On Numerical Schemes in 2D Vortex Element Method for Flow Simulation Around Moving and Deformable Airfoils

Kseniya S. Kuzmina Ilia K. Marchevsky
kuz-ksen-serg@yandex.ru, iliamarchevsky@mail.ru

Abstract

The problem of numerical flow simulation around moving and deformable airfoils using meshless lagrangian vortex methods is considered. The flow assumed to be viscous and incompressible, so the Viscous Vortex Domain method is used for numerical simulation of vortex wake evolution. However the usage of well-known numerical scheme, which often called 'Discrete Vortex Method', on the airfoil surface, can lead to significant errors when calculating intensities for vortex elements.

A new approach is developed for computation of vortex layer intensity with high accuracy. It is based on solving of Fredholm-type integral equation for vortex layer intensity determination instead of Hilbert-type singular integral equation which normally appears in 'classical' Discrete Vortex Method.

The developed approach and appropriate discretization formulae can be implemented into existent numerical procedures and software packages and their usage doesn't require to change any parameters of design models which used in vortex element methods.

1 Introduction

Flow simulation around an airfoil and aerodynamic or hydrodynamic loads computation is a key problem for number of engineering applications. In case of immovable and rigid airfoil various numerical methods can be implemented for numerical simulation of the flow. Most of them are the so-called 'mesh methods' so they presuppose mesh generation in the flow region. But sometimes computational cost of the numerical simulation can be significantly reduced by using meshfree lagrangian numerical methods which don't need mesh in flow region. In case of movable or deformable airfoils mesh reconstruction and numerical solution interpolation are non-trivial procedures and they often lead to degradation of the simulation accuracy. So in such problems meshless methods are preferable, especially for large displacements of the airfoil or its large deformations when is is impossible just to deform the mesh instead of its reconstruction.

Vortex element method [1, 2, 3] is a well-known meshfree lagrangian CFD method. Firstly it was developed for numerical simulation of inviscid flows, but later it was generalized to newtonian viscid flows. There are several approaches for taking viscosity influence into account: particle strength exchange method, combined mesh and meshfree methods et al. We use 'Viscous Vortex Domain' (VVD) method developed by G.Ya. Dynnikova [4] since it is purely lagrangian approach and due to its high accuracy which proved in number of researches.

When using vortex element method the flow around the fixed airfoil is simulated by a thin vortex layer on the airfoil surface while for the flow simulation around moving or deformable airfoil in general case not only vortex layer but also source layer should be introduced on the airfoil surface. Their intensities depend on time, so it should be computed every time step. If law of the airfoil motion is known, source layer intensity and the so-called 'attached' vortex layer intensity can be found explicitly as normal and tangent projections of airfoil surface velocities. At the same time 'free' vortex layer intensity is a'priory unknown.

The accuracy of the vortex layer intensity computation defines the accuracy of the boundary condition satisfaction on the airfoil surface and consequently the accuracy of vortex wake simulation around the airfoil. However, the existent well-known numerical schemes ('Discrete Vortex Method'), normally being used in vortex element method, sometimes lead to significant errors. Number of researchers noted, that such errors arise when simulating the flow around airfoils with angle points or sharp edges (wing airfoils), but it is easy to show that numerical solution can be qualitatively wrong even in case of smooth airfoils in viscid flow.

The aim of this paper is to develop modern numerical scheme for vortex element method based on some different approaches to boundary condition satisfaction for movable and deformable airfoil and to compare its accuracy when flow simulating around smooth airfoils and airfoils with sharp edge.

2 Governing equations

Viscous incompressible media movement is described by continuity equation

$$\nabla \cdot \underline{V} = 0$$

and Navier — Stokes equations

$$\frac{\partial \underline{V}}{\partial t} + (\underline{V} \cdot \nabla)\underline{V} = \nu \Delta \underline{V} - \nabla \left(\frac{p}{\rho} \right)$$

where $\underline{V}(\underline{r}, t)$ is flow velocity, $p(\underline{r}, t)$ — pressure, $\rho = \text{const}$ — density of the media, ν — kinematic viscosity coefficient. No-slip boundary condition on the movable airfoil surface

$$\underline{V}(\underline{r}, t) = \underline{V}_K(\underline{r}, t), \quad \underline{r} \in K$$

and boundary conditions of perturbation decay on infinity

$$\underline{V}(\underline{r}, t) \to \underline{V}_\infty, \quad p(\underline{r}, t) \to p_\infty, \quad |\underline{r}| \to \infty$$

should be satisfied.

