On the inverse coefficient problem for the transversally inhomogeneous elastic layer

Pavel S. Uglich
puglich@inbox.ru

Abstract

A problem of forced antiplane vibrations of the transversally inhomogeneous elastic layer is considered. The numerical method for problem solving is presented. This method may be used to calculate the wave field in an elastic layer with any mechanical parameters distribution law.

The inverse coefficient problem of the mechanical parameters definition using surface field data is also considered. The inverse problem is reduced to iterative sequence of integral equations, the program for its numerical solution is developed.

1 Introduction

Mechanical parameters reconstruction problems often appear in many different fields of mechanics, like geophysics and also when properties identification problems for biological tissues or functionally graded materials are considered. In most recent papers [1], inverse problems for layers, using the Fourier transform, are redused to one-dimentional problems, and for mechanical parameters definition the displacement field data on the whole surface of the layer are used as input data. In present paper for solving the inverse problem the data on the displacement field only of a certain part of the layer boundary is used.

2 Problem statement

We consider the elastic layer antiplane forced vibrations. The layer occupies the domain $\{(x_1, x_2) : -\infty < x_1 < \infty,\ 0 \leq x_2 \leq h\}$, where h is the layer thickness.

The equation of layer vibrations in antiplane shear state is

$$\sigma_{13,1} + \sigma_{23,2} + \rho\omega^2 u = 0, \tag{1}$$

where $\rho = \rho(x_2)$ is the layer density, ω is the frequency of vibrations, σ_{ij} are shear stresses, connected with the displacement function by the governing equation

$$\sigma_{13} = \mu u_{,1},\ \sigma_{23} = \mu u_{,2}, \tag{2}$$

where $\mu = \mu(x_2)$ is the shear modulus (subscript comma denotes differentiation).

The bottom boundary of the layer is restrained, and the shear load is applied on the top surface:

$$u|_{x_2=0} = 0,\ \sigma_{23}|_{x_2=h} = p(x_1) \tag{3}$$

The statement of the problem is completed with the radiation conditions formulated on the basis of the limiting absorption principle [2].

Now we introduce dimentionless values using formulae:

$$x_i = h\bar{x}_i,\ u = h\bar{u},\ \mu = \mu(0)\bar{\mu},\ \rho = \rho(0)\bar{\rho},\ \sigma_{ij} = \mu(0)\bar{\sigma}_{ij},\\ \kappa^2 = \rho(0)\omega^2 h^2 \mu^{-1}(0),\ p(x_1) = \mu(0)\bar{p}(x_1). \tag{4}$$

Overlined values in (4) are dimentionless. The overline will be skipped hereinafter.

3 The direct problem solving

To solve the problem (1)-(3) we use the Fourier transform

$$\tilde{u}(\alpha, x_2) = \int\limits_{-\infty}^{\infty} u(x_1, x_2) e^{i\alpha x_1} dx_1$$

Eliminating $\tilde{\sigma}_{13}$ from the transformed equations (1)-(2), we reduce (1)-(3) to the boundary value problem for the canonical system of ordinary differential equations with respect to $\tilde{u}$, $\tilde{\sigma}_{23}$:

$$\begin{cases} \tilde{u}' = \tilde{\sigma}_{23}\mu^{-1}, \\ \tilde{\sigma}'_{23} = \left(\alpha^2\mu - \rho\kappa^2\right)\tilde{u}, \\ \tilde{u}|_{x_2=0} = 0,\ \tilde{\sigma}_{23}|_{x_2=1} = \tilde{p}(\alpha),\ \tilde{p}(\alpha) = \int\limits_{-\infty}^{\infty} p(x_1) e^{i\alpha x_1} dx_1. \end{cases} \tag{5}$$

The boundary value problem (5) can be easily solved using the shooting method. Let $U_i(\alpha, x_2)$ and $\Sigma_i(\alpha, x_2)$ be the solution of the Cauchy problem for system (5) with following initial conditions:

$$\begin{cases} U_1(\alpha, 0) = 1, \\ \Sigma_1(\alpha, 0) = 0 \end{cases},\ \begin{cases} U_2(\alpha, 0) = 0, \\ \Sigma_2(\alpha, 0) = 1 \end{cases}$$

