Oscillations of self-similar structures in mechanics

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Abstract

Fairly broad class of structures in nature and technology consists from the self-similar structures, in which each cell in a certain scale repeats the structure of previous cell. Here, the translational symmetry is accompanied by a similarity transformation (scaling) of adjacent cells. We study the vibrational properties of such structures both for unidirectional and for branched case, namely the torsional oscillations of self-similar beam with disks and oscillations of dichotomous lattice at different similarity coefficients. A number of dynamic features for these systems at its vibrations are revealed. It is shown that the self-similar structures have a wave solution and this is true for both linear and nonlinear systems. For linear case the dispersion equation is received and it was revealed that self-similar structures are a frequency bandpass filter and their bandwidths are found. In addition in the lattice there are multiple natural frequencies equal to the partial frequency of forming element; the frequency multiplicity depends on the rows quantity (for example the frequency multiplicity of 4-rows lattice equal to 5). These multiple frequencies are within the lattice bandpass and consequently they are very dangerous because increase the system vibroactivity, as well as the onset of instability. The example for calculation natural and forced oscillations of dichotomous lattice is given.

For dichotomous nonlinear lattice with an elastic coupling with an odd type of nonlinearity (e.g. cubic) it was showed that its in-phase natural form are described by a self-similar nonlinear chain.

1 Introduction

Regular structures with translational symmetry constitute rather limited class of structures and do not describe many natural and technical systems. However, it can be significantly expanded to include a class of self-similar structures. Self-similar structures are, in particular, structures in which each cell in a certain scale repeats the structure of previous one. Here, the translational symmetry is accompanied by a similarity transformation (scaling) of neighboring cells. This is one of the types of fractals; Structures of this kind prevail in biology, chemistry of polymers, physics. Growth and formation of polymeric molecules, crystals is accompanied by formation of fractals. Many scientists study fractals, mainly, as a way of a shaping of various structures [1, 2]. The self-similar structures in mechanics are also most widespread, for example: a rod with in steps changing section, a shaft with the disks which parameters change on length, a conic cover with an rigidity edges, cranked shaft (screw symmetry), conic springs, etc. [3]. The branched structures are widespread in nature and the technical systems. In nature, it's airways, human circulatory system, the trees, the table of the genetic code [2]. Branching technical structures were investigated by [4]. He considered two classes of self-similar structures: the piping system in the form of a dichotomous tree and a system of rods. It was found that these structures have certain properties optimality.

2 Oscillations of self-similar beam with disks

But the dynamical properties of self-similar structures are investigated much less, and submitted manuscript is devoted to their studying. We will study the basic dynamic properties of such structures on rather simple systems. At first let's consider a self-similar half-infinity beam with disks (Figure 1) which makes the torsional oscillations around the axis x.

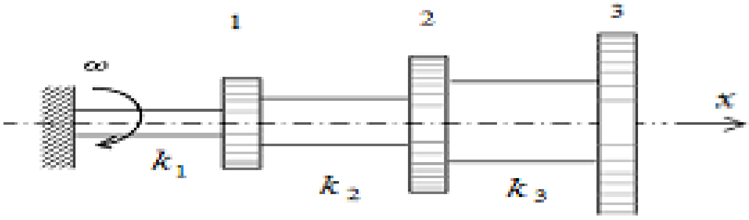


Figure 1: Self-similar beam with concentrated masses.

Lets the radius of the cross section r_s and the discs radius R_s of the portion s change with the same scale $\sqrt[4]{\gamma}$ (Figure 1), while the portion lengths are not changed. Then the partial frequencies for each mass are identical and equal to

$$\nu_1^2 = \nu_2^2 = ...\nu_n^2 = (k_{s-1} + k_s)/J_{ds} \ = (1+\gamma)k_1/J_{d1}, \tag{1}$$

$$k_s = GJ_s/l$$

is a torsional stiffness of the s-th shaft portion, J_{ds} is the polar inertia moment of the s-th disc. Equality condition of the partial frequencies (1) is weaker than the condition of geometric similarity (scaling) required to fractal structures. It does not require scaling of all system parameters, some parameters, (for example in our case the portion length) may remain constant. Therefore, the conditions (1) can be called as incomplete self-similarity conditions.

