Numerical simulation of deformation of a metal foam

Oxana V. Sadovskaya
o_sadov@icm.krasn.ru

Abstract

Mathematical model for the description of deformation of a porous metal is constructed on the basis of generalized rheological method. In this method a new rheological element, so-called rigid contact, is used to describe the uniaxial deformation of materials with different resistance to tension and compression. Change in the resistance of a porous metal to external loads, when the collapse of pores occurs, is taken into account by means of the von Mises–Schleicher strength condition. Irreversible deformation is described with the help of yield condition, modeling the plastic loss of stability of porous skeleton. For a homogeneous porous medium the fields of displacements and stresses in the problem of radial expansion of spherical and cylindrical cavities under the action of internal pressure considering the effect of collapse of pores are constructed in closed form. The algorithm of numerical realization of the model of dynamic deformation of metal foams on multiprocessor computer systems of the cluster type is worked out.

1 Introduction

Porous metals (metal foams) are new artificial materials, which can be widely used in engineering because of their low density and good damping properties [1, 2]. The ability of porous metals to absorb the energy during plastic deformation opens prospect of their use for the production of car bumpers and the so-called "collapsible zones". They also can be applied in gearboxes and drives as destructible fuses, which dissipate the energy of dynamic impact, preventing the destruction of mechanical system.

At present the technologies of production of metal foams based on aluminum, copper, nickel, tin, zinc and other metals are well developed. According to the information published in Internet, professor A. Rabiei from the University of North Carolina (USA) in 2010 created a method of producing the most durable foam in the world. High strength of this material is achieved by ensuring that the surface of a thin-walled skeleton in the foam practically has no dislocations, i.e. defects that are initiators of the destruction. Extensive experimental researches of the mechanical properties of such materials are conducted. The diagrams of uniaxial tension and uniaxial compression on the example of an aluminum foam and of a porous copper were obtained in [3, 4]. The problems of durability and cyclic fatigue of porous metals are considered in [1, 5], etc.

The main difficulties in mathematical modeling of the behavior of porous materials are related to the fact that their deformation properties are significantly different in tension and in compression, before and after the collapse of pores. In tension there are the stage of elastic deformation of a porous skeleton and the stage of plastic flow up to the fracture. In compression there are the stages of elastic and plastic deformation of a skeleton until the collapse of pores, and the subsequent stage of elastic or elastic-plastic deformation of a solid material without pores after the collapse. At small sizes of pores the collapse can occur on the elastic stage with the appearance of plasticity only under a sufficiently high level of load.

Theoretical questions of the constructing constitutive equations and of the analysis on this basis the spatial stress–strain state of structural elements of metal foams are poorly understood. At the level of physical and mechanical representations, the deformation of a metal foam is rather complex process. At high porosity the compression leads to elastic-plastic loss of stability of a metal skeleton, at low porosity the stable mechanism of collapse of pores is realized. The collapse is accompanied by a contact interaction of skeleton walls, which is difficult for modeling at the discrete level. Besides, it is necessary to consider the presence of compressed gas in closed pores. It is rather difficult to describe the process of shear when, according to experiments, the volume of a material changes. Even more difficult to construct a universal model of the spatial stress–strain state of a material under complex loading. The performance of adequate computations, based on discrete models of a metal foam as a structurally inhomogeneous material, is only possible with the use of multiprocessor systems with high productivity and large amount of random-access memory.

2 Mathematical model

The porosity is defined as the ratio of the pore volume to the volume of a porous material: $\theta_0 = V_0/V$. If ρ denotes the density of a source (solid) metal then, ignoring the weight of the gas in the pores, the density of a porous metal can be calculated as follows: $\rho_0 = \rho\,(V - V_0)/V$. Therefore, $\theta_0 = (\rho - \rho_0)/\rho$. The volume strain of a highly porous material, caused by changing the volume of pores, is significantly higher than the volume strain of a solid skeleton, hence, the pores disappear with the volume strain $\theta \approx \big((V - V_0) - V\big)/V = -\theta_0$.