Now the boundary value problem (5) solution is

$$\tilde{u} = \tilde{p}(\alpha)\frac{U_2(\alpha, x_2)}{\Sigma_2(\alpha, 1)},\ \tilde{\sigma}_0 = \tilde{p}(\alpha)\frac{\Sigma_2(\alpha, x_2)}{\Sigma_2(\alpha, 1)} \tag{6}$$

The displacement field can be calculated using the inverse Fourier transform:

$$u(x_1, x_2) = \frac{1}{2\pi}\int\limits_{\sigma} \tilde{p}(\alpha)\frac{U_2(\alpha, x_2)}{\Sigma_2(\alpha, 1)} e^{-i\alpha x_1} d\alpha \tag{7}$$

where σ is the contour chosen according to the limiting absorption principle. The displacement field can be found from the formula (7) using the numeric integrating or the residue theory. It is proved ([2], [3]) that integrand singularities appear as the denumerable set of simple poles. The finite number of these poles are real, the rest of them are imaginary. All these poles can be found from the equation

$$\Sigma_2(\alpha, 1) = 0. \tag{8}$$

Let us consider the displacement field expression. If $p = \delta(x_1)$ and $x_1 > 0$, it turns to

$$u(x_1, x_2) = i \sum_{n=0}^{\infty} \left\{ \frac{U_2(\alpha, x_2)}{\left[\Sigma_2(\alpha, 1)\right]'_\alpha} \right\}\Bigg|_{\alpha=\alpha_n} e^{i\alpha_n x_1} \tag{9}$$

For the inhomogeneous layer the value of the expression $\left[\Sigma_2(\alpha, 1)\right]'_\alpha$ can be found from solving of following Cauchy problem

$$\begin{cases} \dfrac{d}{dx_2}\tilde{u} = \tilde{\sigma}_{23}\mu^{-1}, \\ \dfrac{d}{dx_2}\tilde{\sigma}_{23} = \left(\alpha^2\mu - \rho\kappa^2\right)\tilde{u}, \\ \dfrac{d}{dx_2}\tilde{u}'_\alpha = (\tilde{\sigma}_{23})'_\alpha\mu^{-1}, \\ \dfrac{d}{dx_2}(\tilde{\sigma}_{23})'_\alpha = 2\alpha\tilde{u} + \left(\alpha^2\mu - \rho\kappa^2\right)\tilde{u}'_\alpha, \\ \tilde{u}|_{x_2=0} = \tilde{u}'_\alpha|_{x_2=0} = (\tilde{\sigma}_{23})'_\alpha|_{x_2=0} = 0,\ \tilde{\sigma}_{23}|_{x_2=0} = 1. \end{cases} \tag{10}$$

The equations system (10) was constructed from the system (5). The system (5) was differentiated with respect to the variable α, and obtained equations were added to (5). Initial conditions of the Cauchy problem (5) coincide with initial conditions for U_2, Σ_2 and homogeneous initial conditions for u'_α and $(\tilde{\sigma}_{23})'_\alpha$ are added. Results of calculations show ([4]) that the residual between the displasement field found using (9) and the displasement field found using (7) by the numeric integration is not above 2%.

4 Inverse problem

Now we formulate the inverse problem: determine the mechanical parameters $\mu(x_2)$ и $\rho(x_2)$ distribution law from the data on the displacements within a certain part of the upper boundary. This problem is nonlinear and one of the ways of its investigation is the linearization procedure.

