Various forms of the equations of nonlinear dynamics of the reduced Cosserat continuum

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Abstract

We have obtained full equation system for nonlinear isotropic elastic reduced Cosserat continuum. Lagrangian description was used. We have got 2 equation system: in reference and current configurations. We used two strain tensors wich energy correlated with two asymmetric stress tensors.

In this paper we continue a series of works [1],[2],[3] about reduced Cosserat continuum investigation.

In reduced Cosserat continuum each particle has 6 degrees of freedom, in terms of kinematics its state is described by vector $\mathbf{r}$ and rotation tensor $\mathbf{P}$. Rotation tensor is a orthogonal tensor with determinant equal to 1 and defined by 3 independent parameters. So kinematic state of the medium is described by fields $\mathbf{r}(x^s, t)$ and $\mathbf{P}(x^s, t)$, where $x^s (s = 1, 2, 3)$- coordinates of the medium in the reference configuration (RC), t- time. Usually RC is selected as known position of the body at the initial time $t = 0$. Let $\mathbf{r}(x^s, 0) = \mathbf{R}(x^s)$. Introduce the basis $\mathbf{R}_k(x^s) = \partial\mathbf{R}/\partial x^k$, the mutual basis $\mathbf{R}^k(x^s)$ and the Hamiltonian in RC $\check{\nabla} = \mathbf{R}^s \frac{\partial}{\partial x^s}$.

Current position of the body is called the current configuration (CC). Introduce the basis $\mathbf{r}_k(x^s, t) = \partial\mathbf{r}/\partial x^k$, the dual basis $\mathbf{r}^k(x^s, t)$ and the Hamiltonian in CC $\nabla = \mathbf{r}^k \frac{\partial}{\partial x^k}$. Note that basis in the RC are not time-dependent and in CC they are.

Purpose of this work to obtain the equations of the nonlinear reduced Cosserat continuum as Lagrangian description in RC and in CC.

1 System of equations in the current configuration

Let's use mass conservation law [4]

$$\dot{\rho} + \rho\nabla \cdot \mathbf{v} = 0, \tag{1}$$

the momentum balancing equation

$$\frac{d}{dt}\int\limits_V \rho\mathbf{v}dV = \int\limits_S \mathbf{n} \cdot \tau dS, \tag{2}$$

the kinetic moment balancing equation

$$\frac{d}{dt}\int\limits_V (\rho\mathbf{J} \cdot \omega + \mathbf{r} \times \rho\mathbf{v}) = \int\limits_S \mathbf{r} \times (\mathbf{n} \cdot \tau)dS \tag{3}$$

the energy balancing equation:

$$\frac{d}{dt}\int\limits_V (\frac{1}{2}\rho\mathbf{v}^2 + \frac{1}{2}\omega\cdot\rho\mathbf{J}\cdot\omega + \rho\Pi)dV = \int\limits_S \mathbf{n}\cdot\tau\cdot\mathbf{v}dS. \tag{4}$$

and the rule for differentiating an integral over a moving volume

$$\frac{d}{dt}\int\limits_V \rho A dV = \int\limits_V \rho\dot{A}dV, \tag{5}$$

where ρ- the density in CC, $\mathbf{v}$- the velocity vector ($\mathbf{v} = \dot{\mathbf{r}}$), r- the radius vector,ω - the angular velocity vector ($\dot{\mathbf{P}} = \omega\times\mathbf{P}$), $\mathbf{n}$- an outward unit normal to the surface S, $\mathbf{J}$- the mass density of an inertia tensor, Π- the mass density of the strain energy, $(\dot{...}) = \frac{\partial}{\partial t}(...)$, A - a arbitrary scalar, vector or tensor field, V- volume limited by surface S.

Using (5) and Gauss-Ostrogradskiy formula [4] with equations (2),(3) we get motion equations in CC:

$$\nabla\cdot\tau = \rho\dot{\mathbf{v}} \tag{6}$$

$$\tau_x = \rho(\omega\times\mathbf{J}\cdot\omega + \mathbf{J}\cdot\dot{\omega}),$$

where τ_x is a invariant vector of the tensor τ.

