sense of Euler's turbine equation is clarified.

Let us consider a canal, rotating about immovable point. The canal is supposed to be filled by the fluid which moves through the canal. We must find the moment of forces, acting on the fluid in the canal. It is determined by formula

$$\underline{M} = \int_m \underline{r} \times \underline{a} dm, \tag{1}$$

where

$$\underline{a} = \underline{a}_e + \underline{a}_r + \underline{a}_c, \qquad \underline{a}_e = \frac{d\underline{u}}{dt}, \qquad \underline{a}_r = \frac{d\underline{w}}{dt}, \qquad \underline{a}_c = 2\underline{\omega} \times \underline{w}, \qquad \underline{u} = \underline{\omega} \times \underline{r}. \tag{2}$$

Here $\underline{a}_e$, $\underline{a}_r$ and $\underline{a}_c$ are bulk, relative and Coriolis accelerations; $\underline{u}$ and $\underline{w}$ are bulk and relative velocities; $\underline{\omega}$ is angular velocity; $\underline{r}$ is radius-vector.

Consider next the stationary problem. In this case

$$\underline{u} = \underline{const}, \qquad \underline{w} = \underline{w}(\underline{r}), \qquad \underline{a}_e = \underline{\omega} \times \underline{u}, \qquad \underline{a}_r = \underline{w} \bullet \frac{d\underline{w}}{d\underline{r}}. \tag{3}$$

Formula ($3_4$) defines $\underline{a}_r$ as convective acceleration.

From (1)–(3) it follows that ($\rho$ is density)

$$\underline{M} = \int_V \underline{r} \times (\underline{\omega} \times \underline{u})\rho dV + \int_V \underline{r} \times (\underline{w} \bullet \frac{d\underline{w}}{d\underline{r}})\rho dV + \int_V \underline{r} \times (2\underline{\omega} \times \underline{w})\rho dV. \tag{4}$$

Taking the identity

$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \times \underline{c} + \underline{b} \times (\underline{a} \times c) \tag{5}$$

and formula ($2_5$) into account, we can write

$$\underline{r} \times (\underline{\omega} \times \underline{u}) = (\underline{r} \times \underline{\omega}) \times \underline{u} + \underline{\omega} \times (\underline{r} \times \underline{u}) = \underline{\omega} \times \left[\underline{r} \times (\underline{\omega} \times \underline{r})\right]. \tag{6}$$

By virtue of

$$\underline{r} \times (\underline{\omega} \times \underline{r}) = \underline{\omega}(\underline{r} \bullet \underline{r}) - \underline{r}(\underline{\omega} \bullet \underline{r}) = \underline{\omega} \bullet (r^2 \underline{\underline{E}} - \underline{r} \otimes \underline{r}) \tag{7}$$

and formula (6), for the first term in (4) we have

$$\int_V \underline{r} \times (\underline{\omega} \times \underline{u}) \rho dV = -\underline{\omega} \bullet \underline{\underline{J}} \times \underline{\omega}. \tag{8}$$

Here

$$\underline{\underline{J}} = \int_V (r^2 \underline{\underline{E}} - \underline{r} \otimes \underline{r}) \rho dV \tag{9}$$

is inertia tensor. $\underline{\underline{E}}$ in (7) is unit tensor. Short information about tensor calculus can be found in [1].

Further we have

$$\frac{d(\underline{r} \times \underline{w})}{d\underline{r}} = \underline{\underline{E}} \times \underline{w} - \frac{d\underline{w}}{d\underline{r}} \times \underline{r} \tag{10}$$

and consequently

$$\underline{w} \bullet \frac{d(\underline{r} \times \underline{w})}{d\underline{r}} = \underline{w} \bullet (\underline{\underline{E}} \times \underline{w}) - \underline{w} \bullet \frac{d\underline{w}}{d\underline{r}} \times \underline{r} = \underline{w} \times \underline{w} - \underline{w} \bullet \frac{d\underline{w}}{d\underline{r}} \times \underline{r}, \tag{11}$$

so that

$$\underline{w} \bullet \frac{d\underline{w}}{d\underline{r}} \times \underline{r} = -\underline{w} \bullet \frac{d(\underline{r} \times \underline{w})}{d\underline{r}}. \tag{12}$$