Let $\varepsilon > 0$ be a small formal parameter. Expand the displacement function and functions, describing the density and the shear modulus, into power series of ε.

$$\begin{gathered} \rho(x_2) = \rho_0(x_2) + \varepsilon\rho_1(x_2) + ..., \quad \mu(x_2) = \mu_0(x_2) + \varepsilon\mu_1(x_2) + ..., \\ \tilde{u}(\alpha, x_2) = \tilde{u}_0(\alpha, x_2) + \varepsilon\tilde{u}_1(\alpha, x_2) + ..., \\ \tilde{\sigma}_{23}(\alpha, x_2) = \tilde{\sigma}_0(\alpha, x_2) + \varepsilon\tilde{\sigma}_1(\alpha, x_2) + ... \end{gathered} \tag{11}$$

Substituting expansions (11) into equations (5) and boundary conditions (12), and collecting terms of the same order in ε, yields a boundary problem for $\bar{u}^0$:

$$\begin{cases} \tilde{u}'_0 = \tilde{\sigma}_0\mu_0^{-1}, \\ \tilde{\sigma}'_0 = \left(\mu_0\alpha^2 - \rho_0\kappa^2\right)\tilde{u}_0 \\ \tilde{u}_0(x_1, 0) = 0,\ \tilde{\sigma}_0(x_1, 1) = \tilde{p}(\alpha) \end{cases} \tag{12}$$

For $\bar{u}_1$ we get the following boundary value problem

$$\begin{cases} \tilde{u}'_1 = \tilde{\sigma}_1\mu_0^{-1} - \dfrac{\mu_1}{\mu_0^2}\tilde{\sigma}_0 \\ \tilde{\sigma}'_1 = \left(\mu_0\alpha^2 - \rho_0\kappa^2\right)\tilde{u}_1 + \left(\mu_1\alpha^2 - \rho_1\kappa^2\right)\tilde{u}_0 \\ \tilde{u}_1(x_1, 0) = \tilde{\sigma}_1(x_1, 1) = 0 \end{cases} \tag{13}$$

Solution of the problem (12) has the form (6).

Now we consider the problem (13) for the first approximation. According to the variation of parameters method we construct its solution in the following form:

$$\begin{cases} \tilde{u}_1(x_2) = C_1(x_2)U_1(\alpha, x_2) + C_2(x_2)U_2(\alpha, x_2), \\ \tilde{\sigma}_1(x_2) = C_1(x_2)\Sigma_1(\alpha, x_2) + C_2(x_2)\Sigma_2(\alpha, x_2) \end{cases} \tag{14}$$

Substituting (14) into (13), we obtain the equations system with respect to C_1', C_2'. The solution of the system is

$$C_1' = F_1(x_2)\Sigma_2(\alpha, x_2) - F_2(x_2)U_2(\alpha, x_2),\; C_2' = F_2(x_2)U_1(\alpha, x_2) - F_1(x_2)\Sigma_1(\alpha, x_2),$$

where $F_1(x_2) = -\dfrac{\mu_1}{\mu_0^2}\tilde{\sigma}_0$, $F_2(x_2) = \left(\mu_1\alpha^2 - \rho_1\kappa^2\right)\tilde{u}_0$.

The general solution of the problem (13) now turns to

$$\begin{aligned} \tilde{u}_1 = &\int_0^{x_2} F_1(\xi)\left[U_1(\alpha, x_2)\Sigma_2(\alpha, \xi) - U_2(\alpha, x_2)\Sigma_1(\alpha, \xi)\right] d\xi + \\ &+ \int_0^{x_2} F_2(\xi)\left[U_2(\alpha, x_2)U_1(\alpha, \xi) - U_1(\alpha, x_2)U_2(\alpha, \xi)\right] d\xi + \\ &+ AU_1(\alpha, x_2) + BU_2(\alpha, x_2), \end{aligned} \tag{15}$$

$$\begin{aligned} \tilde{\sigma}_1 = &\int_0^{x_2} F_1(\xi)\left[\Sigma_2(\alpha, \xi)\Sigma_1(\alpha, x_2) - \Sigma_1(\alpha, \xi)\Sigma_2(\alpha, x_2)\right] d\xi + \\ &+ \int_0^{x_2} F_2(\xi)\left[U_1(\alpha, \xi)\Sigma_2(\alpha, x_2) - U_2(\alpha, \xi)\Sigma_1(\alpha, x_2)\right] d\xi + \\ &+ A\Sigma_1(\alpha, x_2) + B\Sigma_2(\alpha, x_2), \end{aligned} \tag{16}$$

Substituting (15) and (16) into boundary conditions (13), we define the constants A and B, and then we find an expression for u if $x_2 = 1$.