From (4),(5) using identity $\nabla\cdot(A\cdot a) = \nabla\cdot A\cdot a + A^T\cdot\cdot\nabla a$ we get

$$\int\limits_V \rho(\dot{\mathbf{v}}\cdot\mathbf{v} + \dot{\omega}\cdot\mathbf{J}\cdot\omega + \dot{\Pi})dV = \int\limits_V (\nabla\cdot\tau\cdot\mathbf{v} + \tau^T\cdot\cdot\nabla\mathbf{v})dV \tag{7}$$

Volume V is an arbitrary volume, so using first equation from (6) from (7) we obtain

$$\rho\dot{\Pi} = \tau^T\cdot\cdot\nabla\mathbf{v} - \rho(\mathbf{J}\cdot\dot{\omega})\cdot\omega \tag{8}$$

Transform the second summand with second equation from (6) and identity $(\omega\times\mathbf{J}\cdot\omega)\cdot\omega = 0$

$$-\rho(\mathbf{J}\cdot\dot{\omega})\cdot\omega = \tau_x^T\cdot\omega = \tau^T\cdot\cdot(\mathbf{I}\times\omega)$$

Last equality were obtained using [5] $(\mathbf{A}\cdot\mathbf{B})_x\cdot\mathbf{a} = \mathbf{A}\cdot\cdot(\mathbf{B}\times\mathbf{a})$ with $\mathbf{B} = \mathbf{I}$. So (8) we can write like

$$\rho\dot{\Pi} = \tau^T\cdot\cdot(\nabla\mathbf{v} + \mathbf{I}\times\omega). \tag{9}$$

2 The compatibility of strains and velocities

Strain state in reduced Cosserat continuum is described by strain tensor $\mathbf{e}(x^k, t)$. Define it in CC:

$$\mathbf{e} = \mathbf{I} - \mathbf{F}^{-T}\cdot\mathbf{P}^T, \tag{10}$$

where $\mathbf{F}^{-1} = \nabla\mathbf{R}^T$.

Let's differentiate (10) with respect to time, use $\dot{\mathbf{F}}^{-T} = -\nabla\mathbf{v}\cdot\mathbf{F}^{-T}$ and $\dot{\mathbf{P}}^T = -\mathbf{P}^T\times\omega$:

$$\dot{\mathbf{e}} + \mathbf{e}\times\omega + \nabla\mathbf{v}\cdot\mathbf{e} = \nabla\mathbf{v} + \mathbf{I}\times\omega \tag{11}$$

We obtain compatibility equation in CC, which holds identically if strains and velocities express by $\mathbf{r}$ and $\mathbf{P}$. Substitute (11) in (9)

$$\rho\dot{\Pi} = \tau^T \cdot\cdot(\dot{\mathbf{e}} + \mathbf{e} \times \omega + \nabla\mathbf{v} \cdot \mathbf{e}) \tag{12}$$

In paper [5] it was shown that (12) provides energy coupling tensors τ and $\mathbf{e}$. It follows constitutive equations in CC:

$$\rho\frac{\partial\Pi}{\partial\mathbf{e}} = \tau. \tag{13}$$

This formulation in CC contains the following unknown functions: 9 stresses τ, 9 strains $\mathbf{e}$, 6 velocities $\mathbf{v}, \omega$ and density ρ. As a result we have 25 unknown functions. Corresponding equation are: 6 motion equations (6), 9 compatibility equations (11), 9 relations of elasticity (13) and 1 mass conservation law (1). In total there are 25 equations. The problem becomes fully set after adding the boundary and initial conditions.

Our statement of the problem in CC does not contain unknown kinematic $\mathbf{r}, \mathbf{P}$ as well as strain gradient $\mathbf{F}$. Unknown $\mathbf{r}, \mathbf{P}$ can be found integrating $(\mathbf{v} = \dot{\mathbf{r}})$,$(\dot{\mathbf{P}} = \omega \times \mathbf{P})$ after solving the system of equation.

3 System of equations in the reference configuration

Easier to use "rotated" velocities vector for the equations in RC

$$\mathbf{V} = \mathbf{P}^T \cdot \mathbf{v} \tag{14}$$

$$\Omega = \mathbf{P}^T \cdot \omega \tag{15}$$

Vector Ω is used in rigid body dynamics [5]. There it was called right angular velocity vector and is defined by equality

$$\dot{\mathbf{P}} = \mathbf{P} \times \Omega \tag{16}$$

Stress state in RC in reduced Cosserat continuum is described by stress tensor

$$\mathbf{T} = \frac{\rho_0}{\rho}\mathbf{F}^{-1} \cdot \tau \cdot \mathbf{P}, \tag{17}$$

where ρ_0 is a density in RC, $\mathbf{F} = \check{\nabla}\mathbf{r}^T$.

Strain state in RC in reduced Cosserat continuum is described by strain tensor

$$\mathbf{E} = \mathbf{F}^T \cdot \mathbf{P} - \mathbf{I} \tag{18}$$

This strain tensor is identically equal to zero when a body moves as a rigid.