But

$$\underline{w} \bullet \frac{d\underline{w}}{d\underline{r}} \times \underline{r} = \left( \underline{w} \bullet \frac{d\underline{w}}{d\underline{r}} \right) \times \underline{r} = -\underline{r} \times \left( \underline{w} \bullet \frac{d\underline{w}}{d\underline{r}} \right). \tag{13}$$

Formulae (13) and (12) yield

$$\underline{r} \times \left( \underline{w} \bullet \frac{d\underline{w}}{d\underline{r}} \right) = \underline{w} \bullet \frac{d(\underline{r} \times \underline{w})}{d\underline{r}}. \tag{14}$$

Bearing in mind that

$$\frac{d}{d\underline{r}} = \underline{\nabla} \tag{15}$$

and taking formula (14) into account we can write

$$\int_V \underline{r} \times \left( \underline{w} \bullet \frac{d\underline{w}}{d\underline{r}} \right) \rho dV = \int_V \underline{w} \bullet \frac{d(\underline{r} \times \underline{w})}{d\underline{r}} \rho dV = \int_V \underline{w} \bullet \underline{\nabla} \otimes (\underline{r} \times \underline{w}) \rho dV. \tag{16}$$

For simplicity let us suppose $\rho$ to be constant. Then according to the law of mass conservation

$$\underline{\nabla} \bullet \underline{w} = 0. \tag{17}$$

Therefore

$$\underline{w} \bullet \underline{\nabla} \otimes (\underline{r} \times \underline{w}) = \underline{\nabla} \bullet \left[ \underline{w} \otimes (\underline{r} \times \underline{w}) \right], \tag{18}$$

and using tensor analog of Gauss-Ostrogradskii's formula we come to the following form of the second term in formula (4) :

$$\int_V \underline{r} \times \left( \underline{w} \bullet \frac{d\underline{w}}{d\underline{r}} \right) \rho dV = \oint_S \underline{r} \times \underline{w} \rho w_n dS, \qquad w_n = \underline{w} \bullet \underline{n}. \tag{19}$$

Now we must transform the last term in (4).

According to identity (5)

$$\underline{r} \times (\underline{\omega} \times \underline{w}) = (\underline{r} \times \underline{\omega}) \times \underline{w} + \underline{\omega} \times (\underline{r} \times \underline{w}) = -\underline{u} \times \underline{w} + \underline{\omega} \times (\underline{r} \times \underline{w}). \tag{20}$$

Let us consider the product

$$\underline{w} \bullet \frac{d(\underline{r} \times \underline{u})}{d\underline{r}} = \underline{w} \bullet \left( \underline{\underline{E}} \times \underline{u} - \frac{d\underline{u}}{d\underline{r}} \times \underline{r} \right). \tag{21}$$

Because of $\frac{d\underline{\omega}}{d\underline{r}} = \underline{\underline{0}}$ derivative

$$\frac{d\underline{u}}{d\underline{r}} = \frac{d}{d\underline{r}}(\underline{\omega} \times \underline{r}) = \frac{d\underline{\omega}}{d\underline{r}} \times \underline{r} - \frac{d\underline{r}}{d\underline{r}} \times \underline{\omega} = -\underline{\underline{E}} \times \underline{\omega}, \tag{22}$$

and formula (21) yields

$$\underline{w} \bullet \frac{d(\underline{r} \times \underline{u})}{d\underline{r}} = \underline{w} \times \underline{u} + \underline{w} \bullet \Big[ (\underline{\underline{E}} \times \underline{\omega}) \times \underline{r} \Big] = -\underline{u} \times \underline{w} + \underline{w} \bullet \Big\{ \big[ (\underline{e}_i \otimes \underline{e}_i) \times \underline{\omega} \big] \times \underline{r} \Big\}. \tag{23}$$

But

$$\underline{w} \bullet \Big\{ \big[ (\underline{e}_i \otimes \underline{e}_i) \times \underline{\omega} \big] \times \underline{r} \Big\} = \underline{w} \bullet \Big\{ \underline{e}_i \otimes \big[ (\underline{e}_i \times \underline{\omega}) \times r \big] \Big\} = w_i (\underline{e}_i \times \underline{\omega}) \times \underline{r} = (\underline{w} \times \underline{\omega}) \times \underline{r}, \tag{24}$$

so that

$$\underline{w} \bullet \frac{d(\underline{r} \times \underline{u})}{d\underline{r}} = -\underline{u} \times \underline{w} + \underline{r} \times (\underline{\omega} \times \underline{w}) \tag{25}$$

and

$$\underline{r} \times (\underline{\omega} \times \underline{w}) = \underline{u} \times \underline{w} + \underline{w} \bullet \frac{d(\underline{r} \times \underline{u})}{d\underline{r}}. \tag{26}$$