Assume that $p(x_1) = \delta(x_1)$. Then we invert the Fourier transform and obtain

$$\begin{aligned} u_1(x_1, 1) = &-\frac{1}{2\pi}\int_0^1 \mu_1(\xi)d\xi \int_\sigma \left\{ \frac{1}{\mu_0^2(\xi)}\left[\frac{\Sigma_2(\alpha, \xi)}{\Sigma_2(\alpha, 1)}\right]^2 + \right. \\ &\left. + \alpha^2\left[\frac{U_2(\alpha, \xi)}{\Sigma_2(\alpha, 1)}\right]^2 \right\} e^{-i\alpha x_1} d\alpha + \frac{\kappa^2}{2\pi}\int_0^1 \rho_1(\xi)d\xi \int_\sigma \left[\frac{U_2(\alpha, \xi)}{\Sigma_2(\alpha, 1)}\right]^2 e^{-i\alpha x_1} d\alpha, \end{aligned} \tag{17}$$

The equation (17) is the linear equation, connecting mechanical characteristics corrections with the displacement correction. We represent the equation (17) in the form:

$$u_1(x_1, 1) = \int_0^1 K_1(x_1, \xi)\mu_1(\xi)d\xi + \int_0^1 K_2(x_1, \xi)\rho_1(\xi)d\xi, \tag{18}$$

$$K_1(x_1, \xi) = -\frac{1}{2\pi}\int_\sigma \left\{ \frac{1}{\mu_0^2(\xi)}\left[\frac{\Sigma_2(\alpha, \xi)}{\Sigma_2(\alpha, 1)}\right]^2 + \alpha^2\left[\frac{U_2(\alpha, \xi)}{\Sigma_2(\alpha, 1)}\right]^2 \right\} e^{-i\alpha x_1} d\alpha$$

$$K_2(x_1,\xi) = \frac{\kappa^2}{2\pi}\int\limits_{\sigma}\left[\frac{U_2(\alpha,\xi)}{\Sigma_2(\alpha,1)}\right]^2 e^{-i\alpha x_1}d\alpha$$

Kernels in the equation (18) are expressed trough the integrals of the form

$$I = \frac{1}{2\pi}\int\limits_{\sigma}\left[\frac{\varphi(\alpha,\xi)}{\psi(\alpha)}\right]^2 e^{-i\alpha x_1}d\alpha,$$

and they can be calculated according to the residue theory using the formula

$$I = i\sum_{n=0}^{\infty} R_n e^{i\alpha_n x_1},\ (x_1 > 0) \tag{19}$$

$$R_n = b_1^{-2}\left\{2a_0(\xi)\left[a_1(\xi) - a_0(\xi)b_1^{-1}b_2\right] - ix_1 a_0^2(\xi)\right\}$$

where $a_0(\xi) = \varphi(\alpha_n,\xi)$, $a_1(\xi) = \varphi'_\alpha(\alpha_n,\xi)$, $b_1 = \psi'(\alpha_n)$, $b_2 = 0.5\psi''(\alpha_n)$, α_n is the root of the dispersion equation (8).

Values a_i, b_i in the case of homogeneous layer can be found analytically, for (17) they can be found from the solution of the Cauchy problem, constructed in a similar manner as the (10). Equations and initial conditions for $(U_2)''_{\alpha\alpha}$ and $(\Sigma_2)''_{\alpha\alpha}$ are added to this Cauchy problem.

