The different types transverse oscillations of thin helical beam

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Abstract

The non-self-conjugated problem of interconnected spatial vibrations thin helical beam was investigated. There are two branches of characteristic equation. Pairs of classical fundamental transverse modes are located near characteristic equation local minimum. But there are pairs of modes which mode shapes are look like spatial fundamental transverse modes and are situated near characteristic equation local maximums.

1 Equations

Interconnected spatial vibrations of the thin helical beam are analyzed on the basis of Kirchhoff-Clebsch's equations [1]. There are 6 equations of motion (1):

$$\begin{aligned} &\frac{\partial Q^*_\chi}{\partial s}+q^*Q^*_\rho-r^*Q^*_\xi = a_1\frac{\partial^2 U}{\partial t^2}; && \frac{\partial M^*_\chi}{\partial s}-Q^*_\xi-r^*M^*_\xi+q^*M^*_\rho = a_4\frac{\partial^2 \theta}{\partial t^2};\\ &\frac{\partial Q^*_\xi}{\partial s}+r^*Q^*_\chi-p^*Q^*_\rho = a_2\frac{\partial^2 V}{\partial t^2}; && \frac{\partial M^*_\xi}{\partial s}+Q^*_\chi+r^*M^*_\chi-p^*M^*_\rho = a_5\frac{\partial^2 \varphi}{\partial t^2};\\ &\frac{\partial Q^*_\rho}{\partial s}-q^*Q^*_\chi+p^*Q^*_\xi = a_3\frac{\partial^2 W}{\partial t^2}; && \frac{\partial M^*_\rho}{\partial s}-q^*M^*_\chi+p^*M^*_\xi = a_6\frac{\partial^2 \gamma}{\partial t^2}; \end{aligned} \tag{1}$$

where all asterisked denotes mean end values of the corresponded terms.

Next 6 equations are geometrical equations (2):

$$\begin{aligned} &\theta = -(\frac{\partial V}{\partial s}+r_0U-p_0W); && \varphi = \frac{\partial U}{\partial s}-r_0V+q_0W;\\ &0 = \frac{\partial W}{\partial s}-q_0U+p_0V; && \delta p = \frac{\partial \theta}{\partial s}-r_0\varphi+q_0\gamma;\\ &\delta q = \frac{\partial \varphi}{\partial s}+r_0\theta-p_0\gamma; && \delta r = \frac{\partial \gamma}{\partial s}-q_0\theta+p_0\varphi. \end{aligned} \tag{2}$$

The Bernoulli–Euler hypothesis:

$$\delta r = C^{-1}\cdot M_\rho; \qquad \delta q = B_\xi^{-1}\cdot M_\xi; \qquad \delta p = B_\chi^{-1}\cdot M_\chi, \tag{3}$$

The system of coordinates is linked to the helix spring line (Fig. 1). For the our coordinate system: χ, ξ, ρ the principal normal, binormal and tangent direction vectors, respectively; u, v, w – the linear displacements; θ, φ, γ — angular displacements; M_χ, M_ξ, M_ρ — components of the internal moment vector; Q_χ, Q_ξ, Q_ρ — components of the internal force vector, s is the arc length along the helix line. B_χ, B_ξ are the bending stiffness with

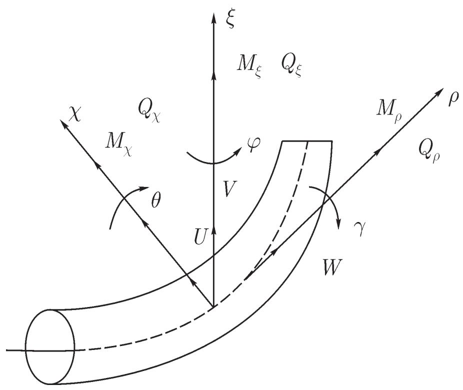

Figure 1: System of coordinates

respect to axis χ and ξ, respectively, C – the torsional rigidity, p, q denotes curvatures and r denote the torsion of the infinitesimal helix element. R_0 is the centerline radius of the helix. α is the helix angle. $C_1 = 1/B_\chi$, $C_2 = 1/B_\xi$, $C_3 = 1/C$ – flexibilities.

We investigate the event with the round coil cross-section, the constant helix angle, static stresses are accepted as constant along the helix spring line.