Let V_0 be a volume in RC which changes in V in CC. Volumes V and V_0 consist of the same particles. We use the momentum balancing equation (2) and ratio [6]

$$\mathbf{n}dS = \frac{\rho_0}{\rho}\mathbf{N} \cdot \mathbf{F}^{-1}dS_0, \tag{19}$$

where $\mathbf{n}$ is an outward unit normal to the surface S, $\mathbf{N}$ is an outward unit normal to the surface S_0, and Gauss-Ostrogradskiy formula. The arbitrariness of V_0 gives us

$$\rho_0\dot{\mathbf{v}} = \check{\nabla} \cdot \mathbf{T} \cdot \mathbf{P}^T + \mathbf{R}^s \cdot \mathbf{T} \cdot \frac{\partial\mathbf{P}^T}{\partial x^s}. \tag{20}$$

We transform summands in (20). For this, we introduce an additional tensor $\mathbf{K}$:

$$\check{\nabla}\mathbf{P}^T = -\mathbf{K} \times \mathbf{P}^T. \tag{21}$$

Hence $\frac{\partial \mathbf{P}^T}{\partial x^s} = -\mathbf{K}_s \times \mathbf{P}^T$, because $\mathbf{K} = \mathbf{r}^s \mathbf{K}_s$.

$$\dot{\mathbf{v}} = (\mathbf{P} \cdot \mathbf{V})^{\cdot} = (\mathbf{P} \times \Omega) \cdot \mathbf{V} + \mathbf{P} \cdot \dot{\mathbf{V}} = (\dot{\mathbf{V}} + \Omega \times \mathbf{V}) \cdot \mathbf{P}^T$$

$$\mathbf{R}^s \cdot \mathbf{T} \cdot \frac{\partial \mathbf{P}^T}{\partial x^s} = -\mathbf{R}^s \cdot \mathbf{T} \cdot (\mathbf{K}_s \times \mathbf{P}^T) = -(\mathbf{R}^s \cdot \mathbf{T} \times \mathbf{K}_s) \cdot \mathbf{P}^T = (\mathbf{K}^T \cdot \mathbf{T})_x \cdot \mathbf{P}^T, \tag{22}$$

where we use identity $\mathbf{A}_x = -\mathbf{A}_x^T$ and $(\mathbf{A}^T \cdot \mathbf{K})_x = \mathbf{R}^k \cdot \mathbf{A} \times \mathbf{K}_k$, that is valid for any tensor $\mathbf{A}$.
Lets return to the equation (20) and multiply by the tensor $\mathbf{P}$ on the right. We obtain a local form of the momentum balance equation in RC

$$\check{\nabla} \cdot \mathbf{T} + (\mathbf{K}^T \cdot \mathbf{T})_x = \rho_0(\dot{\mathbf{V}} + \Omega \times \mathbf{V}) \tag{23}$$

Now we consider the kinetic moment balancing equation (3). In RC tensor $\mathbf{J}$ associated with the tensor $\mathbf{J}_0$ by equation [4] $\mathbf{J} = \mathbf{P} \cdot \mathbf{J}_0 \cdot \mathbf{P}^T$, where $\mathbf{J}$, $\mathbf{J}_0$ are symmetric positive definite tensors. Lets write (3) in RC using (15),(17),(19), Gauss-Ostrogradskiy formula and arbitrariness of the volume V_0

$$\rho_0(\mathbf{P} \cdot \mathbf{J}_0 \cdot \dot{\Omega} + \mathbf{r} \times \mathbf{v}) = -\check{\nabla} \cdot (\mathbf{T} \cdot \mathbf{P}^T \times \mathbf{r}) \tag{24}$$

We transform summands in (24) $\mathbf{r} \times \mathbf{v} = \mathbf{r} \times (\dot{\mathbf{V}} + \Omega \times \mathbf{V}) \cdot \mathbf{P}^T$. Also (22) you can get

$$\check{\nabla} \cdot (\mathbf{T} \cdot \mathbf{P}^T) \times \mathbf{r} = -\mathbf{r} \times (\check{\nabla} \cdot \mathbf{T} + (\mathbf{K}^T \cdot \mathbf{T})_x) \cdot \mathbf{P}^T$$

$$\mathbf{R}^k \cdot \mathbf{T} \cdot \mathbf{P}^T \times \mathbf{r}_k = -\mathbf{P} \cdot ((\mathbf{E} + \mathbf{I})^T \cdot \mathbf{T})_x$$

$$(\mathbf{P} \cdot \mathbf{J}_0 \cdot \Omega)^{\cdot} = \mathbf{P} \cdot (\mathbf{J}_0 \cdot \dot{\Omega} + \Omega \times \mathbf{J}_0 \cdot \Omega)$$