Adding (20) and (26) we have the following formula for the moment of Coriolis acceleration

$$\underline{r} \times (2\underline{\omega} \times \underline{w}) = \underline{\omega} \times (\underline{r} \times \underline{w}) + \underline{w} \bullet \frac{d(\underline{r} \times \underline{u})}{d\underline{r}}. \tag{27}$$

Together with (1) formulae (8), (27) and (19) yield

$$\underline{M} = -\underline{\omega} \bullet \underline{\underline{J}} \times \underline{\omega} + \int_V \underline{\omega} \times (\underline{r} \times \underline{w}) \rho dV + \int_V \underline{w} \bullet \frac{d(\underline{r} \times \underline{u})}{d\underline{r}} \rho dV + \oint_S \underline{r} \times \underline{w} \rho w_n dS. \tag{28}$$

As above (see (14)–(19)) the third integral in (28) may be transformed into integral

$$\oint_S \underline{r} \times \underline{u} \rho w_n dS, \tag{29}$$

so that, taking into account that $\underline{\omega}$ does not depend on $\underline{r}$, we have for $\underline{M}$:

$$\underline{M} = \underline{M}_1 + \underline{M}_2 + \underline{M}_3, \tag{30}$$

where

$$\underline{M}_1 = -\underline{\omega} \bullet \underline{\underline{J}} \times \underline{\omega}, \qquad \underline{M}_2 = \underline{\omega} \times \int_V \underline{r} \times \underline{w} \rho V, \qquad \underline{M}_3 = \oint_S \underline{r} \times \underline{v} \rho w_n dS, \qquad \underline{v} = \underline{u} + \underline{w}. \tag{31}$$

The rate of flow of fluid through the canal is equal to

$$G = \left( \int_S \rho w_n dS \right)_2 = -\left( \int_S \rho w_n dS \right)_1. \tag{32}$$

Here indexes 1 and 2 are numbers of inlet and outlet cross-sections of canal.

On the walls $w_n = 0$. From here it follows that in so-called hydraulic approach

$$\underline{M}_3 = \oint_S \underline{r} \times \underline{v} \rho w_n dS = G(\underline{r}_2 \times \underline{v}_2 - \underline{r}_1 \times \underline{v}_1). \tag{33}$$

Using hydraulic approach for the second term in (30) we can write

$$\underline{M}_2 = \underline{\omega} \times \int_V \underline{r} \times \underline{w} \rho dV = G \underline{\omega} \times \int_0^{\ell} \underline{r} \times \underline{\ell}^o d\ell. \tag{34}$$

Here $\underline{\ell}^o$ and $\ell$ are unit vector and length of axis of the canal.

The moment around the axis of rotation of canal is equal to

$$M_\omega = \underline{M} \bullet \underline{\omega}^o, \tag{35}$$

where $\underline{\omega}^o$ is unit vector of $\underline{\omega}$.

From (35), (30), (31), (33) and (34) it follows that $M_\omega$ is equal to the moment obtained from Euler's turbine equation [3, 4].

Analysis carried out in this work shows that Coriolis inertia forces play important role as a factor, which creates the moment acting on the fluid, moving in rotating canal. At first they produce that part of the moment which depends on circumferential velocities in inlet and outlet sections of the canal. Corresponding formula for this component of the moment is

$$G(\underline{r}_2 \times \underline{u}_2 - \underline{r}_1 \times \underline{u}_1). \tag{36}$$

Besides they are responsible for that part of the moment which is given by formula (34). This formula was first received by another way in unpublished work by professor E.M. Smirnov.

Appearance of formula (34) is caused by space character of the problem. This component of the moment vanishes in plane case: it is absent in Euler's formula for $M_\omega$. From mathematical point of view this result is a consequence of two different formulae for the same $\underline{r} \times (\underline{\omega} \times \underline{w})$ product: see (20) and (26).

Results received above reveal the physical sense of Euler's formula for $M_\omega$ too. One part of this moment has to Coriolis inertia forces, another one is caused by reactive forces, which arise due to relative motion with velocity $w$.

## References

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