Then the equations were linearized by the well-known procedure:

$$\begin{aligned} Q^*_\chi &= Q_{\chi\,0} + Q_\chi; & M^*_\chi &= M_{\chi\,0} + M_\chi; \\ Q^*_\rho &= Q_{\rho\,0} + Q_\rho; & M^*_\rho &= M_{\rho\,0} + M_\rho; \\ Q^*_\xi &= Q_{\xi\,0} + Q_\xi; & M^*_\xi &= M_{\xi\,0} + M_\xi. \end{aligned} \tag{4}$$

With the assumption of invariability internal force factors along helix line, we obtain

$$\frac{\partial\, Q_{\chi\,0}}{\partial s} = \frac{\partial\, Q_{\rho\,0}}{\partial s} = \frac{\partial\, Q_{\xi\,0}}{\partial s} = \frac{\partial M_{\chi\,0}}{\partial s} = \frac{\partial M_{\rho\,0}}{\partial s} = \frac{\partial M_{\xi\,0}}{\partial s} = 0. \tag{5}$$

We neglected of the shear strain. With neglect the inertial terms considering twisting infinitesimal helix element, we obtained $a_4 = a_6 = a_5 = 0$. The inertial terms considering translation infinitesimal helix elemen are $a_1 = a_2 = a_3 = a$.

So the equations [1] may be written as:

$$\begin{aligned} &\frac{\partial Q_\chi}{\partial S} + q_0 Q_\rho + \delta q Q_{\rho_0} - (r_0 Q_\xi + \delta r Q_{\xi_0}) = a_1 \frac{\partial^2 U}{\partial t^2} - \delta q Q_\rho + \delta r Q_\xi; \\ &\qquad \frac{\partial Q_\xi}{\partial S} + r_0 Q_\chi + \delta r Q_{\chi_0} - (\rho_0 Q_\rho + \delta\rho Q_{\rho_0}) = \\ &\qquad = a_2 \frac{\partial^2 V}{\partial t^2} - \delta r Q_\chi + \delta\rho Q_\rho - r_0 Q_{\chi_0} + \rho_0 Q_{\rho_0}; \\ &\qquad \frac{\partial Q_\rho}{\partial S} - (q_0 Q_\chi + \delta q Q_{\chi_0}) + \rho_0 Q_\xi + \delta\rho Q_{\xi_0} = \\ &\qquad = a_3 \frac{\partial^2 W}{\partial t^2} - \delta\rho Q_\xi + \delta q Q_\chi - \rho_0 Q_{\xi_0} + q_0 Q_{\chi_0}; \end{aligned} \tag{6}$$

$$
\begin{aligned}
&\frac{\partial M_\chi}{\partial S} - Q_\xi + q_0 M_\rho + \delta q M_{\rho_0} - (r_0 M_\xi + \delta r M_{\xi_0}) = -\delta\rho M_\rho + \delta r M_\xi;\\
&\frac{\partial M_\xi}{\partial S} + Q_\chi + r_0 M_\chi + \delta r M_{\chi 0} - (\rho_0 M_\rho + \delta\rho M_{\rho_0}) =\\
&= -\delta r M_\chi + \delta\rho M_\rho - r_0 M_{\chi 0} + \rho_0 M_{\rho_0} - Q_{\chi 0};\\
&\frac{\partial M_\rho}{\partial S} - (q_0 M_\chi + \delta q M_{\chi 0}) + \rho_0 M_\xi + \delta\rho M_{\xi_0} =\\
&= -\delta\rho M_\xi + \delta q M_\chi - \rho_0 M_{\xi_0} + q_0 M_{\chi 0}.
\end{aligned}
\tag{7}
$$

In the equations (6)...(7) are took into account the next relations:

$$
q_0 Q_{\rho 0} - r_0 Q_{\zeta 0} = 0; \qquad q_0 M_{\rho 0} - r_0 M_{\zeta 0} - Q_{\zeta 0} = 0. \tag{8}
$$

For the cylindricity is obtain:

$$
p_0 = M_{\chi 0} = Q_{\chi 0} = 0, \tag{9}
$$

original equation became:

$$
\begin{aligned}
&\frac{\partial Q_\chi}{\partial S} + q_0 Q_\rho - r_0 Q_\xi + \delta\, q Q_{\rho_0} - \delta\, r Q_{\xi_0} = a_1 \frac{\partial^2 U}{\partial t^2};\\
&\frac{\partial Q_\xi}{\partial S} + r_0 Q_\chi - \delta\, p \cdot Q_{\rho_0} = a_2 \frac{\partial^2 V}{\partial t^2};\\
&\frac{\partial Q_\rho}{\partial S} - q_0 Q_\chi + \delta\, p \cdot Q_{\xi_0} = a_3 \frac{\partial^2 W}{\partial t^2};\\
&\frac{\partial M_\chi}{\partial S} - Q_\xi + q_0 M_\rho + \delta\, q M_{\rho_0} - r_0 M_\xi - \delta\, r M_{\xi_0} = 0;\\
&\frac{\partial M_\xi}{\partial S} + Q_\chi + r_0 M_\chi - \delta\, p M_{\rho_0} = 0;\\
&\frac{\partial M_\rho}{\partial S} - q_0 M_\chi + \delta\, p M_{\xi_0} = 0.
\end{aligned}
\tag{10}
$$

