Controllability of the Ishlinsky System

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Abstract

In this paper we consider a control problem for the nonholonomic Ishlinsky system [1]. We prove controllability results for this system using the Chow – Rashevsky theorem [2]. We also correct the results about controllability of the Ishlinsky system obtained in the paper [3].

1 Introduction

In 1965 A. Yu. Ishlinsky proposed the example of a low-dimensional nonholonomic non-Chaplygin mechanical system [1]. The Ishlinsky system consists of three cylinders. One of them with the radius R rolls without sliding on top of two other identical cylinders of a radius a, which each rolls without sliding on a fixed horizontal plane. In this paper we formulate and analyze controllability conditions for the system of cylinders. We prove that the Ishlinsky system is completely controllable by the Chow – Rashevsky theorem [2].

It should be noted that the problem of controllability of the Ishlinsky system has been considered previously in the paper [3]. However some of the results obtained in [3] are wrong. In our paper we give the complete and correct analysis of the same problem.

Below we formulate some facts from the control theory which we will use in our investigation.

2 Basic Definitions

Let us consider a driftless control-affine system of the form:

$$\dot{\mathbf{x}} = \sum_{i=1}^{m} u_i g_i(\mathbf{x}), \quad \mathbf{x} \in M, \quad \mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m, \tag{1}$$

where M is an open connected subset of $\mathbb{R}^n$ and $g_1, \ldots, g_m$ are C^∞ vector fields on M. Admissible inputs are $\mathbb{R}^m$-valued measurable functions $u(\cdot)$ defined on some interval $[0, T]$ and a trajectory of (1), corresponding to some $\mathbf{x}_0 \in M$ and to an admissible input $u(\cdot)$, is the (maximal) solution $\mathbf{x}(\cdot)$ in M of the Cauchy problem defined by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{m} u_i(t)\, g_i\big(\mathbf{x}(t)\big),$$

where $t \in [0, T]$ and $\mathbf{x}(0) = \mathbf{x}_0$.

It is possible to formulate the motion planning problem for the system (1) namely to determine a procedure which associates with every pair of points $(\mathbf{p}, \mathbf{q}) \in M \times M$ and admissible input $u(\cdot)$ defined on some interval $[0, T]$ such that the corresponding trajectory of (1) starting from $\mathbf{p}$ at $t = 0$ reaches $\mathbf{q}$ at $t = T$:

$$\mathbf{x}(0) = \mathbf{p}, \quad \mathbf{x}(T) = \mathbf{q}.$$

The system is called completely controllable if for any $(\mathbf{p}, \mathbf{q}) \in M \times M$ the above formulated problem has a solution for some $T > 0$. For the driftless control-affine system (1) the controllability test is given in terms of the Lie algebra of vector fields generated by the vector fields in the right-hand side of the system (1):

$$\operatorname{Lie}(g_1, \ldots, g_m) = \operatorname{span}\Big(g_1, \ldots, g_m, [g_i, g_j], \big[g_i, [g_j, g_k]\big], \ldots\Big).$$

Here span $(\ldots)$ is a linear hull of the vector fields and $[g_i, g_j]$ is the Lie bracket (commutator) of the vector fields g_i and g_j which can be written as follows:

$$[g_i, g_j] = \frac{\partial g_j}{\partial \mathbf{x}} g_i - \frac{\partial g_i}{\partial \mathbf{x}} g_j.$$

System (1) has a full rank at the point $\mathbf{x} \in M$ if

$$\operatorname{Lie}(g_1, \ldots, g_m)(\mathbf{x}) = T_{\mathbf{x}} M.$$

The following theorem is valid.

Theorem. (Chow – Rashevsky [2]). *System* (1) *is completely controllable on M if and only if it has a full rank at every point $\mathbf{x} \in M$.*

Thus the problem of controllability for the driftless control-affine system of the form (1) has a solution. To solve this problem we need to find the dimension of the span of the vector fields $g_1, \ldots, g_m$ and their iterated Lie brackets.

3 The Ishlinsky system. Problem formulation

Let us consider the mechanical system consisting of three homogeneous cylinders. Two cylinders, each of radius a, can roll without sliding on a horizontal plane. We will refer to these cylinders as the lower cylinders. The third cylinder, of radius R, can roll without sliding on top of two lower cylinders. We will refer to it as the upper cylinder. Since the lower cylinders have identical radii the rotation axis of the upper cylinder always remains in a fixed horizontal plane (Fig. 1).

We introduce two inertial reference frames $Oxyz$ and $Ox'y'z'$. The axes Ox and Ox' are parallel to the axes of rotation of the two lower cylinders with the angle α between the axes. As the two lower cylinders are assumed to roll without sliding on the horizontal plane, the angle α is constant.