Navier — Stokes equations can be written down in Helmholtz form using vorticity vector $\underline{\Omega}(\underline{r}, t) = \nabla \times \underline{V}(\underline{r}, t)$:

$$\frac{\partial \underline{\Omega}}{\partial t} + \nabla \times (\underline{\Omega} \times \underline{U}) = 0. \tag{1}$$

Here $\underline{U}(\underline{r}, t) = \underline{V}(\underline{r}, t) + \underline{W}(\underline{r}, t)$, $\underline{W}(\underline{r}, t)$ is the so-called 'diffusive velocity', which is proportional to viscosity coefficient:

$$\underline{W}(\underline{r}, t) = \nu \frac{(\nabla \times \underline{\Omega}) \times \underline{\Omega}}{|\underline{\Omega}|^2}.$$

Equation (1) means that vorticity which exists in the flow moves and its velocity is $\underline{U}$. 'New' vorticity is being generated only on airfoil surface, so we can consider that the vorticity distribution in the flow $\Omega(\underline{r}, t)$ is always known.

The streamlined airfoil influence is equivalent to superposition of the attached vortex $\gamma_{att}(\underline{r}, t)$ and source $q_{att}(\underline{r}, t)$ layer influences and free vortex layer $\gamma(\underline{r}, t)$ influence. All this layers are located on the airfoil surface,

$$\gamma_{att}(\underline{r}, t) = \underline{V}_K(\underline{r}, t) \cdot \underline{\tau}(\underline{r}, t), \quad q_{att}(\underline{r}, t) = \underline{V}_K(\underline{r}, t) \cdot \underline{n}(\underline{r}, t), \quad \underline{r} \in K,$$

where $\underline{n}(\underline{r}, t)$ and $\underline{\tau}(\underline{r}, t)$ are normal and tangent unit vectors on the airfoil surface.

Using generalized Helmholtz decomposition ideas [5] flow velocity can be computed by Biot — Savart law:

$$\underline{V}(\underline{r}, t) = \underline{V}_\infty + \frac{1}{2\pi}\int_{S(t)} \frac{\underline{\Omega}(\underline{\xi}, t) \times (\underline{r} - \underline{\xi})}{|\underline{r} - \underline{\xi}|^2} dS + \frac{1}{2\pi}\oint_{K(t)} \frac{\underline{\gamma}(\underline{\xi}, t) \times (\underline{r} - \underline{\xi})}{|\underline{r} - \underline{\xi}|^2} dl_K + \\ + \frac{1}{2\pi}\oint_{K(t)} \frac{\underline{\gamma}_{att}(\underline{\xi}, t) \times (\underline{r} - \underline{\xi})}{|\underline{r} - \underline{\xi}|^2} dl_K + \frac{1}{2\pi}\oint_{K(t)} \frac{q_{att}(\underline{\xi}, t)(\underline{r} - \underline{\xi})}{|\underline{r} - \underline{\xi}|^2} dl_K. \quad (2)$$

Here $\underline{V}_\infty = \text{const}$ is uniform flow velocity, $S(t)$ is current flow region, $K(t)$ is current airfoil surface; attached and free vortex layer intensities vectors are $\underline{\gamma}_{att} = \gamma_{att}\underline{k}$ and $\underline{\gamma} = \gamma\underline{k}$ and by analogy $\underline{\Omega} = \Omega\underline{k}$, where $\underline{k}$ is unit vector orthogonal to the flow plane; for each point at the airfoil surface $\underline{n}(\underline{r}, t) \times \underline{\tau}(\underline{r}, t) = \underline{k}$.

Vortex layer intensity $\gamma(\underline{\xi}, t)$ can be found from no-slip boundary condition on airfoil surface:

$$\underline{V}(\underline{r}, t) = \underline{V}_K(\underline{r}, t), \quad \underline{r} \in K,$$

where we assume that velocity of the airfoil surface $\underline{V}_K(\underline{r}, t)$ is given by function which is at least continuous.

3 Exact solution for simplest airfoils

In the simplest case when for each point on the airfoil surface $\underline{V}_K = \text{const}$ the exact solution for free vortex layer intensity is the same as in inverse motion case, where $\underline{V}'_\infty = \underline{V}_\infty - \underline{V}_K$ and $\underline{V}'_K = 0$. This approach will be used in order to verify the developed numerical scheme and to estimate its accuracy.