Equations reduced to the double-point boundary problem through the separation of a variables. The original equations in the matrix form, which are used in Izhevsk [1] and MSTU research teams, are presented.

$$
\frac{\partial Y_i}{\partial s} + |A| \cdot Y_i = 0; \tag{11}
$$

$$
\ddot{T} + \omega^2 T = 0, \tag{12}
$$

where

$$
\begin{gathered}
Y_1 = Q_\xi;\ Y_2 = M_\chi;\ Y_3 = M_\rho;\ Y_4 = \theta;\\
Y_5 = \gamma;\ Y_6 = V;\ Y_7 = Q_\chi;\ Y_8 = Q_\rho;\\
Y_9 = M_\xi;\ Y_{10} = \varphi;\ Y_{11} = U;\ Y_{12} = W,
\end{gathered}
$$

$$A = \begin{vmatrix}
0 & -C_1Q_{\rho_0} & 0 & 0 & 0 & aw^2 & r_0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & q_3 & 0 & 0 & 0 & 0 & 0 & -r_2 & 0 & 0 & 0 \\
0 & -q_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -C_1 & 0 & 0 & q_0 & 0 & 0 & 0 & 0 & -r_0 & 0 & 0 \\
0 & 0 & -C_3 & -q_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & r_0 & 0 \\
-r_0 & 0 & -C_3Q_{\xi_0} & 0 & 0 & 0 & 0 & q_0 & C_2Q_{\rho_0} & 0 & aw^2 & 0 \\
0 & C_1Q_{\xi_0} & 0 & 0 & 0 & 0 & -q_1 & 0 & 0 & 0 & 0 & aw^2 \\
0 & r_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & r_0 & 0 & 0 & 0 & 0 & -C_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -r_0 & 0 & 0 & 0 & -1 & 0 & q_0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q_0 & 0
\end{vmatrix} \qquad (13)$$

$$q_1 = q_0 - C_1 M_{\xi_0}; \quad q_3 = q_0 - C_3 M_{\xi_0};$$
$$r_1 = r_0 - C_1 M_{\rho_0}; \quad r_2 = r_0 - C_2 M_{\rho_0}.$$

The preliminary compression and torsion of the thin helical beam are taken into account. The initial compression is defined as $m = \frac{H-H_0}{H_0}$, where H_0 and H are the preliminary height and the current height helix, respectively [1].

$$m = Q_{\xi\,0} \cdot R_0^2 \cdot c_3 \cdot q_0 \cdot r_0^{-1}$$

The preliminary torsion about the helix axis is defined as $m_2 = M_0/(P_0\, R_0)$ [1].

2 Results

All computations are executed in the dimensionless form with respect to the fundamental spatial longitudinal frequency.

The features of the thin helical beam allowed to construct the fundamental solution matrix for the all three variants of the characteristic equation roots.

Two cases of the boundary conditions are used: the fixed ends (the origin placed into the center of the spring) and the special hinged ends. The boundary conditions with the special hinged ends:

$$\begin{aligned}
M_\chi(0) = \gamma(0) &= V(0) = M_\chi(s_0) &= \gamma(s_0) = V(s_0) &= 0 \\
Q_\chi(0) = \varphi(0) &= W(0) = Q_\chi(s_0) &= \varphi(s_0) = W(s_0) &= 0
\end{aligned}$$

enabled to obtain the frequency equation ("generating solution"), Fig 2.

Harmonic number n is associated with wavenumber $k = n{\cdot}\cos alpha/(2i)$ thus frequency spectrum of generating solution also is discreet dispersion relation. So we can obtain phase and group velocities. It is evident from Fig. 2 that with increase of harmonic number, frequencies initially increase then decrease and then increase again. Such behavior frequency spectrum by reason of the non-self-conjugacy represented problem.

All types of special vibrations can be achieved on the basis of the two elastic deformation waves which correspond to the two branches of the characteristic equation. The low branch of the characteristic equation corresponds to spatial longitudinal vibrations. The high branch of the characteristic equation corresponds to spatial torsion vibrations. But spatial longitudinal vibrations correspond with shear waves because helix turn of the compression spring work in torsion.