For the description of the system let us introduce the following generalized coordinates: x, y are coordinates of the center of mass of the upper cylinder (the third vertical coordinate z of the center of mass is constant $z = 2a + R$); θ is the angle between the axis of rotation of the upper cylinder and the coordinate axis Ox; φ is the roll angle of the upper cylinder; φ_1 and φ_2 are the roll angles of the two lower cylinders. Let us denote by $\mathbf{r}_1$ and $\mathbf{r}_2$ the position vectors from the center of mass of the upper cylinder to the points at which the upper cylinder is in contact with the lower cylinders. In the coordinate frame $Oxyz$ these vectors can be written as follows:

$$\mathbf{r}_1 = (a\varphi_1 - y) \operatorname{ctg} \theta \mathbf{e}_x + (a\varphi_1 - y)\, \mathbf{e}_y - R\mathbf{e}_z;$$

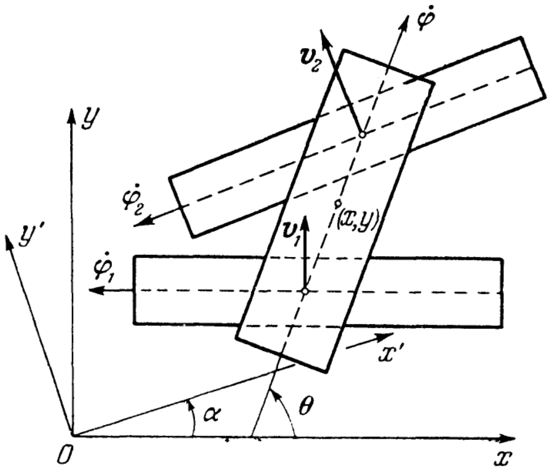

Figure 1: The Ishlinsky System.

$$\mathbf{r}_2 = \frac{\cos\theta}{\sin(\theta-\alpha)}\left(a\varphi_2 + x\sin\alpha - y\cos\alpha\right)\mathbf{e}_x + \frac{\sin\theta}{\sin(\theta-\alpha)}\left(a\varphi_2 + x\sin\alpha - y\cos\alpha\right)\mathbf{e}_y - R\mathbf{e}_z.$$

The velocities of points at which the upper cylinder is in contact with the lower cylinders have a form:

$$\mathbf{v}_1 = 2a\dot{\varphi}_1\mathbf{e}_y, \quad \mathbf{v}_2 = -2a\dot{\varphi}_2\sin\alpha\mathbf{e}_x + 2a\dot{\varphi}_2\cos\alpha\mathbf{e}_y.$$

The velocity of the center of mass of the upper cylinder and its angular velocity have a form:

$$\mathbf{v} = \dot{x}\mathbf{e}_x + \dot{y}\mathbf{e}_y, \quad \boldsymbol{\omega} = \dot{\varphi}\cos\theta\mathbf{e}_x + \dot{\varphi}\sin\theta\mathbf{e}_y + \dot{\theta}\mathbf{e}_z.$$

Condition that the upper cylinder rolls without sliding on the two lower cylinders gives the nonholonomic constraints:

$$\mathbf{v} + [\boldsymbol{\omega}\times\mathbf{r}_1] = \mathbf{v}_1, \quad \mathbf{v} + [\boldsymbol{\omega}\times\mathbf{r}_2] = \mathbf{v}_2.$$

In the scalar form these conditions can be written as follows:

$$\begin{aligned}
&\dot{x} - R\dot{\varphi}\sin\theta - (a\varphi_1 - y)\,\dot{\theta} = 0,\\
&\dot{y} + R\dot{\varphi}\cos\theta + (a\varphi_1 - y)\,\dot{\theta}\,\mathrm{ctg}\,\theta - 2a\dot{\varphi}_1 = 0,\\
&\dot{x}\sin(\theta-\alpha) - R\dot{\varphi}\sin\theta\sin(\theta-\alpha) + 2a\dot{\varphi}_2\sin\alpha\sin(\theta-\alpha) -\\
&- (a\varphi_2 + x\sin\alpha - y\cos\alpha)\,\dot{\theta}\sin\theta = 0,\\
&\dot{y}\sin(\theta-\alpha) + R\dot{\varphi}\cos\theta\sin(\theta-\alpha) - 2a\dot{\varphi}_2\cos\alpha\sin(\theta-\alpha) +\\
&+ (a\varphi_2 + x\sin\alpha - y\cos\alpha)\,\dot{\theta}\cos\theta = 0.
\end{aligned} \tag{2}$$

Thus, the considered system has six generalized coordinates x, y, θ, φ_1, φ_2, φ and two degrees of freedom.