Rheological models of uniaxial stress–strain state of porous materials are constructed in [6]. A simple rheological scheme taking into account the main features of deformation of porous metals is shown in Fig. 1 *a*. The behavior of an elastic-plastic material in tension and in compression before the collapse of pores is simulated by an elastic spring with the compliance modulus a and by a plastic hinge, the increase in stiffness after collapse is simulated by an additional spring with the compliance modulus b. Under the tensile stress σ_s^+ a skeleton goes into the state of plastic flow, and under the compressive stress $-\sigma_s^-$ the plastic loss of stability takes place. The stage of elastic-plastic deformation of a solid material after the collapse of pores is described by the rheological scheme of linear hardening. A diagram of uniaxial tension–compression is represented in Fig. 1 *b* as a four-segment broken line. Dissipation of energy in the process of collapse can be evaluated by the product $\sigma_s^-\theta_0$.

In the case of spatial stress–strain state, according to rheological scheme in Fig. 1, the stress tensor σ is equal to the sum of the tensor σ^p of plastic stresses and the tensor σ^c of additional stresses, acting after the collapse of pores. Elastic compliance of a material is characterized by the fourth-rank tensors a and b, satisfying the usual conditions of symmetry and positive definiteness. Series connection of an elastic spring and a plastic hinge in the scheme corresponds to the Prandtl–Reuss theory of elastic-plastic flow. Constitutive relationships of this theory can be represented in the form of principle of maximum of the energy dissipation rate [6]:

$$(\tilde{\sigma} - \sigma^p) : (a : \dot{\sigma}^p - \dot{\varepsilon}) \geq 0, \qquad \tilde{\sigma},\ \sigma^p \in F. \tag{1}$$

Here ε is the actual strain tensor, $\tilde{\sigma}$ is an arbitrary admissible variation of the stress, F is the convex set in the stress space, limited by the yield surface of a material. The conventional notations and operations of the tensor analysis are used: a colon denotes

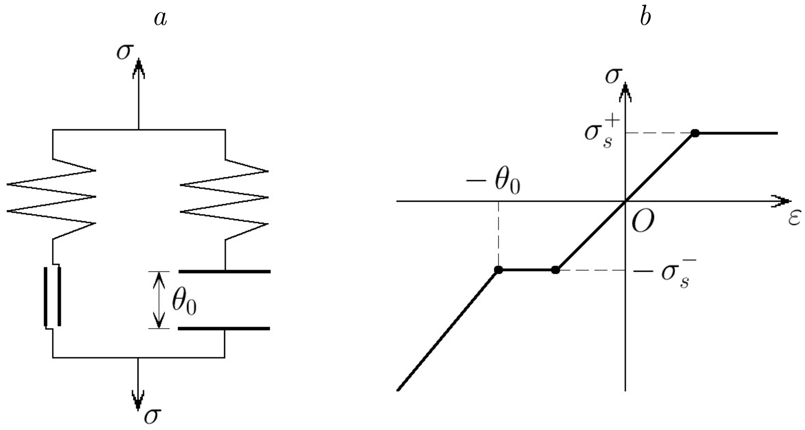

Figure 1: Rheological scheme (a) and diagram (b) of elastic-plastic deformation

double convolution of tensors, a dot over a symbol denotes a derivative with respect to time. The set F of admissible stresses is defined by means of the Tresca–Saint-Venant or von Mises yield condition. So, this set can be approximated by the von Mises cylinder

$$F = \Big\{\sigma \ \Big| \ \tau(\sigma) \leq \tau_s\Big\}, \tag{2}$$

where τ_s is the yield point of a porous material, $\tau(\sigma)$ is the intensity of tangential stresses, calculated via the deviator $\sigma' = \sigma + p(\sigma)\delta$ of stress tensor by the formula: $\tau^2(\sigma) = \sigma' : \sigma'/2$.

Constitutive relationships of a rigid contact are formulated in the form of variational inequality [6]:

$$(\tilde{\sigma} - \sigma^c) : (\varepsilon^c + \varepsilon^0) \leq 0, \qquad \tilde{\sigma},\ \sigma^c \in K. \tag{3}$$

Here $\varepsilon^c = \varepsilon - b : \sigma^c$ is the strain tensor of a porous skeleton, $\varepsilon^0 = \theta_0\,\delta/3$ is the spherical tensor of initial porosity (δ is the Kronecker delta), K is the convex cone in the stress space, which serves for modeling the transition from a porous state to a solid state of a material. Further K is the von Mises–Schleicher circular cone:

$$K = \Big\{\sigma \ \Big| \ \tau(\sigma) \leq æ\, p(\sigma)\Big\}, \tag{4}$$

where æ is the parameter of internal friction, $p(\sigma) = -\sigma : \delta/3$ is the hydrostatic pressure. The variational inequality (3) can be reduced to the next form:

$$(\tilde{\sigma} - \sigma^c) : b : (\sigma^c - s) \geq 0, \qquad \tilde{\sigma},\ \sigma^c \in K. \tag{5}$$