For the fixed ends boundary conditions, the characteristic and boundary equations must simultaneously be inversed into zero. The mode shapes are constructed along the preliminary deformed helix line.

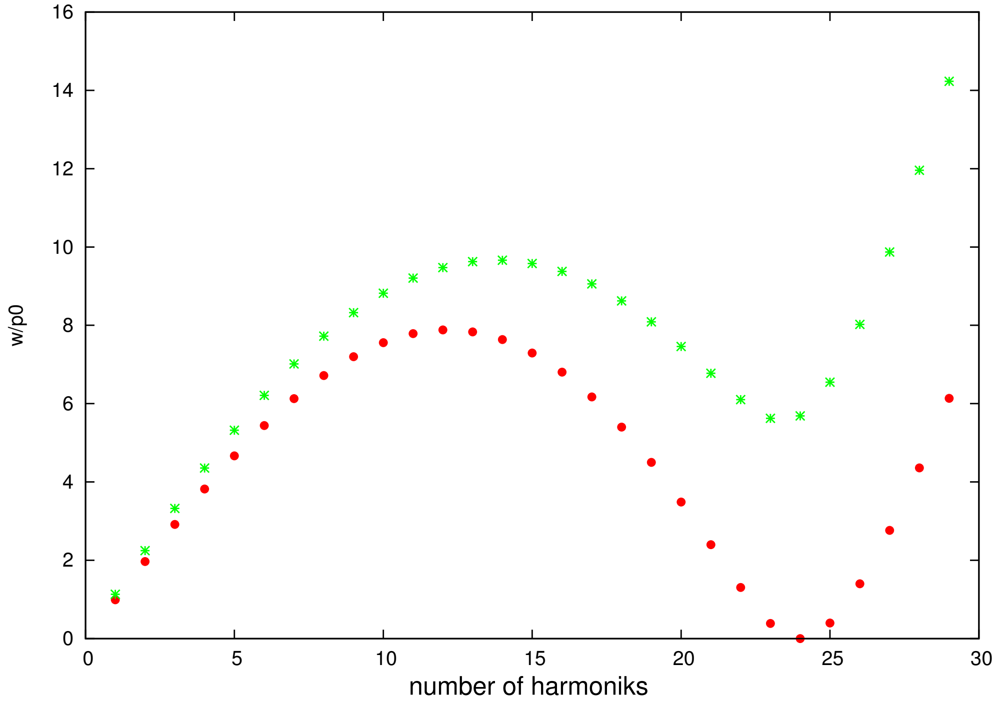

Figure 2: Frequency spectrum of "generating solution" $i = 12$, $m = 0$, $\alpha = 8$, $m_2 = 0$.

There were obtained double modes which correspond to the local extreme points of the characteristic equation Ii the frequency spectrum for fixed ends boundary conditions. The characteristic equation with the local extreme points are shown on the Fig 3. These modes are non-propagating because their group velocity is bound to be equal to zero. We called them "special" frequencies.

In the local maximums half-wave fits the helix turn, in the local minimums wave fits the helix turn. Near the local minimums form pairs of the classical spatial fundamental transverse modes. The static axial (Euler's) instability occurs when one of these modes is equal to zero. The mode shapes of classical spatial fundamental transverse modes are presented in Figs 4, 5. Note that envelops of transversal components spatial fundamental transversal vibration are the half-sinusoids.

Near the spatial fundamental transverse modes their first harmonics allocates. The envelops of transversal components these first harmonic of the transversal vibration are the sinusoids, Fig. 6.

But there are pair modes which envelops of transverse components forms are the half-sinusoids near the local maximums, Figs 7 and 8. They look like a spatial fundamental transversal vibrations. So we called them the "spatial fundamental transverse modes near W11" (W21).

Near the "spatial fundamental transverse modes near W11" (W21) form the pairs modes which envelops of transversal components are the sinusoids, Fig 9. We called these modes "first harmonics spatial transverse modes near W11 (W21) by shapes". These modes have smaller frequency than the frequencies of its fundamental prototypes. It happens because transverse modes near W11 (W21) arranges on the convex parts of the characteristic equation branches.