Consider the equation (18). Assume that $\mu(x_2)$ is known constant and we need to find $\rho(x_2)$. then the equation (18) turns to:

$$f - f^0 = \frac{\kappa^2}{2\pi}\int\limits_0^1 \rho_1(\xi)d\xi\int\limits_{\sigma}\left[\frac{U_2(\alpha,\xi)}{\Sigma_2(\alpha,1)}\right]^2 e^{-i\alpha x_1}d\alpha,\ c \le x_1 \le d, \tag{20}$$

where f is the observed displacement field and f^0 is the "etalon" displacement field corresponding to distribution $\rho_0(x_2)$.

We choose an initial approximation ρ_0. Solving the problem (5) and inverting the Fourier transform, we obtain u_0 and σ_0. Then, assuming that u_1 equals the difference between the observed displacement field and the displacement field u_0, we solve (20) and find ρ_1.

Then we add the correction ρ_1 to the function ρ_0 and repeat the described procedure until correction becomes negligibly small.

The problem of definition of μ can be solved in the similar way.

5 Numeric results

Numeric results of the problem solution are presented further. The real part of the displacement on the segment $[c,d]$ served the initial data for the inverse problem. Formula (9) is used both for observed displacement field calculation and for calculations of iterative equations right parts. For the integral equations kernels calculation formula (19) is used. Integrals in the equation (17) were exchanged for finite sum using the Simpson's rule. For iterative equations solution Tikhonov regularization was used.

Chart 1 presents the iterations number N_{iter} dependent on wave number κ and heterogeneity parameter η , which is necessary for density definition. It also presents the relative error of the density reconstruction if $\mu = const$, $\rho = \rho_0\,(1 + \eta\sin\pi x_2)$. Chart 2

Chart 1

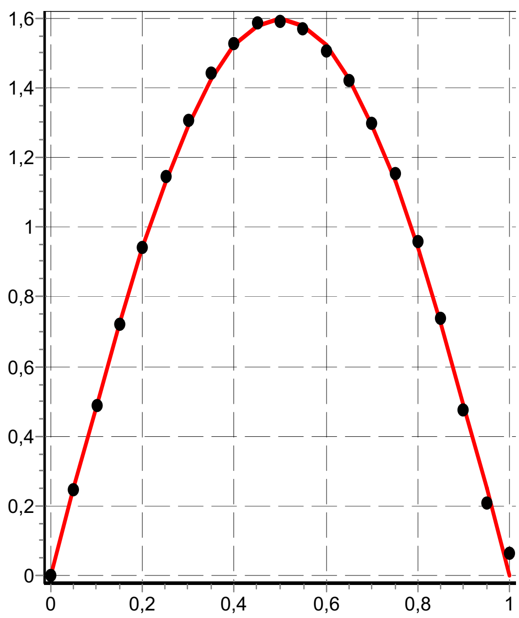
Heterogeneity parameter	Resonant frequencies	Iterations number Relative reconstruction error			
		$\kappa = 1$	$\kappa = 2$	$\kappa = 3$	$\kappa = 5$
$\eta = 0.1$	$\kappa_1 = 1.4935$ $\kappa_2 = 4.5448$	9 9.03%	8 5.6%	7 5.45%	9 4.33%
$\eta = 0.2$	$\kappa_1 = 1.4264$ $\kappa_2 = 4.3965$	10 7.83%	9 5.71%	7 5.52%	11 3.12%
$\eta = 0.3$	$\kappa_1 = 1.3675$ $\kappa_2 = 4.2635$	11 7.46%	10 5.83%	12 3.84%	13 2.75%
$\eta = 0.4$	$\kappa_1 = 1.3153$ $\kappa_2 = 4.1430$	12 7.18%	11 5.88%	13 3.32%	13 2.65%
$\eta = 0.5$	$\kappa_1 = 1.2686$ $\kappa_2 = 4.0330$	12 7.01%	12 5.92%	15 2.58%	14 2.55%
$\eta = 0.75$	$\kappa_1 = 1.1704$ $\kappa_2 = 3.7941$	14 6.74%	17 3.81%	17 1.87%	19 2.08%
$\eta = 1$	$\kappa_1 = 1.0920$ $\kappa_2 = 3.5945$	15 6.52%	20 3.51%	18 1.85%	50 1.65%

shows same parameters for $\rho = const$, $\mu = \mu_0 (1 + \eta \sin \pi x_2)$. Iterations went on while the norm of difference between the observed displacement field, and the displacement field corresponding the reconstructed parameter distribution exceeds 10^{-4} (chart 1), or 10^{-3} (chart 2). Calculations results show that the best reconstruction can be achieved on the vibrations frequency between the first and the second resonant frequencies, and segment $[c, d]$ should be chosen close to vibrations source.