The equation for the s-th node can be written as finite-difference equations with variable coefficients depending on the node number s, i.e. on x.

$$-k_1\gamma^{s-1}x_{s-1} + (-J_{d1}\gamma^{s-1}\omega^2 + k_1\gamma^{s-1}(1+\gamma))x_s - k_1\gamma^s x_{s+1} = 0. \tag{2}$$

Now we make in (2) the change of variables, analogues [5]

$$\mathbf{X}^* = \mathbf{N}\mathbf{X},\ \mathbf{N}^2 = diag[1\ ,\ 1/\gamma\ ,\ 1/\gamma^2]. \tag{3}$$

Kinetic and potential energy for this system in the new coordinates

$$W = T + \Pi = \sum_s m_s \dot{x}_s^2 + \sum_s \left[k_{s-1}(x_s - x_{s-1})^2 + k_s(x_s - x_{s+1})^2\right] =$$
$$\sum_s m_1 \dot{x}_s^{*2} + \sum_s \frac{k_1}{\gamma}(\dot{x}_s^* - \frac{\dot{x}_{s-1}^*}{\gamma})^2 + \sum_s k_1(\dot{x}_s^* - \gamma\dot{x}_{s-1}^*)^2$$

With help of coordinate transformation (3) it can be reduced to the view

$$W = \sum_s m_1 x_s^{*2} + \sum_s [k_1\gamma^{-1/2}(x_s^* - x_{s\pm1}^*)^2 + k_1\left(1-\gamma^{1/2}\right)^2\gamma^{-1}x_s^{*2}].$$

This relation describes now the regular structure with translational symmetry with equal masses m_1, stiffness between them is constant and equal to $k_1\gamma^{-\frac{1}{2}}$. But now there is an additional fastening masses (Figure 2), equal to:

$$k_1(1+\gamma)\gamma^{-1} - 2k_1\gamma^{-1/2} = k_1\left(1-\gamma^{1/2}\right)^2\gamma^{-1} = k^*.$$

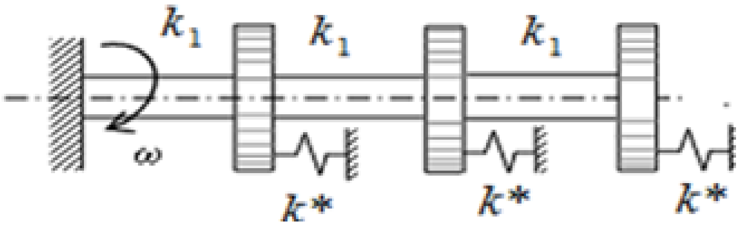


Figure 2: Equivalent periodic structure with the same natural frequencies.

3 Self-similar chain as mechanical band pass

Such structure is known as mechanical band pass filter. Since the linear coordinate's transformation (3) does not change the frequency properties, thus the structure on Figures 1-2 have the same frequency. Consequently, the self-similar system Figure 1 is a mechanical band filter. Its pass band can be found from the dispersion equation. To find it lets write the equation for the s-th mass on Figure 2. It is the finite-difference equation, but now with the constant coefficients.

$$-\frac{k_1}{\sqrt{\gamma}}x_{s-1} + (-m_1\omega^2 + \frac{k_1(1+\gamma)}{\gamma})x_s - \frac{k_1}{\sqrt{\gamma}}x_{s+1} = 0.$$

Dispersion equation for the system in Figure 1 has the form:
- for realμ:

$$-m_1\omega^2 + \frac{k_1(1+\gamma)}{\gamma} - 2\frac{k_1}{\sqrt{\gamma}}\cos\mu = 0.$$

-for purely imaginaryμ:

$$-m_1\omega^2 + \frac{k_1(1+\gamma)}{\gamma} - 2\frac{k_1}{\sqrt{\gamma}}\,\mathrm{c}\,h\mu' = 0.$$

The bandwidth of harmonic signal is

$$\omega_0 < \omega < \omega*,$$

where

$$\omega_0^2 = k^*/m = k_1\left(1-\sqrt{\gamma}\right)^2/m\gamma \ at\ \mu = 0$$

$$\omega^{*2} = k_1\left(1+\sqrt{\gamma}\right)^2/m\gamma \ at\ \mu = \pi$$

and bandwidth is

$$\Delta\omega^2 = \omega^{*2} - \omega_0^2 = 4k/m\gamma^{\frac{1}{2}}.$$

It is seen that bandwidth is inversely proportional to the scaling coefficient.