The physical model that the spatial longitudinal vibrations transfer with the increase of harmonic number through the "spatial transversal vibrations near W21" to the spatial

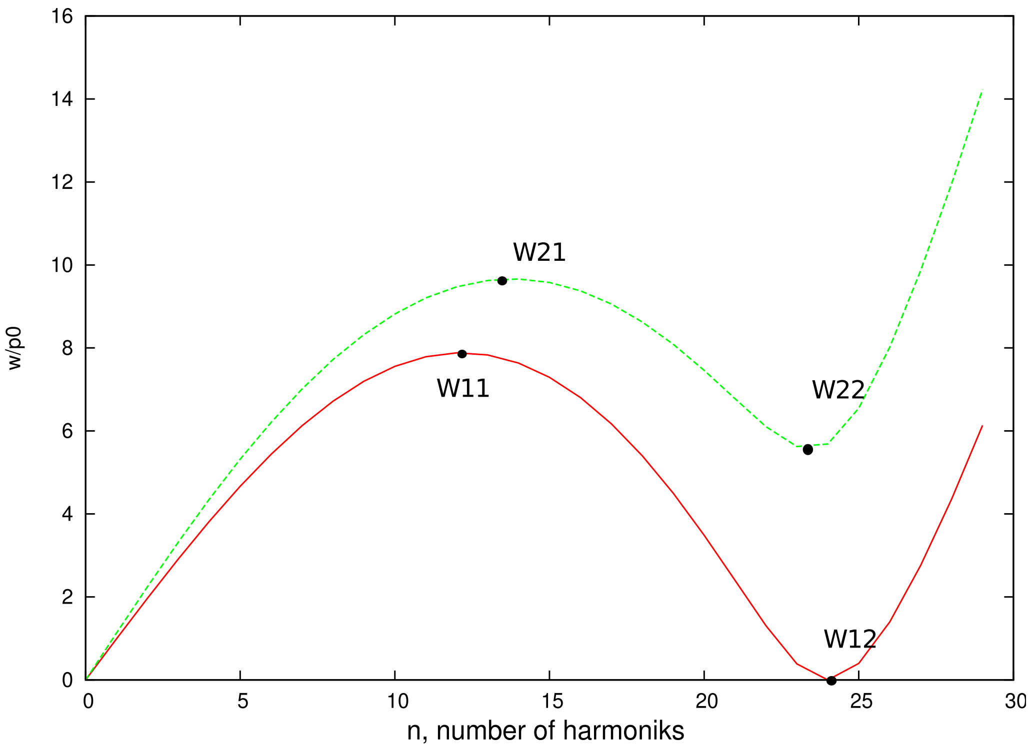

Figure 3: "Special" frequencies

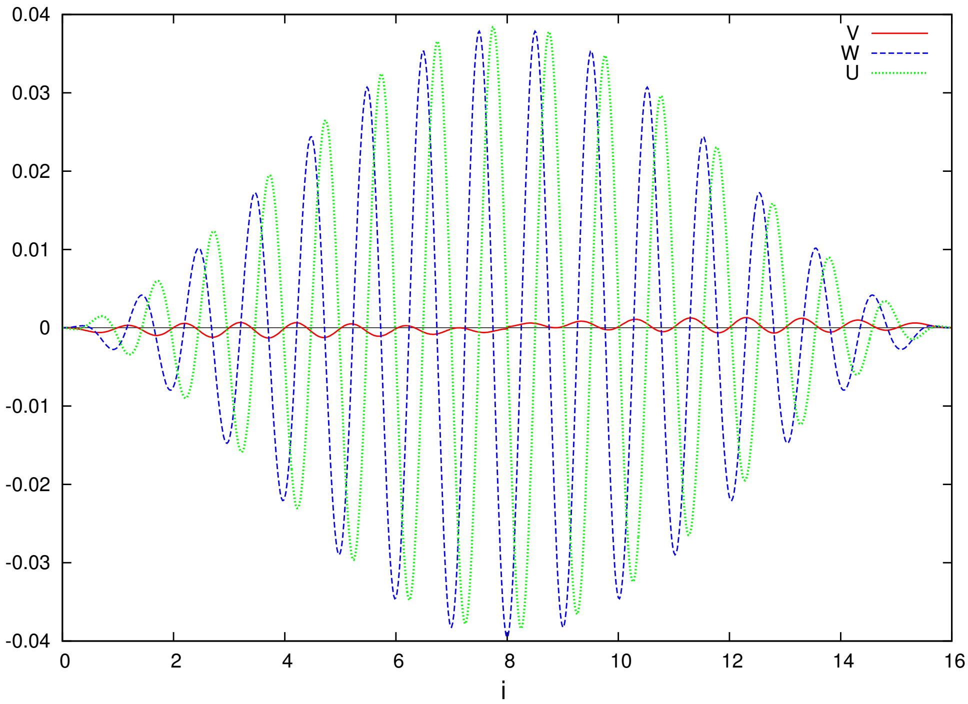

Figure 4: Mode shapes of fundamental transverse antisymmetric modes ($i = 16$, $\alpha = 6^0$, $m = -0,1$ $\omega = 0,60$)

