# Motion of a Rotationally Symmetric Paraboloid on a Perfectly Rough Plane

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**Abstract**

The problem of motion of a rotationally symmetric paraboloid on a fixed perfectly rough horizontal plane is considered. The qualitative description of motion of the paraboloid on the plane is given. Steady motions of the paraboloid (permanent rotations and regular precessions) are described. It is proved that all the steady motions of the paraboloid are stable.

## 1 Problem formulation and equations of motion

Let us consider the general problem of motion of a rigid body bounded by an axisymmetric surface on a fixed perfectly rough horizontal plane. Suppose that the center of mass $G$ of the body is situated on the symmetry axis $G\zeta$, and moments of inertia about principal axes of inertia $G\xi$ and $G\eta$ perpendicular to $G\zeta$ are equal to each other. The body moves in presence of the homogeneous gravity field.

Let $Oxyz$ be the fixed coordinate frame with the origin in the supporting plane $Oxy$. Let $\theta$ be the angle between the symmetry axis and the vertical. The distance $GQ$ of the center of mass over the plane $Oxy$ is a function of angle $\theta$ ([1, 2]):

$$GQ = f(\theta). \tag{1.1}$$

Let $\beta$ be the angle between meridian $M\zeta$ of the body and a certain fixed meridian plane, and $\alpha$ is the angle between horizontal tangent $MQ$ of the meridian $M\zeta$ and the $Ox$-axis. The position of the body will be completely determined by the angles $\alpha$, $\beta$ and $\theta$ and the $x$ and $y$ coordinates of the point $M$.

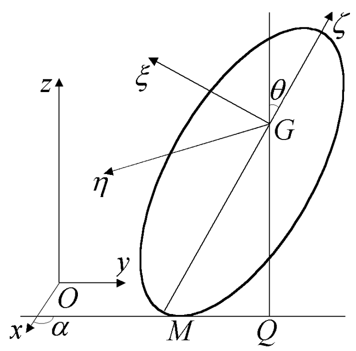


Figure 1: Motion of body of revolution: basic coordinate systems.

Let us specify now the position of the coordinate system $G\xi\eta\zeta$. Suppose that the $G\xi$-axis always lies in the plane of vertical meridian $M\zeta$ while the $G\eta$-axis is perpendicular to this plane (Fig. 1). In this case the coordinate frame $G\xi\eta\zeta$ moves both in the space and in the body. Denote by $\xi$, $\eta$, $\zeta$ the coordinates of the point of contact $M$ of the body with the supporting plane in the coordinate frame $G\xi\eta\zeta$. Then ([1, 2]):

$$\xi = -f(\theta)\sin\theta - f'(\theta)\cos\theta, \quad \eta = 0, \quad \zeta = -f(\theta)\cos\theta + f'(\theta)\sin\theta, \tag{1.2}$$

where $()'$ is a derivative of function $f(\theta)$ with respect to $\theta$. Therefore function $f(\theta)$ completely characterizes the shape of the surface of the moving body.

Let the velocity $\mathbf{v}$ of the center of mass $G$ and the angular velocity vector $\boldsymbol{\omega}$ of the body are specified in the coordinate frame $G\xi\eta\zeta$ by the components $v_\xi$, $v_\eta$, $v_\zeta$ and $p$, $q$, $r$ respectively. Let $m$ be the mass of the body, $A_1$ – its moment of inertia about axes $G\xi$ and $G\eta$, and $A_3$ – its moment of inertia about the symmetry axis. The velocity of the point of contact $M$ is zero therefore

$$v_\xi + q\zeta = 0, \quad v_\eta + r\xi - p\zeta = 0, \quad v_\zeta - q\xi = 0,$$

and for four unknown functions $p$, $q$, $r$ and $\theta$ we obtain closed system of equations [1]:

$$\begin{aligned}
&\left[A_1 + m\left(\xi^2+\zeta^2\right)\right]\frac{dq}{dt} = mgf'(\theta) + (A_3 r - A_1 p \operatorname{ctg}\theta)\,p-\\
&-mp\,(\zeta\operatorname{ctg}\theta + \xi)\,(p\zeta - r\xi) - mq\left(\xi\frac{d\xi}{dt} + \zeta\frac{d\zeta}{dt}\right),\\
&A_1\frac{dp}{dt} + A_3\frac{\zeta}{\xi}\frac{dr}{dt} = (A_1 p\operatorname{ctg}\theta - A_3 r)\,q,\\
&\frac{d}{dt}(p\zeta - r\xi) - \frac{A_3}{m\xi}\frac{dr}{dt} = (\zeta\operatorname{ctg}\theta + \xi)\,pq, \quad q = -\frac{d\theta}{dt}.
\end{aligned} \tag{1.3}$$

The system (1.3) admits the energy integral:

$$A_1p^2 + \left(A_1 + m\left(\xi^2+\zeta^2\right)\right)q^2 + A_3r^2 + m\,(p\zeta - r\xi)^2 + 2mgf(\theta) = c_0^2 = \text{const}. \tag{1.4}$$

Assume $\theta \neq \text{const}$. Then using the last equation of the system (1.3) we change the independent variable $t$ in the second and in the third equation of the system to new independent variable $\theta$. As a result we obtain:

$$\begin{aligned}
&A_1\frac{dp}{d\theta} + A_3\frac{\zeta}{\xi}\frac{dr}{d\theta} = -A_1 p\operatorname{ctg}\theta + A_3 r,\\
&\zeta\frac{dp}{d\theta} - \frac{A_3 + m\xi^2}{m\xi}\frac{dr}{d\theta} = -(\zeta\operatorname{ctg}\theta + \xi + \zeta')p + \xi' r.
\end{aligned} \tag{1.5}$$

Integration of the system (1.5) gives the expressions for $p$ and $r$ as functions of $\theta$ with two arbitrary constants. Further integration of the problem finishes in quadratures. By now the considered problem has been completely solved in the case when the moving body is a nonhomogeneous dynamically symmetric sphere [1, 2] or circular disk [2, 3]. Recently (see [4, 5]) it was proved that this problem can be completely solved also in the case when the moving rigid body is a rotationally symmetric paraboloid. The present paper deals with the detailed study of motion of a rotationally symmetric paraboloid on a fixed perfectly rough horizontal plane.

## 2 Motion of a rotationally symmetric paraboloid

Suppose that the moving rigid body is a rotationally symmetric paraboloid with the focal length $2\lambda$. By formulae (1.1) and (1.2) we obtain:

$$f(\theta) = \frac{\lambda}{\cos\theta}, \quad \xi = -\frac{2\lambda\sin\theta}{\cos\theta}, \quad \zeta = \frac{\lambda\sin^2\theta}{\cos^2\theta} - \lambda, \quad \xi^2 = 4\lambda\zeta + 4\lambda^2. \tag{2.1}$$

In this case the solution of the system (1.5) has the form ([4, 5]):

$$r(\theta) = \sqrt{\frac{K_1(\theta)}{K_2(\theta)}}\left(c_1\cos\Phi(\theta) + c_2\sin\Phi(\theta)\right),$$

$$p(\theta) = -\frac{\cos\theta}{2A_1\sqrt{K_1(\theta)}\sin\theta}\left[\left(c_2\cos\Phi(\theta) - c_1\sin\Phi(\theta)\right)D\cos\theta + \right.$$

$$\left. + \frac{2A_3}{\sqrt{K_2(\theta)}}\left(K_1(\theta) + (2A_1 - A_3)\,m\lambda^2\left(1 - 2\cos^2\theta\right)\right)\left(c_1\cos\Phi(\theta) + c_2\sin\Phi(\theta)\right)\right],$$