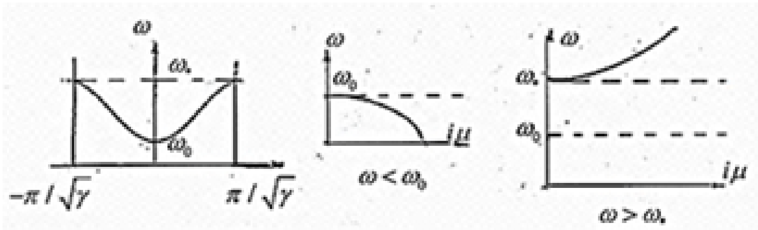


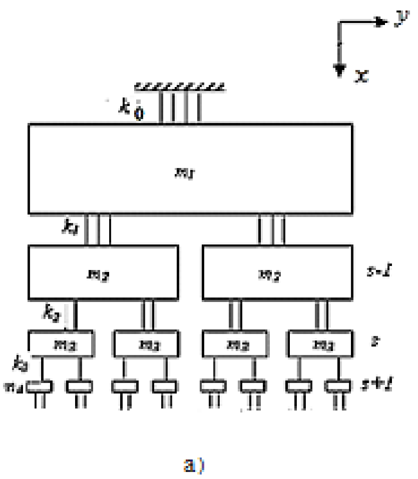
Figure 3: The bandwidth of self-similar chain.

As for natural forms, in accordance with the coordinates transformation (3) they can be easily obtained from the corresponding natural form of a regular structure (Figure 2), by proportionally changing of the vibration amplitude of each mass in time $\sqrt{\gamma}$. Let's say the higher vibration mode for a regular structure Figure 2 is known a sinusoid with nearby masses is in ant phase. So, the corresponding natural form for the original self-similar structure (Figure 1) is obtained by constant decreasing of the each mass amplitude in $\sqrt{\gamma}$ times. The envelope of natural forms is a straight with inclination angle equal to$\sqrt{\gamma}$, and the maximum amplitude point of each mass shifted along the x-axis.

Self-similar structure with nonlinear supports, the rigidity of which varies with a certain scale can similarly lead to a regular. Indeed, let the support stiffness has a cubic nonlinearity $k'_s = k_s + \alpha_s x^3$. If $\alpha_s = \alpha_0 \gamma^{-s/2}$ then after the variables change (1) we get on a regular structure on nonlinear supports with stiffness equal to $kx = k^* x + \alpha_0 x^3$.

4 Dichotomous lattice

The other widespread self-similar structure is dichotomous lattice (Figure 4a).



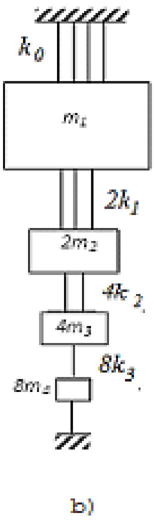


Figure 4: The dichotomous lattice (a), its inphase substructure, corresponding to inphase lattice oscillations(b)

Dynamic self-similarity conditions for the lattice are that its elastic and inertial parameters vary with the same scale γ for each cell:

$$k_s = k_1/\gamma^s, m_s = m_1/\gamma^s \qquad (4)$$

By (4) the partial frequencies of forming subsystems are equal to each other

$$\nu_1^2 = \nu_2^2 = ... = \nu_n^2 = (k_{s-1} + 2k_s)/m_s = \frac{k_0}{m_0}(\gamma + 2).$$

5 Natural oscillations of lattice

We now consider the vibrations for dichotomous lattice. The partial forming subsystem has a double numbering (s, j). The equation for (s, j)-th mass, has the form

$$\begin{aligned} &m_{sj}\ddot{x}_{sj} + k_{sj}x_{sj} - k_{s-1,j/2}(x_{s-1,j/2}) - k_{s+1,2j-1}(x_{s+1,2j-1} + x_{s+1,2j}) = 0, \\ &k_{sj} = k_{s-1,j/2} + 2k_{s+1,2j-1}, \end{aligned} \tag{5}$$

(here we take an even value of the pair $(j, j+1)$; on further results , as we shall see below, this choice does not affect) .

We seek a solution (5) in the form of a product

$$x_{s+h,j+r} = X_t Y_j e^{i(\mu p + qr + \omega t)} \tag{6}$$

Here, q is the phase shift between an adjacent elements in a given row, and the phase shift p between the adjacent rows. Substituting (6) into (5), we find

$$\begin{aligned} &m_{sj}\lambda^2 X_s Y_j + k_{s,j} X_s Y_j - k_{s,j/2}(X_{s-1}Y_{j/2}) - k_{s+1,2j-1}(X_{s+1}Y_{2j-1} + X_{s+1}Y_{2j}) = 0 \\ &k_{sj} = k_{s,j/2} + 2k_{s+1,2j-1}. \end{aligned}$$

Then the dispersion equation is

$$(-m_{sj}\lambda^2 + k_{sj}) - k_{s,j/2}(e^{-\mu}e^{-qj/2}) - k_{s+2,2j-1}e^{\mu}(e^{q(j-1)} + e^{qj}) = 0 \tag{7}$$

In-phase vibrations forms obtained from (7) if the phase shift between adjacent elements $q = 0$

$$\begin{aligned} &(-m_{sj}\lambda^2 + k_{sj}) - k_{s,j/2}(e^{-\mu}) - 2k_{s+1,2j-1}e^{\mu} = 0, \\ &k_{sj} = k_{s,j/2} + 2k_{s=2,2j-1}. \end{aligned}$$

Due to the high degree of the lattice symmetry, its in-phase oscillations, as is easily seen, correspond to oscillations of subsystem Figure 4 b. It is the 4- mass chain self-similar subsystem in which its elastic and inertial elements are equal to the sum of the corresponding elements of each row. The total mass of each row is equal to

$$M_s = m_s 2^s = m_1(2/\gamma)^s$$

For brevity we call it as inphase subsystem.