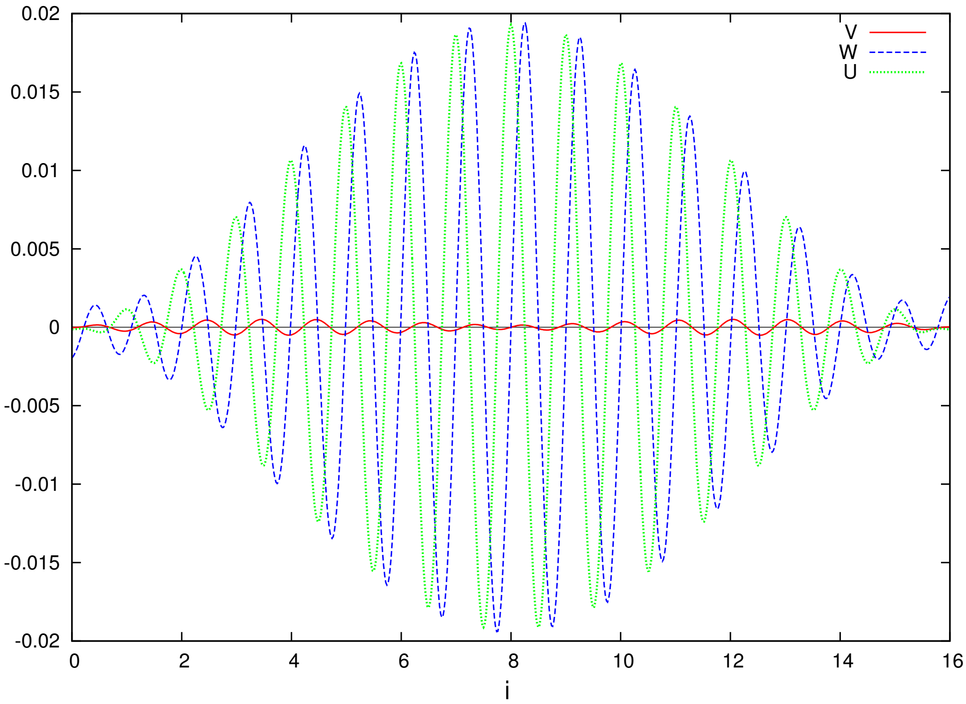

Figure 5: Mode shapes of fundamental transverse symmetric modes ($i = 16$, $\alpha = 6^0$, $m = -0,1$ $\omega = 0,60$)