$$\Phi(\theta) = 2m\lambda^2 D\int\limits_0^\theta \frac{\sin^3\varphi\cos^2\varphi d\varphi}{K_1(\varphi)\sqrt{K_2(\varphi)}}, \quad D = \sqrt{2A_1A_3\left(A_3 + 4m\lambda^2\right)(2A_1 - A_3)},$$

$$K_1(\theta) = \left(A_1A_3 + 4A_1m\lambda^2\right)\cos^4\theta - 2A_3m\lambda^2\cos^2\theta + A_3m\lambda^2,$$

$$K_2(\theta) = \left(A_1A_3 + 4m\lambda^2(A_3 - A_1)\right)\cos^4\theta - 4m\lambda^2(A_3 - A_1)\cos^2\theta + A_3m\lambda^2.$$

Here $c_1$ and $c_2$ are arbitrary constants and function $\Phi(\theta)$ is expressed through the elliptic functions. It is easy to see that the functions $p(\theta)$ and $r(\theta)$ have a very complicated form. They can be simplified only in the case when the moments of inertia of the paraboloid are connected by the relation $A_3 = 2A_1$. For this particular case we have:

$$D = 0, \quad \Phi(\theta) = 0, \quad r(\theta) = c_1 = \text{const}, \quad p(\theta) = -\frac{2\cos\theta}{\sin\theta}c_1.$$

This particular case has been completely investigated by Kh. M. Mustari [6]. However the motion of a paraboloid when $D \neq 0$ and consequently $\Phi(\theta) \neq 0$ is a question of some interest. Except the case $A_3 = 2A_1$ there are no any other cases when we can simplify the elliptic integrals, which are included in the expression for $\Phi(\theta)$. Therefore the investigation of motion of a rotationally symmetric paraboloid on a perfectly rough horizontal plane is realized similarly for all values of moments of inertia which are not connected by relation $A_3 = 2A_1$. Therefore we will consider further the case of a homogeneous paraboloid. By the plane perpendicular to the symmetry axis of the paraboloid we can cut off the parabolic segment such that its center of mass will be in the focus of the corresponding generating parabola. Trivial calculations show that the cutting plane should be defined by equation $\zeta = \lambda/2$. For the considered parabolic segment its equatorial and axial principal central moments of inertia equal:

$$A_1 = \frac{9}{8}m\lambda^2, \quad A_3 = 2m\lambda^2. \tag{2.2}$$

Further we will refer to this parabolic segment as the paraboloid. While rolling on the plane angle $\theta$ varies in the following limits:

$$-\theta_* \le \theta \le \theta_*, \quad \theta_* = \arccos\sqrt{\frac{2}{5}}.$$

Functions $p(\theta)$, $r(\theta)$ and $\Phi(\theta)$ have the following form:

$$r(\theta) = \sqrt{\frac{27\cos^4\theta - 16\cos^2\theta + 8}{23\cos^4\theta - 14\cos^2\theta + 8}}\left(c_1\cos\Phi(\theta) + c_2\sin\Phi(\theta)\right),$$

$$p(\theta) = -\frac{4\cos\theta}{\sin\theta\sqrt{27\cos^4\theta - 16\cos^2\theta + 8}}\left[\frac{\sqrt{3}\cos\theta}{3}\left(c_2\cos\Phi(\theta) - c_1\sin\Phi(\theta)\right) + \right.$$

$$\left. + \frac{4\left(3\cos^4\theta - 2\cos^2\theta + 1\right)}{\sqrt{23\cos^4\theta - 14\cos^2\theta + 8}}\left(c_1\cos\Phi(\theta) + c_2\sin\Phi(\theta)\right)\right],$$

$$\Phi(\theta) = \int\limits_0^\theta \frac{24\sqrt{3}\sin^3\varphi\cos^2\varphi d\varphi}{\left(27\cos^4\varphi - 16\cos^2\varphi + 8\right)\sqrt{23\cos^4\varphi - 14\cos^2\varphi + 8}}.$$