Figures 1-2 show the results of numerical experiments for mechanical parameters reconstruction. Figure 1 shows a density reconstruction, figure 2 shows a shear modulus reconstruction. The solid line represents the exact value of a mechanical parameter, and the dots show the parameter distribution reconstructed from the data given on the segment $[c, d] = [0.01, 1]$. On Figure 1 on the left the density reconstruction result is shown for $\mu = const$, $\rho = \rho_0 (1 + 1,6 \sin \pi x_2)$, on the right the result is shown for $\mu = const$, $\rho = \rho_0 e^{x_2}$. Figure 2 shows reconstruction results for $\rho = const$, $\mu = \mu_0$ if $x_2 < 0.5$, $\mu = 1,2\mu_0$ if $x_2 \geq 0.5$ (on the left) and for $\rho = const$, $\mu = \mu_0(1 + 0.3 \sin 2\pi x_2)$ (on the right).

Chart 2

Heterogeneity parameter	Resonant frequencies	Iterations number Relative reconstruction error			
		$\kappa = 1$	$\kappa = 2$	$\kappa = 3$	$\kappa = 4$
$\eta = 0.1$	$\kappa_1 = 1.6193$ $\kappa_2 = 4.85289$	6 4.4%	7 2.59%	6 2.76%	5 5.83%
$\eta = 0.2$	$\kappa_1 = 1.6650$ $\kappa_2 = 4.9989$	7 4.05%	10 2.82%	7 2.99%	7 4.87%
$\eta = 0.3$	$\kappa_1 = 1.7084$ $\kappa_2 = 5.1334$	6 3.99%	10 3.04%	9 2.6%	9 3.52%
$\eta = 0.4$	$\kappa_1 = 1.7497$ $\kappa_2 = 5.2628$	8 4.69%	13 3.28%	— —	9 3.16%
$\eta = 0.5$	$\kappa_1 = 1.7822$ $\kappa_2 = 5.3878$	9 4.61%	14 3.93%	11 2.7%	13 2.91
$\eta = 0.75$	$\kappa_1 = 1.8812$ $\kappa_2 = 5.6834$	11 5.11%	14 3.93%	14 6.15%	11 3.22%
$\eta = 1$	$\kappa_1 = 1.9653$ $\kappa_2 = 5.9587$	12 4.5%	11 11.43%	13 4.82%	17 4.54%



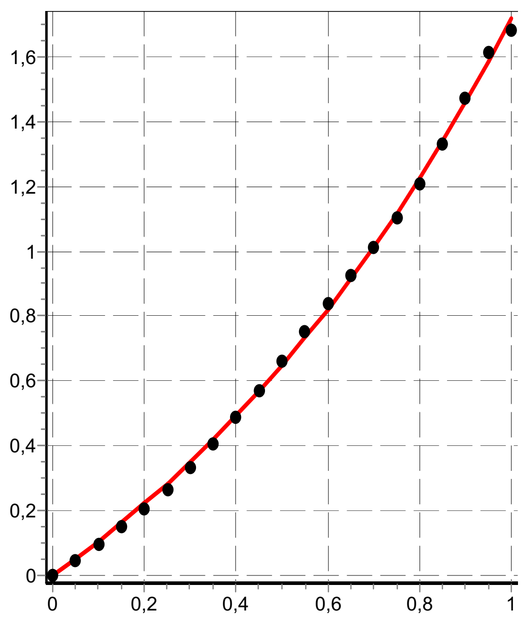


Figure 1: Density reconstruction results for $\mu = const$, $\rho = \rho_0\,(1 + 1,6\sin \pi x_2)$ (on the left), for $\mu = const$, $\rho = \rho_0 e^{x_2}$ (on the right)