Here s is the conditional stress tensor, which is calculated by the law of linear elasticity taking into account initial strains, namely $b : s = \varepsilon + \varepsilon^0$. If this tensor is admissible, i.e. if $s \in K$, then by (5) the tensor σ^c is equal to s. If s is not admissible ($s \notin K$) and for any $\tilde{\sigma} \in K$ the inequality $\tilde{\sigma} : b : s \leq 0$ is valid, which means that the sum $\varepsilon + \varepsilon^0$ of tensors belongs to the cone C of admissible strains, conjugate to K, then $\sigma^c = 0$, as follows from (5). In the general case the variational inequality (5) allows to determine the tensor $\sigma^c = \pi_K(s)$ as the projection of s onto the cone K with respect to the Euclidean norm $|s| = \sqrt{s : b : s}$, and the considered above two variants are particular cases, when the projection coincides with the original tensor and when the projection is the vertex of the cone. Third variant, when the projection belongs to the conical surface, is realized under the fulfillment of two conditions: $s \notin K$ and $\varepsilon + \varepsilon^0 \notin C$. For an isotropic material the elastic compliance tensor b is characterized by two independent parameters: the bulk modulus k_0 and the shear modulus μ_0. Formulas for calculating the projection onto the conical surface are as follows [6]:

$$p(\sigma^c) = \frac{\mu_0\, p(s) + æ\, k_0\, \tau(s)}{\mu_0 + æ^2 k_0}, \qquad (\sigma^c)' = æ\, p(s)\, \frac{s'}{\tau(s)}. \tag{6}$$

The cone C, conjugate to the von Mises–Schleicher cone K, is defined as

$$C = \Big\{ \varepsilon \;\Big|\; æ\, \gamma(\varepsilon) \le \theta(\varepsilon) \Big\}, \tag{7}$$

where $\theta(\varepsilon) = \varepsilon : \delta$ is the volume strain, $\gamma(\varepsilon) = \sqrt{2\, \varepsilon' : \varepsilon'}$ is the shear intensity. The condition $\varepsilon + \varepsilon^0 \in C$ means that a rigid contact in rheological scheme is open, i.e. that pores are in the open state. The limit condition of the collapse of pores $æ\, \gamma(\varepsilon) = \theta_0 + \theta(\varepsilon)$ describes the dilatancy of a porous material due to the shear deformation.

Mathematical model, describing the dynamic deformation of a porous material at small strains and rotations of elements, can be written in the following form:

$$\begin{aligned} &\rho_0\, \dot v = \nabla \cdot \sigma + \rho_0\, f, \\ &(\tilde\sigma - \sigma^p) : \big(a : \dot\sigma^p - \nabla v\big) \ge 0, \qquad \tilde\sigma,\ \sigma^p \in F, \\ &b : \dot s = \frac{1}{2}\, \big(\nabla v + \nabla v^*\big), \qquad \sigma = \sigma^p + \pi_K(s). \end{aligned} \tag{8}$$

Here v is the velocity vector, f is the vector of body forces, ∇ is the vector of gradient with respect to spatial variables, the asterisk denotes transposition. This system consists of the equation of motion in vector form and the constitutive relationships, which follow from the inequalities (1) and (3) taking into account the kinematic equation $2\, \dot\varepsilon = \nabla v + \nabla v^*$. The vector v, the tensors σ^p and s are unknown functions in this model. The initial conditions, describing the natural (unstressed) state of a material, are as follows:

$$v\Big|_{t=0} = 0, \qquad \sigma^p\Big|_{t=0} = 0, \qquad s\Big|_{t=0} = b^{-1} : \varepsilon^0. \tag{9}$$

The boundary conditions can be given in velocities or stresses:

$$v\Big|_{\Gamma} = v^0(x), \qquad \sigma\Big|_{\Gamma} \cdot \nu(x) = q(x), \tag{10}$$

where ν is the normal vector to the boundary, v^0 and q are given functions. Mathematical correctness of the boundary conditions follows from the integral estimates of the difference between two solutions of the system (8). For a more general situation a priori estimates, guaranteeing the uniqueness of solution and the continuous dependence on the initial data for the Cauchy problem and for the boundary–value problems with dissipative boundary conditions, were obtained in the monograph [6]. Dissipativity of boundary conditions in this case means that at each point of the boundary the inequality

$$(\bar\sigma - \sigma) \cdot \nu(x) \cdot (\bar v - v) \le 0 \tag{11}$$

is fulfilled. This inequality holds automatically for the mentioned above main types of boundary conditions.

