Antisymmetric higher order edge waves in plates with fixed faces

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Abstract

The paper describes the propagation of surface waves localized near the edge of a plate (edge waves). Antisymmetric waves are considered in a plate with fixed faces. Three-dimensional theory of elasticity is used to study higher order edge waves. Asymptotic analysis is performed, which shows that there is an infinite spectrum of higher order edge waves. Asymptotics of phase velocities are obtained for the large values of the wave number. It is demonstrated that in the short-wave limit the phase velocities of all higher order edge waves tend to the velocity of Rayleigh wave. Numerical investigations are performed by using the modal expansion method. Numerical results are presented for the first four higher order edge waves . All these waves except the first one are damped by propagating modes, but for each wave there is a critical wave number, after which the wave becomes non-damped. These critical wave numbers are estimated asymptotically. By numerical investigation the additional resonance peaks were discovered which also correspond to the higher order edge waves. The reason of this phenomenon is discussed.

1 Introduction

Surface waves in plates (edge waves) are well known in two-dimensional approximate structural theories, which describe long-wave low-frequency motions. In the theory of generalized plain stress the edge wave is described by classical Rayleigh wave solution [1] with only difference in the longitudinal wave speed. Edge wave in the classical theory of plate bending was derived in [2]. Existence and uniqueness of bending edge wave in anisotropic plates is studied in [3]. Free vibration of shells connected with edge waves are considered in [4],[5]. But approximate structural theories describe only the first, so-called fundamental wave. By using three-dimensional theory of elasticity we will have an infinite family of edge waves, which can be called higher order edge waves. In [6] a simple analytical solution for such waves in a plate with mixed boundary conditions on the faces is obtained. In [7],[8] the higher order edge waves in a plate with free or fixed faces are studied for the case of plate motions which are symmetric with respect to its mid-plane. In [9] the antisymmetric higher order edge waves are considered in the plate with free faces. This study deals with antisymmetric higher order edge waves in plates with fixed faces.

2 Basic equations

Let us consider the harmonic vibrations of an isotropic homogeneous elastic plate, occupying the region $0 \le x < \infty,\ |y| \le h,\ -\infty < z < \infty$ (see fig. 1). The small harmonic

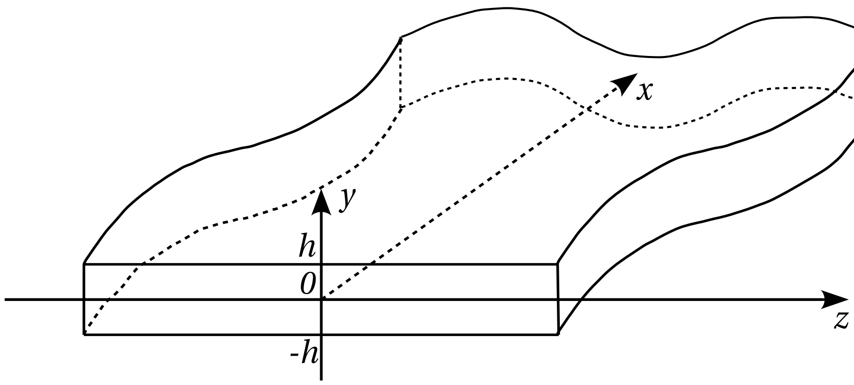

Figure 1: The geometry of the problem

vibrations of the plate are governed by the 3D classical equation of linear elasticity. Let us introduce dimensionless variabilities as follows:

$$x = h\pi^{-1}\widetilde{x},\ y = h\pi^{-1}\widetilde{y},\ z = h\pi^{-1}\widetilde{z},\ \{u_x, u_y, u_z\} = h\pi^{-1}\{\widetilde{u}_x\widetilde{u}_y, \widetilde{u}_z\}, \tag{1}$$

$$\widetilde{\omega} = h\pi^{-1}\omega c_2^{-1}, \{\sigma_x, \sigma_y,\sigma_z, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}\} = \\ = E[2(1+\nu)]^{-1}\{\widetilde{\sigma}_x, \widetilde{\sigma}_y, \widetilde{\sigma}_z, \widetilde{\sigma}_{xy}, \widetilde{\sigma}_{xz}, \widetilde{\sigma}_{yz}\},$$

where u is the displacement vector, σ_x, σ_y, σ_z, σ_{xy}, σ_{xz}, σ_{yz} are the stress components, E is Young's modulus, ν is Poisson ratio, c_2 is the transverse waves speed, ω is the circular frequency. The factor $e^{i\omega t}$ is implied but omitted everywhere, as well as the sign «$\sim$». The displacement vector can be expressed as

$$\mathbf{u} = \text{grad}\ \varphi + \text{rot}\ \psi, \tag{2}$$

where φ and ψ are Lame potentials. The stress components can be also expressed in Lame potentials (see [8]). In addition to the representation (2) we write down the equation for the potential ψ:

$$\text{div}\ \psi = 0. \tag{3}$$

From the three-dimensional equations of elasticity we have

$$\Delta\varphi + \kappa^2\omega^2\varphi = 0,\ \ \Delta\psi + \omega^2\psi = 0, \tag{4}$$

where $\kappa = \sqrt{(1-2\nu)/2(1-\nu)}$. On the faces $y = \pm\pi$ we impose the following boundary conditions

$$u_x = u_y = u_z = 0. \tag{5}$$

We assume that the stress-strain state depends on z as $\cos sz$ or $\sin sz$. On the edge $x = 0$ the boundary conditions are

$$\sigma_x = T_{\text{g}}(y)\cos sz, \quad \sigma_{xy} = 0, \quad \sigma_{xz} = 0, \tag{6}$$

where $T_{\text{g}}(y)$ is a given antisymmetrical function. The resonance frequencies correspond to eigenvalues of the problem with free edge, i.e. with

$$\sigma_x = 0, \quad \sigma_{xy} = 0, \quad \sigma_{xz} = 0 \tag{7}$$

at $x = 0$. The solution of this problem can be presented as a sum of two edge waves, propagating in the opposite directions along the axis z. For such waves the parameter s is the wave number. The phase velocity of the edge wave can be calculated as

$$c = \frac{\omega_0}{s}, \tag{8}$$

where ω_0 is some eigenvalue of the problem (4), (5), (7) or some resonance frequency of the problem (4), (5), (6).

3 Asymptotical analysis

Let us consider the problem (4), (5), (7). The approximate eigenform for this problem can be presented as standing three-dimensional surface wave, which is analogous to the classical Rayleigh wave in 3D case. This wave is described by potentials (see [8])

$$\begin{aligned}
\varphi &= -CK_R\frac{\gamma'}{\gamma}\sin(\gamma y)\cos(sz)\exp(-r_1^R x),\\
\psi_x &= 0,\\
\psi_y &= C\frac{s}{\gamma}\sin(\gamma y)\sin(sz)\exp(-r_2^R x),\\
\psi_z &= C\cos(\gamma y)\cos(sz)\exp(-r_2^R x),
\end{aligned} \tag{9}$$

where

$$r_1^R = \sqrt{\gamma'^2 - \kappa^2\omega^2},\ r_2^R = \sqrt{\gamma'^2 - \omega^2},\ \gamma' = \sqrt{\gamma^2 + s^2} = \vartheta\omega,\ K_R = \frac{2\vartheta\sqrt{\vartheta^2 - 1}}{2\vartheta^2 - 1},$$

θ is the root of equation

$$\left[\vartheta^2 - 0.5\right]^2 - \vartheta^2\sqrt{\vartheta^2 - \kappa^2}\sqrt{\vartheta^2 - 1} = 0. \tag{10}$$

Let us notice that $\vartheta = 1/c_R$, where c_R is the ratio of Rayleigh wave speed to c_2. The solution described by (9) satisfies the boundary conditions (7). If the parameter γ is fixed and $s \to \infty$, the displacements tend to

$$\begin{aligned}
u_x^R &= C\frac{1}{\gamma}\left[\left[-\frac{K_R\vartheta_1}{\vartheta}s^2\left(1 + O\left(\frac{\gamma^2}{s^2}\right)\right)e^{-s\tilde{r}_1 x} + \left(s^2 + \gamma^2\right)e^{-s\tilde{r}_2 x}\right]\right]\sin\gamma y\cos sz,\\
u_y^R &= C\left[-K_R s\left(1 + O\left(\frac{\gamma^2}{s^2}\right)\right)e^{-s\tilde{r}_1 x} + \frac{\vartheta_2}{\vartheta}s\left(1 + \frac{\gamma^2}{s^2}\right)e^{-s\tilde{r}_2 x}\right]\cos\gamma y\cos sz,\\
u_z^R &= C\frac{1}{\gamma}\left[-K_R s^2 e^{-s\tilde{r}_1 x} + \frac{\vartheta_2}{\vartheta}s^2 e^{-s\tilde{r}_2 x}\right]\left(1 + O\left(\frac{\gamma^2}{s^2}\right)\right)\sin s\gamma y\sin sz.
\end{aligned} \tag{11}$$

The formulae (11) show that the displacement u_y^R is asymptotically small. By setting

$$\gamma = n,\ n = 1, 2, \dots \tag{12}$$

we satisfy the boundary conditions (5) in the leading approximation. From the relation $\gamma' = \vartheta\omega$ we have

$$\omega_n^{(\infty)} = \frac{1}{\vartheta}\sqrt{s^2 + n^2}. \tag{13}$$

By using of (8) we derive asymptotics for the phase velocities of the edge wave:

$$c_n^{(\infty)} = c_R\sqrt{1 + \frac{n^2}{s^2}}. \tag{14}$$

Thus, there is an infinite spectrum of higher order edge waves. Their speeds tend to Rayleigh wave speed as $s \to \infty$.