If there is no vortex wake near the airfoil, exact solutions for the vortex layer intensity in incompressible steady flow can be found for some simplest airfoils (circular, elliptical, Zhukovsky airfoils) using methods of complex analysis. The vortex layer intensity is equal to velocity tangential component on the airfoil surface. Complex value of flow velocity can be found using the following formula [7]:

$$V^*(p) = \frac{R|\underline{V}'_\infty| \sin(\phi + \beta - p) + G/(2\pi)}{\dfrac{iRe^{i(p-\phi)}}{2}\left(1 - \dfrac{a^2}{(Re^{i(p-\phi)} + H)^2}\right)}.$$

Here V^* means complex conjugate quantity to velocity V, $p \in [0, 2\pi)$ defines the point on airfoil surface, β — angle of incidence.

For elliptical airfoil

$$a = \sqrt{a_1^2 - b_1^2}, \quad R = a_1 + b_1, \quad \phi = 0, \quad H = 0,$$

a_1 and b_1 are major and minor semiaxes of the ellipse.

For Zhukovsky airfoil

$$R = \sqrt{(a + d\cos\phi)^2 + (h + d\sin\phi)^2}, \quad \phi = \arctan\frac{h}{a}, \quad H = ih - de^{-i\phi},$$

a, d and h are arbitrary parameters, which correspond to length, width and curvature of the airfoil.

The flow velocity circulation G for elliptical airfoil can be chosen arbitrarily (from mathematical point of view); we assume it to be equal to zero independently on angle of incidence, while for Zhukovsky airfoil it is proportional to uniform flow velocity and depends on the airfoil shape and its angle of incidence:

$$G = -2\pi|\underline{V}'_\infty|\sin(\beta + \phi)\big(\sqrt{h^2 + a^2} + d\big).$$

Using the previous formulae we can obtain the exact solution for the vortex layer intensity.

Vortex wakes normally are being simulated by discrete vortex-type singularities (vortex elements), so the vorticity distribution can be written in the following form:

$$\Omega(\underline{r}, t) = \sum_{i=1}^{N_V} \Gamma_i \delta(\underline{r} - \underline{r}_i),$$

where N_V is number or vortex elements, Γ_i and $\underline{r}_i$ are intensities and positions of vortex elements correspondingly, δ is Dirac delta function.

Velocity field generated by such vorticity field is the following:

$$V_\omega(\underline{r}, t) = \frac{1}{2\pi}\int_S \frac{\Omega(\underline{\xi}, t) \times (\underline{r} - \underline{\xi})}{|\underline{r} - \underline{\xi}|^2} dS = \sum_{i=1}^{N_V} \frac{\Gamma_i}{2\pi} \frac{\underline{k} \times (\underline{r} - \underline{r}_i)}{|\underline{r} - \underline{r}_i|^2}.$$

It is also possible to find some exact solution for vortex layer intensity in this case: the simplest way is to consider circular cylinder with radius ρ and use well-known 'reflection method':

$$V^*(p) = \frac{2|\underline{V}'_\infty|\sin(\beta - p)}{ie^{ip}} = \sum_{j=1}^{N_V}\left(\frac{\Gamma_j}{2\pi i}\left(\frac{1}{z - z_j} - \frac{1}{z - w_j} + \frac{1}{z}\right)\right),$$

where $w_j = z_j \rho^2/|z_j|^2$.

These exact solutions will be used for numerical schemes comparison and their accuracy estimation.

4 The numerical scheme for vortex layer intensity computation

According to (2) and taking into account that the unknown vortex layer intensity $\gamma(\xi, t)$ concerns to free vortex layer which is part of vortex wake, it could be easily shown that limit value of flow velocity on the airfoil surface is equal to ($\underline{r} \in K$, time dependence hereafter is omitted)

$$\underline{V}_-(\underline{r}) = \underline{V}_\infty + \frac{1}{2\pi}\int_S \frac{\Omega(\underline{\xi}) \times (\underline{r} - \underline{\xi})}{|\underline{r} - \underline{\xi}|^2} dS + \frac{1}{2\pi}\oint_K \frac{\underline{\gamma}(\underline{\xi}) \times (\underline{r} - \underline{\xi})}{|\underline{r} - \underline{\xi}|^2} dl_K +$$

$$+\frac{1}{2\pi}\oint_K \frac{\underline{\gamma}_{att}(\underline{\xi}) \times (\underline{r}-\underline{\xi})}{|\underline{r}-\underline{\xi}|^2} dl_K + \frac{1}{2\pi}\oint_K \frac{q_{att}(\underline{\xi})(\underline{r}-\underline{\xi})}{|\underline{r}-\underline{\xi}|^2} dl_K - \left(\frac{\underline{\gamma}(\underline{r})}{2} \times \underline{n}(\underline{r})\right). \quad (3)$$