4 Formulation of the control problem

According to the paper [3] let us denote the introduced generalized coordinates as follows:

$$x = q_1, \quad y = q_2, \quad \theta = q_3, \quad \varphi_1 = q_4, \quad \varphi_2 = q_5, \quad \varphi = q_6$$

and denote also

$$a\varphi_1 - y = aq_4 - q_2 = z,$$

$$a\varphi_2 + x\sin\alpha - a\varphi_1\cos\alpha = aq_5 + q_1\sin\alpha - aq_4\cos\alpha = w.$$

Let us solve the nonholonomic constraints (2) with respect to generalized velocities $\dot{q}_1$, $\dot{q}_2$, $\dot{q}_3$ and $\dot{q}_5$. Let the roll rates of the first lower cylinder and of the upper cylinder will be the control variables: $\dot{q}_4 = u_1$ and $\dot{q}_6 = u_2$. Then the nonholonomic constraints (2) can be transformed into the nonlinear control system of the form (1):

$$\begin{aligned}
\dot{q}_1 &= \frac{2az\sin q_3\sin\alpha}{w\sin q_3 + z\sin\alpha\cos q_3}u_1 + Ru_2\sin q_3,\\
\dot{q}_2 &= \frac{2aw\sin q_3}{w\sin q_3 + z\sin\alpha\cos q_3}u_1 - Ru_2\cos q_3,\\
\dot{q}_3 &= \frac{2a\sin\alpha\sin q_3}{w\sin q_3 + z\sin\alpha\cos q_3}u_1,\\
\dot{q}_4 &= u_1,\\
\dot{q}_5 &= \frac{\sin q_3}{\sin(q_3-\alpha)}u_1,\\
\dot{q}_6 &= u_2.
\end{aligned} \tag{3}$$

The two control vector fields g_1 and g_2 are determined according to the nonlinear control equations (3) and are defined on the open subset of $\mathbb{R}^6$ where equations (3) are well defined:

$$g_1 = \begin{pmatrix} \dfrac{2az\sin q_3\sin\alpha}{w\sin q_3 + z\sin\alpha\cos q_3} \\ \dfrac{2aw\sin q_3}{w\sin q_3 + z\sin\alpha\cos q_3} \\ \dfrac{2a\sin\alpha\sin q_3}{w\sin q_3 + z\sin\alpha\cos q_3} \\ 1 \\ \dfrac{\sin q_3}{\sin(q_3-\alpha)} \\ 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} R\sin q_3 \\ -R\cos q_3 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore it is possible to formulate for the Ishlinsky system the control problem described at the beginning of the paper. Let us prove now that the Ishlinsky system are completely controllable.

5 Controllability

To prove controllability of the Ishlinsky system we need to choose from the vector fields g_1, g_2 and their iterated Lie brackets

$$g_3 = [g_1, g_2] \text{ – Lie bracket of the first level;}$$

$$\left.\begin{aligned} g_4 &= [g_2, [g_1, g_2]] = [g_2, g_3] \\ g_5 &= [[g_1, g_2], g_1] = [g_3, g_1] \end{aligned}\right\} \text{ – Lie brackets of the second level;}$$

$$\left.\begin{aligned} g_6 &= \Big[g_2, [g_2, [g_1, g_2]]\Big] = [g_2, g_4] \\ g_7 &= \Big[g_2, [[g_1, g_2], g_1]\Big] = [g_2, g_5] \\ g_8 &= \Big[g_1, [g_2, [g_1, g_2]]\Big] = [g_1, g_4] \\ g_9 &= \Big[g_1, [[g_1, g_2], g_1]\Big] = [g_1, g_5] \end{aligned}\right\} \text{ – Lie brackets of the third level;}$$

the system of vector fields which has a full rank at every point of the configuration space M. In this case the Ishlinsky system will be completely controllable everywhere on the configuration space M by the Chow – Rashevsky theorem [2].

For the first time the problem of controllability of the Ishlinsky system has been investigated in the paper [3]. The proof of controllability proposed in [3] also relies on the Chow – Rashevsky theorem. One of the main results obtained in [3] states that for the specific configurations of the Ishlinsky system it is possible to choose the full rank distributions defined by Lie brackets up to level three.

However this statement is wrong. It is possible to prove by the direct calculations that the Lie brackets g_4, g_6, g_7 and g_8 are expressed through the other Lie brackets according to the formulae:

$$g_4 = -\frac{2R\sin\alpha}{w\sin q_3 + z\sin\alpha\cos q_3} g_3, \qquad g_6 = -\frac{6R^2\sin^2\alpha}{(w\sin q_3 + z\sin\alpha\cos q_3)^2} g_3,$$

$$g_7 = -\frac{2R\sin\alpha}{w\sin q_3 + z\sin\alpha\cos q_3} g_5 + \frac{2aR\left(\sin^2(q_3-\alpha) - \sin^2 q_3\right)\sin\alpha}{(w\sin q_3 + z\sin\alpha\cos q_3)^2\sin(q_3-\alpha)} g_3, \quad g_8 = -g_7.$$