Return to (24), using (23) we get a local form of the kinetic moment balancing equation in RC.

$$((\mathbf{E} + \mathbf{I})^T \cdot \mathbf{T})_x = \rho_0 \cdot (\mathbf{J}_0 \cdot \dot{\Omega} + \Omega \times \mathbf{J}_0 \cdot \Omega) \tag{25}$$

4 The compatibility of strains and velocities in RC

Lets differentiate with respect to time (18). In RC basis does not depend on time, so we get $\dot{\mathbf{F}}^T = \check{\nabla}\mathbf{v}$ using the opportunity to reshuffle $\partial/\partial t$ and $\partial/\partial x$.

Considering (16) and $\dot{\mathbf{I}} = 0$ we get

$$\dot{\mathbf{E}} = \check{\nabla}\mathbf{v} \cdot \mathbf{P} + \mathbf{F}^T \cdot \mathbf{P} \times \Omega \tag{26}$$

We had transformed first summand from (26) using (21) and (14):

$$\check{\nabla}\mathbf{v} \cdot \mathbf{P} = \check{\nabla}(\mathbf{v} \cdot \mathbf{P}) - \check{\nabla}\mathbf{P}^T \cdot \mathbf{v} = \check{\nabla}\mathbf{V} + \mathbf{K} \times \mathbf{P}^T \cdot \mathbf{v} = \check{\nabla}\mathbf{V} + \mathbf{K} \times \mathbf{V} \tag{27}$$

We had transformed second summand from (26) using (18) and finally we get

$$\dot{\mathbf{E}} = \check{\nabla}\mathbf{V} + \mathbf{K} \times \mathbf{V} + (\mathbf{E} + \mathbf{I}) \times \Omega. \tag{28}$$

Lets derive an equation relating the **K** and Ω. Transpose both sides of the (16), we get $\check{\nabla}\dot{\mathbf{P}}^T = -\check{\nabla}\Omega \times \mathbf{P}^T - \mathbf{R}^s\Omega \times \frac{\partial \mathbf{P}^T}{\partial x^s} = -\check{\nabla}\Omega \times \mathbf{P}^T + \mathbf{R}^s\Omega \times (\mathbf{K}_s \times \mathbf{P}^T)$. To transform the second term we use the identity [4] $\mathbf{a} \times (\mathbf{b} \times \mathbf{A}) = \mathbf{b} \times (\mathbf{a} \times \mathbf{A}) + (\mathbf{a} \times \mathbf{b}) \times \mathbf{A}$ which is valid for any $\mathbf{a}$,$\mathbf{b}$,$\mathbf{A}$. Then

$$\check{\nabla}\dot{\mathbf{P}}^T = -\check{\nabla}\Omega \times \mathbf{P}^T + \mathbf{K} \times (\Omega \times \mathbf{P}^T) - (\mathbf{K} \times \Omega) \times \mathbf{P}^T \tag{29}$$

Lets differentiate with respect to time (21), so we get

$$(\check{\nabla}\mathbf{P}^T)^{\cdot} = -\dot{\mathbf{K}} \times \mathbf{P}^T + \mathbf{K} \times (\Omega \times \mathbf{P}^T) \tag{30}$$

In RC basis does not depend on time, so $(\check{\nabla}\mathbf{P}^T)^{\cdot} = \check{\nabla}(\dot{\mathbf{P}}^T)$. We equate (29) and (30) $\dot{\mathbf{K}} \times \mathbf{P}^T = (\check{\nabla}\Omega + \mathbf{K} \times \Omega) \times \mathbf{P}^T$. Hence

$$\dot{\mathbf{K}} = \check{\nabla}\Omega + \mathbf{K} \times \Omega \tag{31}$$

Equations (28) and (31) are compatibility equations.
In [5] it shown that constitutive equations in RC are

$$\rho_0 \frac{\partial \Pi}{\partial \mathbf{E}} = \mathbf{T}. \tag{32}$$

Unknowns number increases to 33 in RC: 9 stresses **T**, 9 strains **E**, 6 velocities $\mathbf{V}, \Omega$ and 9 components of the additional tensor **K**. Corresponding equation are: 6 motion equations (23),(25), 18 compatibility equations (28),(31), 9 relations of elasticity (32). In total there are 33 equations.

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