Classical approach which is normally being used in vortex element method presupposes that the unknown function $\gamma(\underline{r})$ should be found from the equality between *normal* components of the flow velocity limit value and the airfoil surface velocity:

$$\underline{V}_-(\underline{r}) \cdot \underline{n}(\underline{r}) = \underline{V}_K(\underline{r}) \cdot \underline{n}(\underline{r}), \quad \underline{r} \in K. \quad (4)$$

This integral equation is singular and the principal value of the corresponding integral should be understood in Cauchy sense. This approach sometimes leads to significant errors and even qualitatively wrong solution can be obtained.

In order to solve such problems another approach can be implemented. It is shown [5] that 'boundary condition' (4) is equivalent from a mathematical point of view to the following condition

$$\underline{V}_-(\underline{r}) \cdot \underline{\tau}(\underline{r}) = \underline{V}_K(\underline{r}) \cdot \underline{\tau}(\underline{r}), \quad \underline{r} \in K, \quad (5)$$

which corresponds to the equality between *tangent* components of the flow velocity limit value and the airfoil surface velocity.

It should be noted that in case of smooth airfoils (5) leads to Fredholm-type integral equation with bounded kernel. Both equations (4) and (5) have infinitely many solutions; in order to select the unique solution an additional equation should be solved together with (4) and (5):

$$\oint_K \gamma(\underline{\xi}) dl_K = \Gamma. \quad (6)$$

Total circulation Γ of the vorticity layer on the airfoil can be found from problem statement; it depends on angular velocity and angular acceleration of the airfoil and on time step [6].

In order to find numerical solution of (4), (6) equations in classical Vortex Element Method normally the so-called 'Discrete Vortex-type' quadrature formula is used, which allows to compute numerically Cauchy principal values but imposes strong constraints on airfoil discretization. For example, polygon legs (hereafter we call them 'panels') which approximate curvilinear airfoil surface should have nearly the same legs, quadrature formula for integral approximation is in fact central rectangular formula which accuracy is not very high due to singularity in integral kernel and integral equation is satisfied only in 'collocation' points $\underline{k}_i$ placed precisely in centers of panels. All the vorticity in vortex layer assumed to be concentrated in vortex elements with circulations Γ_j placed in vertexes $\underline{c}_j$ of the polygon (i.e., at the end of the panels). So the algebraic equations which approximate (4) have the following form:

$$\left(\sum_{j=1}^{N} \frac{\Gamma_j}{2\pi} \frac{\underline{k} \times (\underline{k}_i - \underline{c}_j)}{|\underline{k}_i - \underline{c}_j|^2}\right) \cdot \underline{n}_i = V_{norm,i}, \quad i = 1, \dots, N.$$

Here N is number of panels, $\underline{n}_i$ is unit normal vector at the i^{th} collocation point on the airfoil, $V_{norm,i}$ is the normal component of velocity vector at the i^{th} collocation point influenced by the uniform flow, vortex wake, attached vortex and source layers. Such an approach is hereinafter called 'N-scheme'. It should be noted that vortex wake influences at collocation points can be computed straightforward, however, attached vortex and source layers influences computation is not a trivial procedure.

Another approach, based on solution of (5) and (6) equation, doesn't have such restrictions. Nevertheless, if we use nearly the same ideas as in Discrete Vortex Method, the accuracy will remain very low. So it seems to be rationally to use some other problem definition: firstly, not to concentrate vorticity into vortex elements and secondly solve integral equation (5) in weak formulation (i.e., on average along the panels) instead of its satisfying at certain collocation points.