From the energy integral (1.4) taking into account (2.2) we obtain:

$$\frac{m\lambda^2\left(8 + 9\cos^4\theta\right)}{8\cos^4\theta}q^2 = \left(c_0^2 - \frac{2mg\lambda}{\cos\theta}\right) -$$

$$-m\lambda^2\left[\frac{9}{8}p^2 + 2r^2 + \left(\frac{2\sin\theta}{\cos\theta}r + \frac{\left(1 - 2\cos^2\theta\right)}{\cos^2\theta}p\right)^2\right].$$

Multiplying this equation by $\sin^2\theta$ and taking into account the last equation of system (1.3) we will have:

$$\frac{m\lambda^2}{8}\left(9 + 8u^4\right)\left(\frac{du}{dt}\right)^2 = F(u) = u^2\left(u^2 - 1\right)\left(c_0^2 - 2mg\lambda u\right) - K_0, \qquad (2.4)$$

where $u = 1/\cos\theta$ and the function $K_0$ is determined as follows:

$$K_0 = m\lambda^2u^2\left[\frac{9}{8}u^2p_1^2 + 2\left(u^2 - 1\right)r^2 + \left(2\left(u^2 - 1\right)r + u\left(u^2 - 2\right)p_1\right)^2\right], p_1 = p\sin\theta.$$

If $c_1 = c_2 = 0$ then the paraboloid moves such that $p = r = 0$. In other words it moves such that its symmetry axis $G\zeta$ is situated in the fixed vertical plane. Evolution of the angle $\theta$ is determined by (2.4) where we should set $K_0 = 0$:

$$\frac{m\lambda^2}{8}\left(9 + 8u^4\right)\left(\frac{du}{dt}\right)^2 = u^2\left(u^2 - 1\right)\left(c_0^2 - 2mg\lambda u\right).$$

When $u = 1$ the center of mass of the paraboloid is in the lowest position. Therefore when $c_0^2 < 2mg\lambda$ the motion is impossible. In the case $c_0^2 > 2mg\lambda$ the symmetry axis of the paraboloid oscillates in the vertical plane. The amplitude of these oscillations will be less than $\theta_*$ and the trajectory of the point of contact on the supporting plane will be a straight line segment. Since we suppose that the paraboloid permanently touches the plane by its surface during the motion hence the following inequalities should be valid:

$$1 \le u \le \sqrt{\frac{5}{2}} \qquad (2.5)$$

and therefore for the actual motion we have $2mg\lambda \le c_0^2 \le \sqrt{10}mg\lambda$.

Paraboloid can perform steady motion when its symmetry axis remains at a constant angle $\theta_0$ with the vertical. Below we will consider steady motions of the paraboloid in details.

Let us study the motion of the paraboloid in general case. Note that for the actual motion inequalities (2.5) are valid and the right part of equation (2.4) should be nonnegative, i.e. for the actual motion we have (see (2.4)):

$$F(u) > 0, \quad F\left(\sqrt{\frac{5}{2}}\right) \le 0.$$

Analysis of the function $F(u)$ shows that

$$F(-\infty) > 0, \quad F(-1) < 0, \quad F(0) = 0, \quad F(1) < 0, \quad F(+\infty) < 0.$$

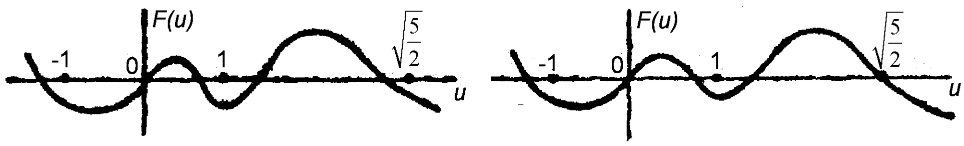


Figure 2

Hence, two possible plots of function $F(u)$ can exist. They are shown on Fig. 2. Thus, the function $F(u)$ is a single-valued function having two zeros in the interval (2.5). Therefore the function $u(t) = 1/\cos\theta(t)$ is a periodic function of $t$.