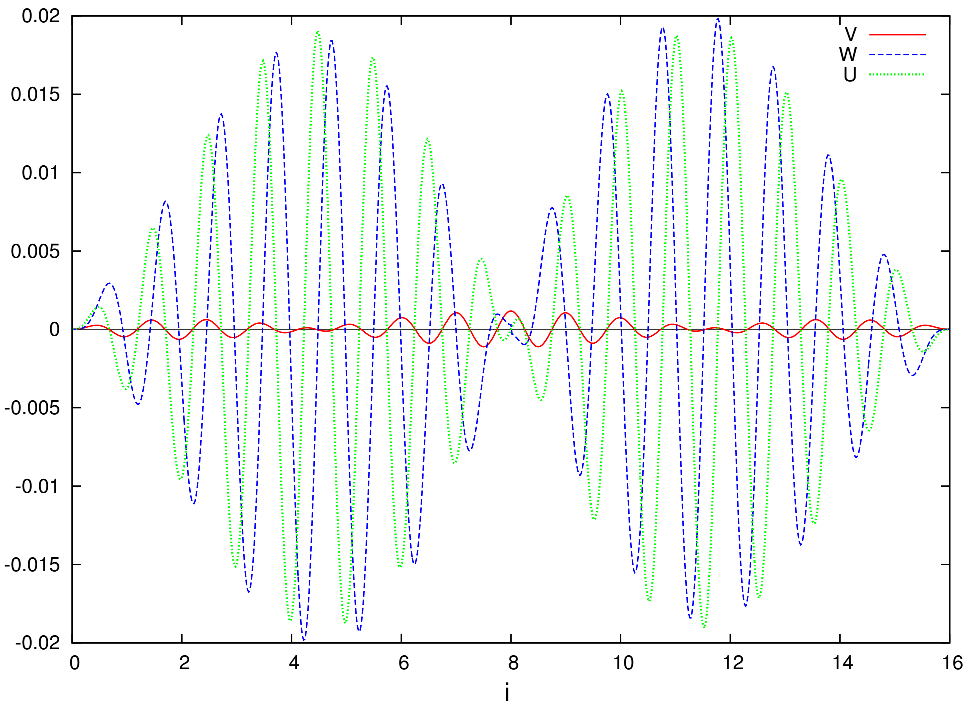

Figure 6: Mode shapes of first harmonic transversal antisymmetric mode ($i = 16$, $\alpha = 6^0$, $m = -0,1$ $\omega = 1,54$)

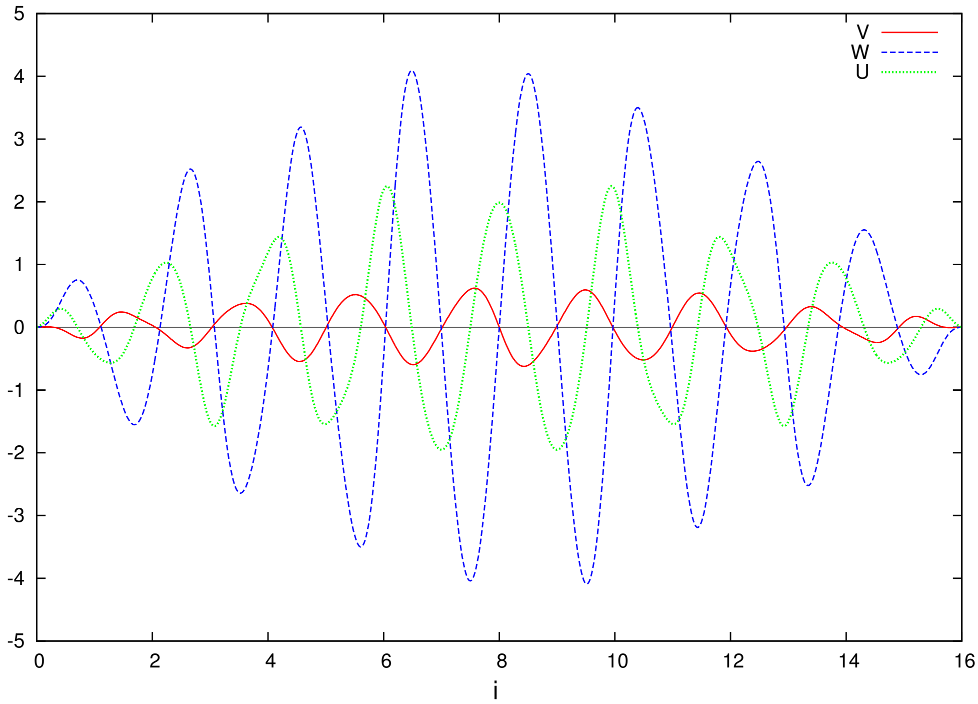

Figure 7: Mode shapes of fundamental transverse symmetric modes “near W21” ($i = 16$, $\alpha = 6^0$, $m = -0,1$ $\omega = 10,47$)