Vortex layer intensity is supposed to be piecewise constant function, we denote its unknown value γ_i on the i^{th} panel and attached vortex and sources layers are similarly should be approximated by piecewise constant functions $\gamma_{att,i}$ and $a_{att,i}$ correspondingly. So the following algebraic equations approximate integral equation (5):

$$\frac{1}{L_i}\left(\int_{K_i} dl_r \sum_{j=1}^{N} \frac{\gamma_j}{2\pi}\left(\int_{K_j} \frac{\underline{k}\times(\underline{r}-\underline{\xi})}{|\underline{r}-\underline{\xi}|^2} dl_\xi\right)\right)\cdot \tau_i - \frac{\gamma_i}{2} =$$
$$= -\frac{1}{L_i}\left(\int_{K_i} dl_r \sum_{w=1}^{Nv} \frac{\Gamma_w}{2\pi}\left(\frac{\underline{k}\times(\underline{r}-\underline{r}_w)}{|\underline{r}-\underline{r}_w|^2}\right) + \underline{\int_{K_i} dl_r \sum_{j=1}^{N} \frac{\gamma_{att,j}}{2\pi}\left(\int_{K_j} \frac{\underline{k}\times(\underline{r}-\underline{\xi})}{|\underline{r}-\underline{\xi}|^2} dl_\xi\right)} + \right.$$
$$\left. + \underline{\underline{\int_{K_i} dl_r \sum_{j=1}^{N} \frac{q_{att,j}}{2\pi}\left(\int_{K_j} \frac{\underline{r}-\underline{\xi}}{|\underline{r}-\underline{\xi}|^2} dl_\xi\right)}}\right)\cdot \underline{\tau}_i + (\underline{V}_i - \underline{V}_\infty)\cdot \underline{\tau}_i, \quad i = 1,\ldots,N. \quad (7)$$

Here L_i is the i^{th} panel length which denoted as K_i; Γ_w and $\underline{r}_w$ are circulations and positions of vortex elements which simulate vortex wake; $\gamma_{att,j}$ and $q_{att,j}$ are average values of attached vortex and sources layers along the j^{th} panel correspondingly; $\underline{V}_i$ is average velocity along the i^{th} panel.

Coefficients of this system

$$A_{ij} = \frac{1}{L_i}\int_{K_i} dl_r \left(\int_{K_j} \frac{\underline{k}\times(\underline{r}-\underline{\xi})}{2\pi|\underline{r}-\underline{\xi}|^2} dl_\xi\right)\cdot \underline{\tau}_i, \quad i,j = 1,\ldots,N.$$

can be computed analytically [8]. In order to write down final formula for A_{ij}, we note that this coefficient is average tangent velocity on i^{th} panel, which is induced by j^{th} panel in assumption that j^{th} panel has unit vortex layer intensity. In order to compute it we firstly derive formula for average velocity vector on i^{th} panel, which is induced by j^{th} panel

$$\underline{V}_{ij} = \frac{1}{L_i}\int_{K_i} dl_r \left(\int_{K_j} \frac{\underline{k}\times(\underline{r}-\underline{\xi})}{2\pi|\underline{r}-\underline{\xi}|^2} dl_\xi\right), \quad \text{then} \quad A_{ij} = \underline{V}_{ij}\cdot \underline{\tau}_i.$$

On fig. 1 some auxiliary vectors are introduced: vectors $\underline{d}$ and $\underline{d}_0$ are codirectional with i^{th} and j^{th} panels, their lengths are L_i and L_j correspondingly; vectors $\underline{s}_1$ and $\underline{s}_2$ join the beginning of j^{th} with the beginning and the ending of i^{th} panel; vectors $\underline{p}_1$ and $\underline{p}_2$ join the ending of j^{th} with the beginning and the ending of i^{th} panel.

After some transformations we can obtain the following formula:

$$\underline{V}_{ij} = \frac{1}{2\pi|\underline{d}_0||\underline{d}|^2}\left[q_1^{(1)}\underline{c}_1 + q_2^{(1)}\underline{c}_2 + q_3^{(1)}\underline{c}_3 + \left(q_1^{(2)}\underline{c}_1 + q_2^{(2)}\underline{c}_2 + q_3^{(2)}\underline{c}_3\right)\times \underline{k}\right].$$