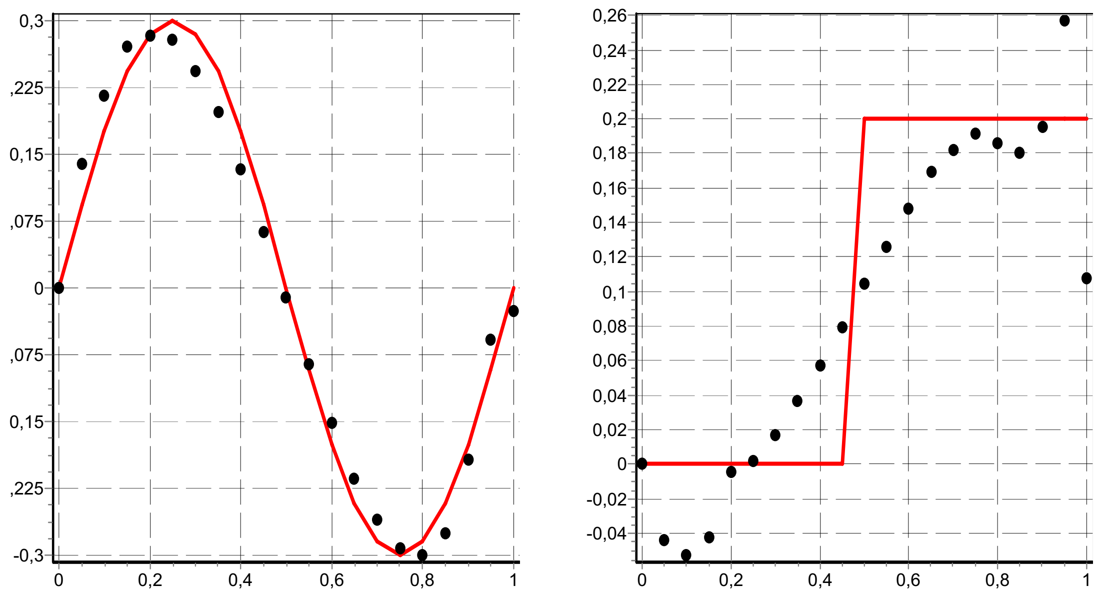


Figure 2: Shear modulus reconstruction results for $\rho = const$, $\mu = \mu_0$ if $x_2 < 0.5$, $\mu = 1,2\mu_0$ if $x_2 \geq 0,5$ (on the left) and for $\rho = const$, $\mu = \mu_0(1 + 0.3 \sin 2\pi x_2)$ (on the right).

6 Conclution

Thus, the inverse coefficient problem for mechanical parameters reconstruction in elastic layer, performing steady-state vibrations in antiplane shear state, is reduced to an iterative sequence of integral equations with respect to functions describing material parameters distribution law. Results of its solution show that the best parameter reconstruction can be achieved on vibrations frequency between the first and the second resonant frequencies.

Acknowledgements

This work is supported by RFBR.

References

[1] A. O. Vatulyan, O. V. Yavruyan, I. V. Bogachev. Acoustical physics. **57**, No. 3, (2011) p. 723-730.

[2] I. I. Vorovich, V. A. Babeshko, Dynamic mixed problems of the elasticity theory for non-classical domains. - M.: Nauka, 1979. 320 p. (in russian)

[3] A. O. Vatulyan, M. A. Dvoskin, P. S. Satunovski. Journal of Applied Mechanics and Technical Physics. **47**, No. 3, (2006) p. 157-164.

[4] M. V. Abramovich, P. S. Uglich. Proceedings of XVIth international conference "Modern continuum mechanics problems". Rostov-na-Donu, 2012, vol. 2. p. 6-10.

Pavel S. Uglich, Vladikavkaz, Russia