There are two different cases: $\gamma \neq 2, \gamma = 2$. When $\gamma \neq 2$, we obtain a self-similar chain in which the total mass and stiffness of each raw varies proportionally to the ratio $(\gamma/2) = \gamma^*$, which is the similarity coefficient for each row. But the self-similar chain, as shown above, is a band pass filter. The limiting frequencies λ_0, λ^* for in-phase self-similar chain analogues to chain on Figure 2 are, the limit frequencies for dichotomous lattice Figure 4a [6]. Its bandwidth

$$\Delta\lambda = \lambda^{*2} - \lambda_0^2 = 4\frac{k_0}{m_1}\sqrt{\gamma'}$$

$$\lambda_{\grave{a}}^2 = \frac{k^*}{m_1} = \frac{k_0}{m_1}(1 - \sqrt{\gamma'})^2, \quad \mu = 0,$$
$$\lambda^{*2} = \frac{k_0}{m_1}\left(1 + \sqrt{\gamma'}\right)^2, \quad \mu = \pi.$$

When $\gamma = 2$ then the total mass and stiffness of each row are the same and equal to m_1 and k_0, and the in-phase chain becomes as periodic because its similarity coefficient $\gamma^* = 1$, and therefore its lower limit of the bandwidth becomes equal to 0

$$\lambda_0^2 = 0, \lambda^{*2} = 4\frac{k_0}{m_1}$$

In [6] it was shown that in self-similar lattice there are multiple roots equal to partial frequency of forming lattice element. Its quantity depends on rows quantity and they locate within the lattice pass band. These frequencies are dangerous in terms of increased system vibroaktivity, as well as the onset of instability. For 4- row chain, for example there are 5 multiple roots.

Note that the in-phase natural forms of lattice have also the self-similar structure. Indeed, the in-phase natural forms in the transformed coordinates $x*$ describes the self-similar chain of Figure 4b. In [5] it was shown that natural forms for such a chain obtained from natural forms of regular chain by changing of masses amplitude in each subsequent node in γ times (γ - scaling factor). For the lattice must also take into account the coordinate transformation (2.5), which implies that s-th element of the chain corresponds to 2s lattice coordinates. Thus, the natural forms of lattice have a line character where the length of each subsequent section increases in 2 times, and amplitude changes in γ times.

6 Numerical results. Natural oscillations of dichotomous lattice

Let's consider the 4-th row lattice:

$$m_1 = 27, \ k_0 = 2,7.10^5/, \gamma = 3, \quad \nu = \sqrt{5/3} = 1.291$$

Natural frequencies for this 4-th row dichotomous lattice are: 58.8 71.6 92.2 92.2 107.8 129.1 129.1 129.1 129.1 129.1 147.3 157.6 157.6 168.0 172.8

It is seen that lattice has a 5 -fold root $\nu = 129.1\, s^{-1}$ Similarity coefficient for inphase chain is $\gamma' = 3/2$. Lowest possible frequency lattice is $\lambda_0 = 31.2s^{-1}$ and limit frequency is $\lambda^* = 285^{-1}$.

It is clear that there is an abrupt change in the oscillations form at low frequency change in the neighborhood of partial frequency $\nu = 129.1\, s^{-1}$ of forming element.

7 Conclusions

* The self-similar chain is a mechanical bandpass filter. Its bandwidth is inversely proportional to the scaling coefficient.

* Dichotomous lattice is a frequency bandpass filter. When $\gamma \neq 2$ the bandwidth does not depend on the rows quantity in the lattice and equal to $\Delta\lambda = \nu\sqrt{\gamma/2}$. At $\gamma = 2$ the lower limit bandwidth disappears.

* In the lattice there are multiple natural frequencies equal to the partial frequency of forming lattice element; the multiplicity of these frequencies depends on the rows quantity. So for 4-th row lattice the root multiplicity is equal to 5.

Acknowledgements

The author is deeply grateful to D.A. Indeitzev, and I.I.Blekhman for the useful discussion.

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