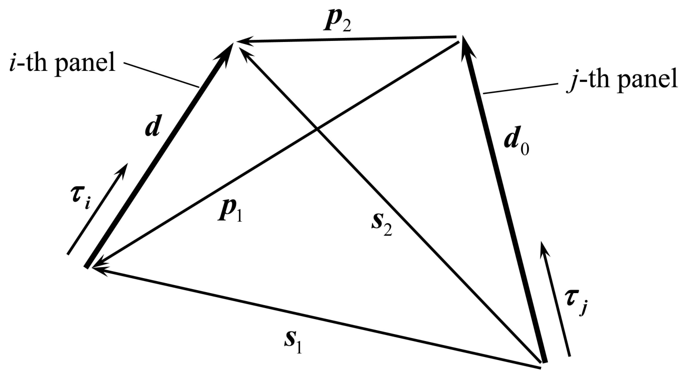

Figure 1: Two panels from airfoil surface and auxiliary vectors

Here we denote:

$$q_1^{(1)} = \arctan \frac{\underline{d} \cdot \underline{p}_1}{z_1} - \arctan \frac{\underline{d} \cdot \underline{p}_2}{z_1},$$

$$q_2^{(1)} = \arctan \frac{\underline{d} \cdot \underline{s}_2}{z_2} - \arctan \frac{\underline{d} \cdot \underline{s}_1}{z_2},$$

$$q_3^{(1)} = \arctan \frac{\underline{d}_0 \cdot \underline{p}_2}{z_3} - \arctan \frac{\underline{d}_0 \cdot \underline{s}_2}{z_3};$$

$$z_1 = (\underline{p}_1 \times \underline{p}_2) \cdot \underline{k}, \qquad z_2 = (\underline{s}_1 \times \underline{s}_2) \cdot \underline{k}, \qquad z_3 = (\underline{s}_2 \times \underline{p}_2) \cdot \underline{k};$$

$$q_1^{(2)} = \ln \frac{|\underline{p}_2|}{|\underline{p}_1|}, \quad q_2^{(2)} = \ln \frac{|\underline{s}_1|}{|\underline{s}_2|}, \quad q_3^{(2)} = \ln \frac{|\underline{p}_2|}{|\underline{s}_2|};$$

$$\underline{c}_1 = (\underline{d}_0 \cdot \underline{p}_1)\underline{d} + (\underline{d} \cdot \underline{s}_1)\underline{d}_0 - (\underline{d} \cdot \underline{d}_0)\underline{s}_1,$$

$$\underline{c}_2 = (\underline{d}_0 \cdot \underline{s}_1)\underline{d} + (\underline{d} \cdot \underline{s}_1)\underline{d}_0 - (\underline{d} \cdot \underline{d}_0)\underline{s}_1 = \underline{c}_1 + (\underline{d}_0 \cdot \underline{d}_0)\underline{d},$$

$$\underline{c}_3 = (\underline{d} \cdot \underline{d})\underline{d}_0.$$

If $i = j$ then we should assume that $\underline{V}_{ii} = \underline{0}$; for neighboring panels $\underline{p}_1 = 0$, $\underline{s}_2 \neq 0$, coefficients $q_1^{(1)}$ and $q_1^{(2)}$ vanish. For neighboring panels $\underline{s}_2 = 0$, $\underline{p}_1 \neq 0$ vectors $\underline{d}_0$ and $\underline{d}$ should be replaced with $(-\underline{d}_0)$ and $(-\underline{d})$ correspondingly; then we obtain the previous case when $\underline{p}_1 = 0$, $\underline{s}_2 \neq 0$. If $\underline{s}_1 = 0$ or $\underline{p}_2 = 0$ only one of vectors $\underline{d}_0$ or $\underline{d}$ should be replaced with its opposite in order to obtain the first case.

Integrals in the underlined term in right side of equation (7) have the same form as integrals in left side, so they can be computed using the previous formulae. For the twice-underlined term analytical formula for integrals has similar form:

$$B_{ij} = \frac{1}{L_i} \int_{K_i} dl_r \left(\int_{K_j} \frac{\underline{r} - \underline{\xi}}{2\pi |\underline{r} - \underline{\xi}|^2} dl_\xi \right) \cdot \underline{\tau}_i = \underline{Q}_{ij} \cdot \underline{\tau}_i, \quad i, j = 1, \ldots, N,$$

where

$$\underline{Q}_{ij} = \frac{1}{2\pi |\underline{d}_0| |\underline{d}\,|^2} \left[\left(q_1^{(1)} \underline{c}_1 + q_2^{(1)} \underline{c}_2 + q_3^{(1)} \underline{c}_3 \right) \times \underline{k} - \left(q_1^{(2)} \underline{c}_1 + q_2^{(2)} \underline{c}_2 + q_3^{(2)} \underline{c}_3 \right) \right].$$