3 Radial expansion of cavities

Let us describe the expansion of a cavity of the radius r_0 in an infinite porous medium under the action of slowly increasing internal pressure p_0 within the framework of suggested model. In the case of spherical symmetry at elastic stage of the process the following system of equations is valid:

$$\begin{aligned} &\frac{d\sigma_r}{dr} + 2\,\frac{\sigma_r - \sigma_\varphi}{r} = 0, \qquad \varepsilon_r = \frac{du_r}{dr}, \qquad \varepsilon_\varphi = \frac{u_r}{r}, \\ &\sigma_r + 2\,\sigma_\varphi = 3\,k\,(\varepsilon_r + 2\,\varepsilon_\varphi), \qquad \sigma_\varphi - \sigma_r = 2\,\mu\,(\varepsilon_\varphi - \varepsilon_r). \end{aligned} \tag{12}$$

The solution of (12), satisfying the boundary condition on the surface of a cavity, is given by the formulas:

$$\sigma_r = -2\,\sigma_\varphi = -p_0\left(\frac{r_0}{r}\right)^3, \qquad u_r = \frac{p_0\,r_0^3}{4\,\mu\,r^2}. \tag{13}$$

The plastic stage is described by a system, which is obtained from (12) by replacing the last equation on the Treska–Saint-Venant yield condition: $\sigma_\varphi - \sigma_r = 2\,\tau_s$. For this system one can find:

$$\sigma_r = -p_0 + 4\,\tau_s \ln\frac{r}{r_0}, \qquad \sigma_\varphi = 2\,\tau_s + \sigma_r, \tag{14}$$

$$3\,k\,u_r = -p_0\,r + 4\,\tau_s\,r\ln\frac{r}{r_0} + \frac{C_1}{r^2}, \quad \text{where} \quad C_1 = \left(\frac{k}{\mu} + \frac{4}{3}\right)\tau_s\,r_s^3. \tag{15}$$

At the interface between a zone of plastic deformation and an elastic zone the conditions of continuity of radial stress and radial displacement and the additional condition of limit elastic state are fulfilled. The solution in an elastic zone is given by the formulas (13), in which the radius r_0 of a cavity is replaced by the radius $r_s = r_0 \exp\bigl((p_0 - p_s)/(4\,\tau_s)\bigr)$ of a plastic zone, and the acting pressure p_0 is replaced by the limit elastic pressure $p_s = 4\,\tau_s/3$.

The expression for displacement allows to determine the total strain in a plastic zone up to the moment of the pores collapse. Via the stresses (14) by means of Hooke's law one can find the elastic components ε_j^e of strains ($j = r, \varphi$), and then their plastic components $\varepsilon_j^p = \varepsilon_j - \varepsilon_j^e$. The critical pressure $p_c = p_s + k\,\theta_0$ of the collapse of pores on the surface of a cavity is calculated by the equality $\theta = -\theta_0$. This equality describes the collapse in the case of æ $= 0$, when a porous material has not the dilatancy.

Further increase in pressure leads to the formation near the cavity of a zone of compressed, non-porous material ($r < r_c$) within a zone of plastic compaction. According to the constitutive relationships (3), $\theta(\varepsilon^c) = -\theta_0$ and $\sigma_r^c = \sigma_\varphi^c = -q \le 0$ in this zone. Thus, the total stresses in a zone of collapse are calculated via the stresses σ_j^p in a plastic element as $\sigma_r = \sigma_r^p - q$, $\sigma_\varphi = \sigma_\varphi^p - q$, where q can be interpreted as the additional pressure, caused by the collapse.