For the actual motion $u$ oscillates between the values $u_1$ and $u_2$ that is $u_1 \le u \le u_2$ with $u \ne 1$ if $p_0 \ne 0$. It is easy to see that the trajectory of the point of contact on the surface of paraboloid is a curve built up from periodically repeated waves touching two parallels of paraboloid. The trajectory of the point of contact on the supporting plane is a similar curve bounded between two concentric circles that are touched by point $M$ alternately while the paraboloid moves on the plane. Fig. 3 illustrates that fact.

## 3 Steady motions of the paraboloid and their stability

Equations (1.3) have a particular solution

$$p = 0, \quad r = \omega = \text{const}, \quad q = 0, \quad \theta = 0. \tag{3.1}$$

In this motion paraboloid rotates about its vertically situated symmetry axis with arbitrary constant angular velocity $\omega$. It can be proved (see [1]) that for the rotationally symmetric rigid body the condition of stability of the corresponding steady motion has the form:

$$\left(A_3 + mf_0\left(f_0 + f_0''\right)\right)^2 \omega^2 + 4mgf_0''\left(A_1 + mf_0^2\right) > 0, \tag{3.2}$$

where index 0 denotes values of function and its second derivative with respect to $\theta$ when $\theta = 0$. Taking into account that in the case of the paraboloid

$$f''(\theta)\big|_{\theta=0} = \left.\frac{\lambda\left(1+\sin^2\theta\right)}{\cos^3\theta}\right|_{\theta=0} = \lambda > 0,$$

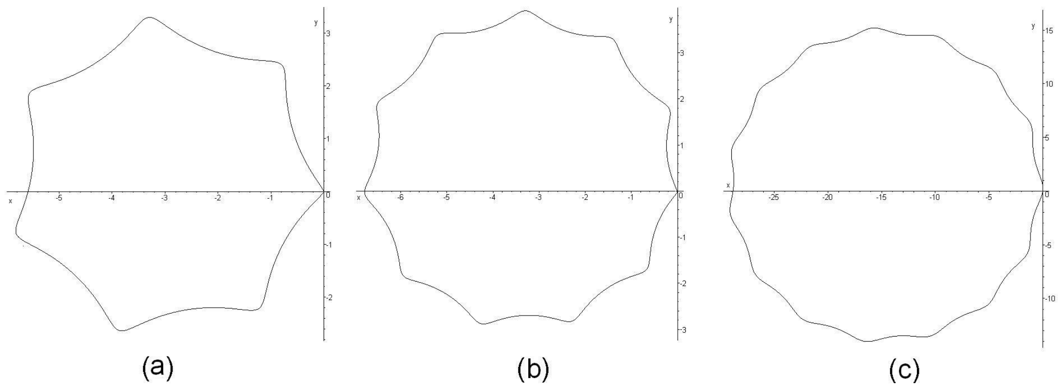


Figure 3: Trajectories of the point contact of paraboloid with the supporting plane for various initial conditions: (a) $\theta(0) = \pi/6$, $q(0) = 0.05$, $p(0) = 0.2$, $r(0) = 0.7$; (b) $\theta(0) = \pi/8$, $q(0) = 0.08$, $p(0) = 0.2$, $r(0) = 1$; (c) $\theta(0) = \pi/4$, $q(0) = 0.05$, $p(0) = 0.2$, $r(0) = 1.3$.

we conclude that the left hand side of inequality (3.2) is a positive expression and solution (3.1) is stable for all values of $\omega$.