The exact solution can be found by using of modal expansion

$$f = \sum_m C_m f^{(m)}, \tag{15}$$

where $f^{(m)} = \left(\varphi^{(m)}, \psi_x^{(m)}, \psi_y^{(m)}, \psi_z^{(m)}\right)^{\mathrm{T}}$ describes an elementary solution satisfying (5) (for details see [8]). Among modes $f^{(m)}$ there are propagating ones that can damp the edge waves. The lowest cut-off frequency has asymptotic approximation

$$\Omega_1^{(\infty)} = \sqrt{s^2 + \frac{1}{4}}. \tag{16}$$

From (14) and (16) we conclude that the nth edge wave is not damped at

$$s > s_{cr}^{(\infty)} = \sqrt{\frac{4n^2 - \theta^2}{4(\theta^2 - 1)}}. \tag{17}$$

4 Numerical results and discussion

Numerical investigation was carried out on the basis of the modal expansion (15) and confirmed all the results of the asymptotical analysis.

Dispersion curves of the first four edge waves in a plate with fixed faces are presented in fig. 2. Here the thin solid curves are asymptotics (13), dash-dot curves are numerical results, thick solid line is $\omega = c_{\mathrm{R}}s$. The critical values s_{cr} are pointed with the circles.

In some regions two resonance peaks are found near asymptotic approximations. The curves for the second peak are also shown in fig. 2. They should be also considered as the higher order edge waves. These branches, not observed in the other cases mentioned in introduction, arise because of influence of the cut-off frequencies, i.e. thickness resonances. The curves for the latter are presented in fig. 2 as dash lines. The resonances which are nearer to the cut-off frequency are related with the shear wave, the influence of the Rayleigh wave is secondary. For the resonances which are nearer to the thin solid lines we have opposite situation: they are resonances of the Rayleigh wave influenced by bulk waves.

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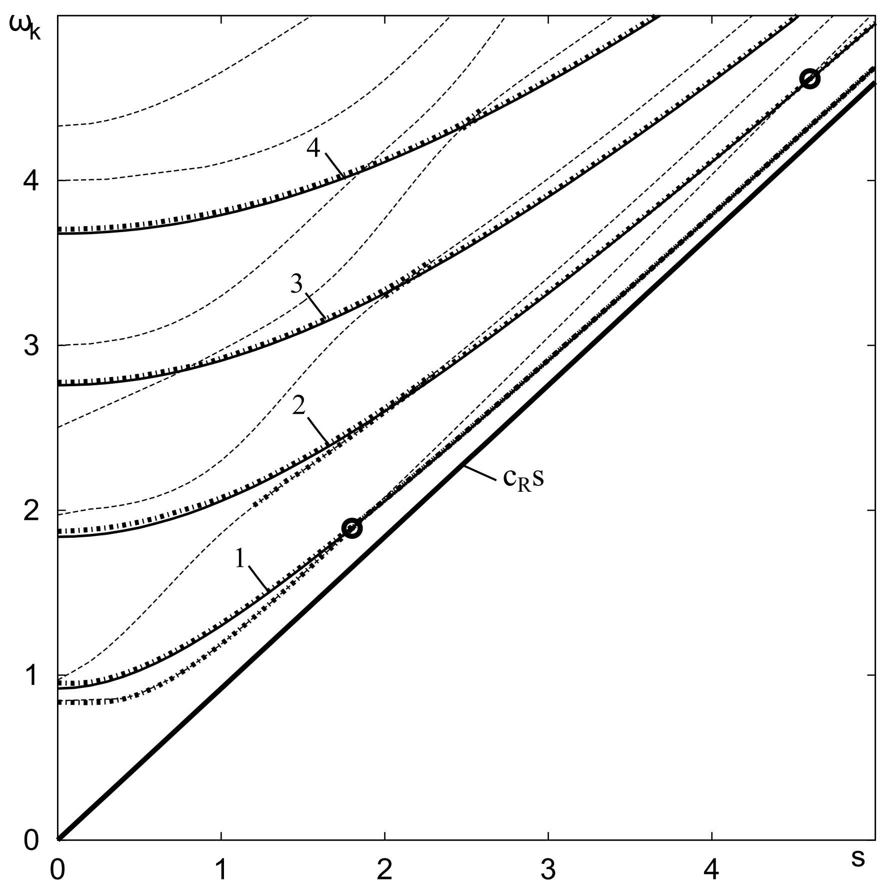

Figure 2: Dispersion curves of the higher order edge waves

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