Since $\sigma_\varphi - \sigma_r = \sigma_\varphi^p - \sigma_r^p$, then the continuity of radial stress in a plastic zone leads to the continuity of stresses σ_r and σ_φ, hence, the stresses in a whole space are defined by the formulas (14). The difference is only in formulas for strains. According to the rheological scheme, in a zone of collapse $\theta = -\theta_0 - q/k_0$, where k_0 is the bulk modulus of an elastic spring, which is in series connection with a rigid contact. On the other hand, $\theta = (\sigma_r^p + 2\,\sigma_\varphi^p)/(3k)$. Hence, $(1/k + 1/k_0)\,q = -\theta_0 - (\sigma_r + 2\,\sigma_\varphi)/(3k)$. The displacement in a zone of the collapse of pores is calculated by the formula:

$$3\,(k + k_0)\,u_r = -(p_0 + k_0\,\theta_0)\,r + 4\,\tau_s\,r\ln\frac{r}{r_0} + \frac{C_2}{r^2}, \tag{16}$$

where $k\,C_2 = -k_0\,p_s\,r_c^3 + (k + k_0)\,C_1$. The equation (16) allows to find reversible and irreversible components of strains in this zone. The constant C_2 and the radius $r_c = r_0\exp\big((p_0 - p_c)/(4\,\tau_s)\big)$ of a zone of collapse are obtained from the condition of continuity of displacement and from the condition $\theta = -\theta_0$ of collapse on the boundary of a plastic zone. The displacement in a plastic zone can be found from (15) after the replacing p_0 by p_c and r_0 by r_c.

Similarly one can obtain the solution of the problem of expansion of a cylindrical cavity in an infinite porous medium. It differs from the solution for a spherical cavity, because in the case of cylindrical symmetry after the appearance of a plastic zone of incomplete plasticity at the neighbourhood of a cavity appears a zone of full plasticity. With increasing pressure a zone of the collapse of pores can occupy some part of a zone of full plasticity or all points of this zone with some part of a zone of incomplete plasticity.

The engineering formulas for calculation of critical pressures of the elastic limit state and the limit state of a medium with open pores, and also the formulas for determining the radiuses of interfaces of zones of plasticity and compaction are obtained. The formulas of displacements, strains and stresses allow us to determine the stress–strain state around spherical and cylindrical cavities for an arbitrary set of phenomenological parameters of a porous material and for an arbitrary pressure inside the cavity both in the case of monotone loading, and in the presence of unloading.

Computer program in the Matlab system was worked out, which automatically selects the appropriate variant of location of the zones of plasticity and compaction according to input parameters of the problem. Typical graphs of distribution of the residual stresses, displacements, reversible and irreversible strains around spherical and cylindrical cavities for a porous (cellular) aluminum were constructed.

4 Computational algorithm

Within the framework of proposed mathematical model the parallel computational algorithm was worked out for numerical solution of the problems of dynamic deformation of porous materials. The system (8) is solved by means of the splitting method with respect to spatial variables. Variational inequality is solved by the splitting with respect to physical processes, which leads to a special procedure of solution correction in each node of the computational domain at each time step.

The technology of parallelization of computational algorithm is based on the method of two-cyclic splitting with respect to spatial variables. For the solution of 1D systems of equations an explicit monotone finite-difference ENO–scheme of the "predictor–corrector" type with piecewise-linear distributions of velocities and stresses over meshes, based on the principles of grid-characteristic methods [7], is applied. The parallelizing of computations is carried out using the MPI library, the programming language is Fortran. The data exchange between processors occurs at step "predictor" of the finite-difference scheme by means of the function MPI_Sendrecv. At first each processor exchanges with neighboring processors the boundary values of their data, and then calculates the required quantities in accordance with the explicit finite-difference scheme. Mathematical models are embedded in programs by means of software modules that implement the constitutive relationships, the initial data and boundary conditions of problems. The universality of programs is achieved by a special packing of the variables, used at each node of the cluster, into large one-dimensional arrays. Computational domain is distributed between the cluster nodes by means of 1D, 2D or 3D decomposition so as to load the nodes uniformly and to minimize the number of passing data. Detailed description of the parallel algorithm one can found in [6].

Parallel version of programs, describing the behaviour of porous materials, is included in the previously developed and registered by Rospatent parallel program systems 2Dyn_Granular [8] and 3Dyn_Granular [9], intended for numerical analysis of dynamic processes in elastic-plastic and granular media. Testing and verification of parallel programs, simulating the deformation of a metal foam in 2D and 3D cases, are performed at present on the MVS–100K cluster of JSCC RAS.

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Oxana V. Sadovskaya, Akademgorodok 50/44, Krasnoyarsk 660036, Russia