Equations (1.3) have another particular solution for which the angle $\theta$ between symmetry axis and the vertical remains constant:

$$\theta = \theta_0 \neq 0, \quad q = 0, \quad p = p_0, \quad r = r_0, \quad \theta_0 \in (-\theta_*, \theta_*) \tag{3.3}$$

if three constants $\theta_0$, $p_0$ and $r_0$ satisfy to the equation [1]:

$$a_{11} p_0^2 + a_{12} p_0 r_0 - mg f_0' = 0, \tag{3.4}$$

$$a_{11} = \left( A_1 - \frac{m\zeta_0}{\cos\theta_0} f_0 \right) \operatorname{ctg}\theta_0, \quad a_{12} = -\left( A_3 - \frac{m\xi_0}{\sin\theta_0} f_0 \right).$$

The solution (3.3) corresponds to the regular precession of paraboloid. The condition of stability of solution (3.3) has the form [1]:

$$b_{11} p_0^2 + b_{12} p_0 r_0 + b_{22} r_0^2 + mg f_0'' > 0, \tag{3.5}$$

$$b_{11} = \frac{\left(A_1 + m\zeta_0^2\right)\left(1 + 2\cos^2\theta_0\right)}{\sin^2\theta_0} + \frac{m\xi_0\left(\xi_0 \sin\theta_0 + 3\zeta_0\cos\theta_0\right)}{\sin\theta_0} +$$

$$+ \frac{A_3 m\zeta_0\left(\xi_0 + \zeta_0'\right)\left(\left(A_1 + m\zeta_0^2\right)\cos\theta_0 + m\xi_0\zeta_0\sin\theta_0\right)}{\left(A_1A_3 + A_1 m\xi_0^2 + A_3 m\zeta_0^2\right)\sin\theta_0},$$

$$b_{12} = -\left(3A_3 + 3m\xi_0^2 + m\xi_0'\zeta_0\right)\frac{\cos\theta_0}{\sin\theta_0} - \frac{m\xi_0\zeta_0\left(1 + \cos^2\theta_0\right)}{\sin^2\theta_0} +$$

$$+ \frac{m\xi_0\left(2A_3 + 2m\xi_0^2 + m\xi_0'\zeta_0\right)}{A_1A_3 + A_1 m\xi_0^2 + A_3 m\zeta_0^2}\left(A_1\xi_0\frac{\cos\theta_0}{\sin\theta_0} - A_3\zeta_0\right) -$$

$$- \frac{A_3 m\zeta_0\zeta_0'}{A_1A_3 + A_1 m\xi_0^2 + A_3 m\zeta_0^2}\left(A_3 + m\xi_0^2 + \frac{m\xi_0\zeta_0\cos\theta_0}{\sin\theta_0}\right),$$

$$b_{22} = \left(A_3 + m\xi_0^2 + \frac{m\xi_0\zeta_0\cos\theta_0}{\sin\theta_0}\right)\frac{A_3\left(A_3 + m\xi_0^2 + m\xi_0'\zeta_0\right)}{A_1A_3 + A_1 m\xi_0^2 + A_3 m\zeta_0^2}.$$

It is easy to see that the coefficients $b_{ij}$ are very complicated. One can show that for homogeneous paraboloid with moments of inertia determined by (2.2) and with angle $\theta$ limited by (2.3) the coefficients $b_{ij}$ satisfy the following inequalities:

$$b_{11} > 0, \quad b_{22} > 0, \quad b_{12}^2 - 4b_{11}b_{22} < 0.$$

This means that the quadratic form $b_{11}p_0^2 + b_{12}p_0r_0 + b_{22}r_0^2$ is positive definite for all values of constants $p_0$, $r_0$. Taking into account that $mgf_0''$ is also positive we conclude that the regular precessions (3.3) of the homogeneous paraboloid are stable.

In conclusion, the problem of motion of the homogeneous paraboloid on the perfectly rough horizontal plane is completely investigated. Qualitative description of motion of the paraboloid is given. All steady motions of the paraboloid (permanent rotations and regular precessions) are described and their stability is studied.

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