In order to take into account vortex wake influence tangent velocity on i^{th} panel, which is induced by w^{th} vortex element of the wake, the following analytical formula can be used:

$$\frac{1}{L_i}\left(\int_{K_i} dl_r \sum_{w=1}^{Nv} \frac{\Gamma_w}{2\pi}\left(\frac{\underline{k}\times(\underline{r}-\underline{r}_w)}{|\underline{r}-\underline{r}_w|^2}\right)\right)\cdot\underline{\tau}_i = \sum_{w=1}^{Nv}\Gamma_w \underline{V}_{iw}^{(wake)}\cdot\underline{\tau}_i.$$

On fig. 2 by analogy with fig. 1 some auxiliary vectors are introduced for velocity $\underline{V}_{iw}^{(wake)}$ computation: vectors $\underline{s}_0$ and $\underline{s}$ join the beginning and the ending of i^{th} panel with w^{th} vortex element placed in point $\underline{r}_w$.

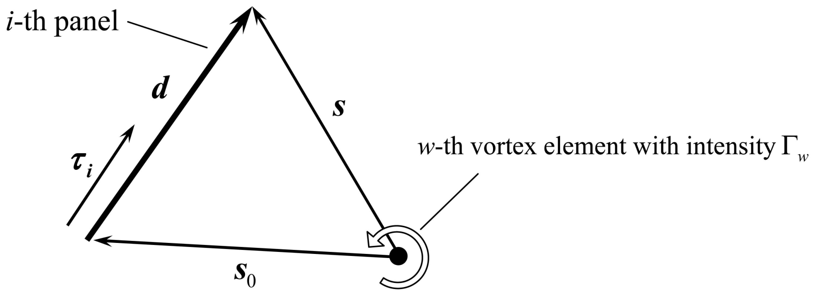

Figure 2: i^{th} panel on airfoil surface, w^{th} vortex element in vortex wake and auxiliary vectors

Using these vectors after some transformations we can write down formula for vortex wake influence in the following form:

$$\underline{V}_{iw}^{(wake)} = -\frac{1}{2\pi|\underline{d}|^2}\left[\alpha\underline{d} + \beta(\underline{d}\times\underline{k})\right]. \tag{8}$$

Here we denote

$$\alpha = \arctan\frac{\underline{s}\cdot\underline{d}}{z_0} - \arctan\frac{\underline{s}_0\cdot\underline{d}}{z_0}, \quad \beta = \ln\frac{|\underline{s}|}{|\underline{s}_0|}, \quad z_0 = (\underline{d}\times\underline{s}_0)\cdot\underline{k}. \tag{9}$$

The developed approach to vortex layer intensity computation is called 'T-scheme'.

5 Numerical experiment

Now we compare numerical solutions which can be obtained using 'classical' Discrete Vortex Method scheme and the developed numerical schemes. We consider two test problems for which exact analytical solution for the vortex layer intensity is known. We consider airfoils which move in the flow with constant velocity $V = 1.0$ in horizontal direction as well as the same airfoils being immovable in the horizontal constant incoming flow $V = 1.0$: stationary solutions of these problems are the same.

In order to compare accuracies of the classical N-scheme and the developed T-scheme we calculate the vortex circulation error:

$$\|\Delta\Gamma\|_N = \frac{1}{2}\max_i\big[(|\gamma_i^0 - \gamma_i|)(L_{i-1}+L_i)\big], \quad L_0 \equiv L_n,$$

$$\|\Delta\Gamma\|_T = \max_i\big[(|\gamma_i^0 - \gamma_i|)L_i\big].$$

Here γ_i^0 is exact solution (average vortex layer intensity on i^{th} panel), γ_i is computed value for it, L_i is i^{th} panel length. $\Delta\Gamma$ is the error of vortex elements circulations computation, and it is the most important value for unsteady flow simulation because the vortex elements generated on the airfoil surface form vortex wake around the airfoil and the circulations of vortex elements in the wake remain constant.

5.1 Flow around an elliptical airfoil

An elliptical airfoil with major and minor semiaxes equal to $a_1 = 1.0$ and $b_1 = 0.1$ for angle of incidence $\beta = \pi/6$ is considered. In table 16 the obtained errors of the mentioned schemes are shown for different values of N — numbers of panels on the airfoil surface.