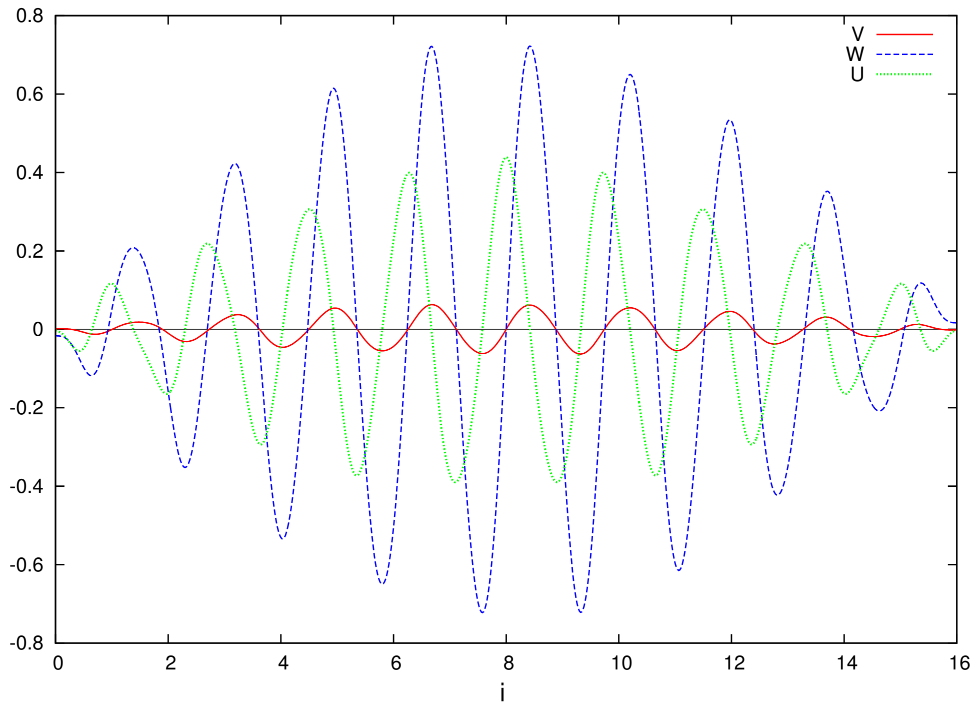

Figure 8: Mode shapes of fundamental transverse symmetric mode “near W11” ($i = 16$, $\alpha = 6^0$, $m = -0,1$ $\omega = 10,47$)

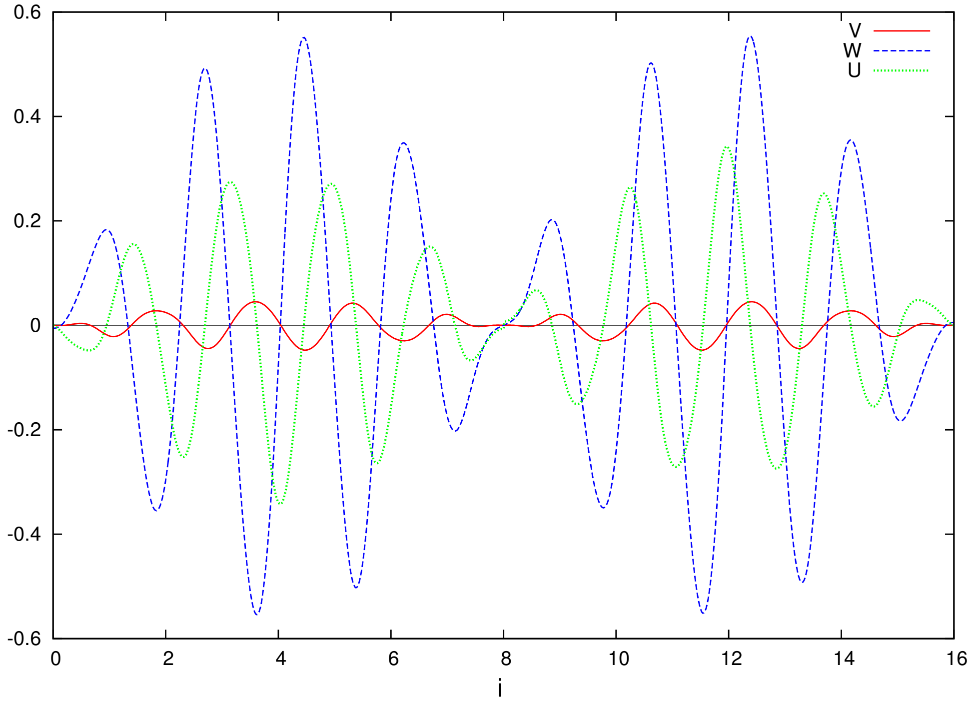

Figure 9: Mode shapes of first harmonic transversal symmetric mode near W21 ($i = 16$, $\alpha = 6^0$, $m = -0,1$ $\omega = 12,820$)

transversal vibration could be assumed. It is a similar situation to the high branch of characteristic equation.

3 Conclusions

The non-self-conjugated problem of interconnected spatial vibrations thin helical beam was investigated. There are two branches of characteristic equation. In the frequency spectrum for fixed ends boundary conditions were obtained double non-propagating modes which corresponds to local extreme points of characteristic equation. Near the local minimums form pairs of the classical spatial fundamental transverse modes. Near the spatial fundamental transverse modes form their first harmonics. But near local maximums forms pair modes which mode shapes looks like a mode shapes spatial fundamental transverse modes. We called these modes “spatial fundamental transverse modes near W11” (W21). The physical model that the spatial longitudinal vibrations transfer with the increase of harmonic number through the “spatial transversal vibrations near W21” to the spatial transversal vibration could be assumed. It is a similar situation to the high branch of characteristic equation.

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