Thus among the vector fields g_1, g_2 and their iterated Lie brackets up to level three only the following vector fields

$$g_1, \quad g_2, \quad g_3, \quad g_5, \quad g_9$$

will be independent. This system of vector fields has a rank 5 and it is not a system of the full rank (=6). Therefore to prove controllability of the Ishlinsky system we need to take into consideration the Lie brackets of the fourth level:

$$g_{10} = [g_2, g_6],\ g_{11} = [g_2, g_7],\ g_{12} = [g_2, g_8],\ g_{13} = [g_2, g_9],$$

$$g_{14} = [g_1, g_6],\ g_{15} = [g_1, g_7],\ g_{16} = [g_1, g_8],\ g_{17} = [g_1, g_9]$$

For these Lie brackets the following conditions take place:

$$g_{10} = -\frac{24R^3\sin^3\alpha}{\left(w\sin q_3 + z\sin\alpha\cos q_3\right)^3} g_3,$$

$$g_{11} = \frac{6R^2\sin^2\alpha}{\left(w\sin q_3 + z\sin\alpha\cos q_3\right)^2} g_5 - \frac{12aR^2\left(\sin^2\left(q_3-\alpha\right) - \sin^2 q_3\right)\sin\alpha}{\left(w\sin q_3 + z\sin\alpha\cos q_3\right)^3\sin\left(q_3-\alpha\right)} g_3,$$

$$\begin{aligned} g_{13} &= \frac{6Ra^2\left(\sin^3 q_3\cos\left(q_3-\alpha\right) - \cos q_3\sin^3\left(q_3-\alpha\right) + \sin\alpha\sin^2 q_3\right)\sin^2\alpha}{\left(w\sin q_3 + z\sin\alpha\cos q_3\right)^3\sin^2\left(q_3-\alpha\right)} g_3 - \\ &- \frac{3R\sin\alpha}{w\sin q_3 + z\sin\alpha\cos q_3} g_9, \\ g_{15} &= \frac{4Ra^2\left(\sin^3 q_3\cos\left(q_3-\alpha\right) - \cos q_3\sin^3\left(q_3-\alpha\right) + \sin\alpha\sin^2 q_3\right)\sin^2\alpha}{\left(w\sin q_3 + z\sin\alpha\cos q_3\right)^3\sin^2\left(q_3-\alpha\right)} g_3 - \\ &- \frac{4aR\left(\sin^2\left(q_3-\alpha\right) - \sin^2 q_3\right)\sin\alpha}{\left(w\sin q_3 + z\sin\alpha\cos q_3\right)^2\sin\left(q_3-\alpha\right)} g_5 - \frac{2R\sin\alpha}{w\sin q_3 + z\sin\alpha\cos q_3} g_9, \end{aligned}$$

$$g_{12} = -g_{11}, \qquad g_{14} = -g_{11}, \qquad g_{16} = -g_{15}.$$

Therefore among the Lie brackets of the level four only the bracket

$$g_{17} = \left[g_1, g_9\right] = \left[g_1, \left[g_1, \left[\left[g_1, g_2\right], g_1\right]\right]\right].$$

is independent with g_1, g_2, g_3, g_5 and g_9.

Finally we can conclude that the Ishlinsky system is completely controllable everywhere on the configuration space M. The vector fields

$$g_1, \quad g_2, \quad g_3 = \left[g_1, g_2\right], \quad g_5 = \left[\left[g_1, g_2\right], g_1\right],$$

$$g_9 = \left[g_1, \left[\left[g_1, g_2\right], g_1\right]\right], \quad g_{17} = \left[g_1, \left[g_1, \left[\left[g_1, g_2\right], g_1\right]\right]\right]$$

are independent and form the spanning distribution of the full rank 6. Therefore the Ishlinsky system is completely controllable by the Chow – Rashevsky theorem.

6 Discussion

After we proved the controllability of the Ishlinsky system the problem of construction of the explicit control $u_1(t)$, $u_2(t)$, steering the system (3) from the initial state $\mathbf{p}$ to a target state $\mathbf{q}$ at time $T > 0$ with any desired degree of accuracy is arisen. However this problem is highly nontrivial for such systems due to the anisotropy of the state space. In other words these systems are characterized by different shifts in different directions. The value of displacement in the direction of the fields g_1 and g_2 in a small time t is $O(t)$, in the direction of a commutator $g_3 = [g_1, g_2]$ is $O\left(t^2\right)$, in the direction $g_4 = [g_2, g_3]$ and $g_5 = [g_3, g_1]$ is $O\left(t^3\right)$ and so on. Further we hope to study in details the problem of construction of the explicit control for the Ishlinsky system.

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