N	N-scheme		T-scheme
	immovable airfoil	movable airfoil	immovable/movable
25	0.147975	0.062758	0.060417
50	0.046236	0.026013	0.011163
100	0.012713	0.008921	0.001828
200	0.003331	0.002653	0.000246
500	0.000548	0.000477	0.000015
1000	0.000138	0.000123	0.000002

Table 16: Errors in test problem for the elliptical airfoil for N- and T-schemes

It is seen that both approaches (the classical N-scheme and the developed T-scheme) allows to obtain sufficiently accurate results and their errors have orders $O(h^2)$ and $O(h^3)$ respectively, where h is average panel length on the airfoil. It's interesting to note that the accuracy of the classical N-scheme for moving airfoil in the immovable media is slightly higher than in case of immovable airfoil in the flow. However, the developed T-scheme is 'symmetric' and the results for 'direct' and 'inverse' motions are the same.

5.2 Flow around Zhukovsky airfoil

Results of the vortex layer intensity computation for steady flow around symmetrical Zhukovsky airfoil with thickness ratio 15 % ($a = 3.5$, $d = 0.4$, $h = 0.3$) for angle of incidence $\beta = \pi/6$ are shown in table 17.

N	N-scheme		T-scheme
	immovable airfoil	movable airfoil	immovable/movable
25	4.97	21.77	0.066143
50	5.08	20.29	0.010718
100	4.59	19.39	0.001294
200	3.72	18.75	0.000226
500	2.80	17.91	0.000025
1000	2.22	17.46	0.000005

Table 17: Errors in test problem for Zhukovsky airfoil for N- and T-schemes

Because of the sharp edge on Zhukovsky airfoil the obtained results are differ from the previous case of smooth (elliptical) airfoil. The error of N-scheme is unacceptably large so it's impossible to get qualitatively and quantitatively correct solution and consequently to simulate flow around the airfoils with sharp edge. The developed T-scheme allows to get solution which is close to exact one and the error becomes smaller when number of panels grows. The obtained results for number of model problems shows that the error of T-scheme has order approximately $O(h^{2,5})$, where h is average panel length on the airfoil.

6 Conclusion

The problem of 2D flow numerical simulation around movable or deformable airfoils is considered. For its solution two numerical schemes can be used: classical N-scheme which is suitable only for smooth airfoils, and the developed T-scheme which suit both for smooth airfoils and airfoils with sharp edges. T-scheme can be used for unsteady flow simulation and also for solving complicated aeroelastic problems.

Acknowledgements

The work was supported by Russian Federation Grant for young scientists [proj. MK-3705.2014.08].

References

[1] I. K. Lifanov, S. M. Belotserkovskii. Methods of Discrete Vortices. CRC Press, 1993.

[2] G.-H. Cottet, P. D. Koumoutsakos. Vortex Methods: Theory and Practice.Cambridge University Press, 2008.

[3] R. I. Lewis. Vortex Element Methods For Fluid Dynamic Analysis Of Engineering Systems. Cambridge University Press, 2005.

[4] G. Ya. Dynnikova. Vortex Motion in Two-Dimensional Viscous Fluid Flows // Fluid Dynamics. 2003. Vol. 38/5. P. 670–678.

[5] S. N. Kempka, M. W. Glass, J. S. Peery, J. H. Strickland. Accuracy Considerations for Implementing Velocity Boundary Conditions in Vorticity Formulations. SANDIA REPORT SAND96-0583 UC-700, 1996.

[6] P. R. Andronov, S. V. Guvernyuk, G. Ya. Dynnikova. Vortex Methods for Unsteady Hydrodynamic Loads. Moscow, MSU, 2006. [in Russian]

[7] M. Ye. Makarova. Ideal incompressible steady flow around an airfoil computation // VESTNIK. Journal of the Bauman Moscow State Technical University. Natural Sciences & Engineering. 2011. Special issue 'Applied Mathematics'. P. 63–74. [in Russian]

[8] I. K. Marchevsky, V. S. Moreva. Vortex Element Method for 2D Flow Simulation with Tangent Velocity Components on Airfoil Surface // ECCOMAS 2012 — European Congress on Computational Methods in Applied Sciences and Engineering, e-Book Full Papers. 2012. P. 5952–5965.

Kseniya S. Kuzmina, Ilia K. Marchevsky, 2-nd Baumanskaya st., 5, Bauman University, Applied Mathematics dep., Moscow, Russia