Finite Element Method for Solving Boundary Value Problems of Bending of Micropolar Elastic Thin Bars

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Abstract

Currently finite element method (FEM) is widely used to solve problems of continuous media mechanics. This is explained by wide versatility of the FEM and possibility of representing the most complex structures by finite elements of a simple configuration. The method is very useful when computer is used, as all of its algorithms are written in matrix form.

In the present paper the finite element method is developed to solve specific problems of determination the stress-strain state of bending deformation of micropolar elastic thin bars with different boundary conditions, which is implemented on personal computer. On the basis of the numerical results effective manifestations of micropolar materials in terms of strength and rigidity characteristics are discussed.

1 Introduction

Micropolar theory of elastic thin bars, plates and shells is one of the key models of studying the mechanical behavior of nanostructures [1-3] in the structural mechanics of deformable solid body.

In papers [4-7] general applied theories of statics and dynamics of micropolar elastic thin bars, plates and shells are constructed. In papers [4,5,7-9], based on these theories, for some of the simplest static and dynamic problems analytical solutions in closed form are constructed, which are reduced to final numerical results. The influence of material micropolarity of bars, plates and shells is studied on characteristics of the stress-strain state.

Currently finite element method (FEM) is more universally used and practical method to solve different difficult applied problems of statics, stability and dynamics of mechanics of deformable solid body [10]. From this point of view, development of appropriate finite element models is actual for determination of the stress-strain state of micropolar elastic thin bars, plates and shells, which will reflect the features of deformation of micropolar materials.

In the present paper the basic relations of finite element method are constructed to solve boundary value problems of applied theory of bending deformation of micropolar elastic thin bars with independent fields of displacements and rotations. The analytical expressions are obtained for calculation stiffness matrix and efforts-moments vector. The results of numerical calculation of bending deformation of micropolar elastic thin bars are given in case of various external influences and boundary conditions.

2 Problem statement. Mathematical model of bending deformation of micropolar elastic thin bars

An isotropic micropolar elastic parallelepiped of constant height $2h$, length a and constant thickness $2h_1 = 1$ is considered. The coordinate plane x_1x_2 is placed in the middle plane of the parallelepiped. The axis x_2 is directed along the height and x_1-along the length of the parallelepiped, which divides the height $2h$ in half. It is assumed that plane stress state is realized in direction of axis x_3. Basic equations of the static plane problem (in rectangle $0 \leqslant x_1 \leqslant a,\ -h \leqslant x_2 \leqslant h)$) of micropolar theory of elasticity with independent fields of displacements and rotations (or otherwise, Cosserat continuum) are given in papers [11,12]. In paper [13] asymptotic properties of solutions of the plane problem of the theory of elasticity are studied in thin micropolar rectangle. In paper [4] these asymptotic properties are assumed as hypotheses, on the basis of which applied one-dimensional theory of micropolar elastic thin bars with independent fields of displacements and rotations is constructed.

One of the main hypotheses [4] of construction the mathematical model of micropolar bars is so-called Tymoshenko's kinematic hypothesis generalized for micropolar case, where points' displacement and free rotation of normal element in case of bending are expressed as follows:

$$V_2 = w(x_1),\ \ V_1 = x_2\psi(x_1),\ \ \omega_3 = \Omega_3(x_1). \tag{1.1}$$

Here w is bar deflection, Ω_3 is angle of free rotation, ψ is total angle of normal element rotation.

The basic system of one-dimensional equations of micropolar bar finite element is the following [4]:

Equilibrium equations

$$\frac{\partial N_{12}}{\partial x_1} = -2q,\quad \frac{\partial M_{11}}{\partial x_1} - N_{21} = -h\cdot 2q_1,$$
$$\frac{\partial L_{13}}{\partial x_1} + N_{12} - N_{21} = -2m_2. \tag{1.2}$$

Elasticity relations

$$N_{12} = 2h[(\mu+\alpha)\Gamma_{12} + (\mu-\alpha)\Gamma_{21}],\ \ N_{21} = 2h[(\mu-\alpha)\Gamma_{12} + (\mu+\alpha)\Gamma_{21}],$$
$$M_{11} = \frac{2Eh^3}{3}K_{11},\ \ L_{13} = 2Bhk_{13}. \tag{1.3}$$

Geometrical relations

$$\Gamma_{12} = \frac{\partial w}{\partial x_1} - \Omega_3,\ \ \Gamma_{21} = \psi + \Omega_3,$$
$$K_{11} = \frac{\partial \psi}{\partial x_1},\ \ k_{13} = \frac{\partial \Omega_3}{\partial x_1}. \tag{1.4}$$

Here N_{12}, N_{21} are averaged forces along the bar thickness; M_{11}, L_{13} are averaged moments of power stress σ_{11} and moment stress μ_{13} along the bar thickness; Γ_{12}, Γ_{21} are shear deformations; K_{11} is bar axis bending (connected with transfer moment M_{11}), and k_{13} is bar axis bending (connected with transfer moment L_{13}); $2q$ is intensity of the load distributed normally to the bar axis; $2q_1$ is intensity of the load distributed parallel to the bar axis; $2m_2$ is intensity of external moment; E and μ are classical modules of elasticity and shear of bar material; α and B are new elastic constants of bar micropolar material.

Boundary conditions on the edge (on $x_1 = 0$ or $x_1 = a$) of the bare are the followings:

$$M_{11} = M_{11}^*, \text{ or } \psi = \psi^*,$$
$$N_{12} = N_{12}^*, \text{ or } w = w^*, \tag{1.5}$$
$$L_{13} = L_{13}^*, \text{ or } \Omega_3 = \Omega_3^*.$$

General form of the potential energy functional of micropolar elastic thin bars deformation is expressed as follows:

$$U = \int_0^a (W - 2q_1 h\psi - 2qw - 2m_2\Omega_3)\, dx_1 -$$
$$- \left((M_{11}\psi + N_{12}w + L_{13}\Omega_3)_{x_1=a} - (M_{11}\psi + N_{12}w + L_{13}\Omega_3)_{x_1=0}\right), \tag{1.6}$$

where

$$W = E\frac{h^3}{3}K_{11}^2 + h(\mu+\alpha)(\Gamma_{12}^2 + \Gamma_{21}^2) + 2h(\mu-\alpha)\Gamma_{12}\Gamma_{21} + Bhk_{13}^2\,. \tag{1.7}$$

W is linear density of deformation potential energy of micropolar bar during the bending.

Minimizing the functional (1.6) basic differential equations (1.2)-(1.4) and natural boundary conditions (1.5) will be obtained for bending deformation of micropolar bar.

3 Stiffness matrix of finite element of micropolar bar

Let's consider determination of stiffness matrix of micropolar bar finite element.

Following expansions in the form of cubic polynomials are chosen for deflection w, complete rotation ψ of normal element and for free rotation Ω_3 of normal element:

$$w(x) = a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3,$$
$$\psi(x) = b_0 + b_1x_1 + b_2x_1^2 + b_3x_1^3, \tag{2.1}$$
$$\Omega_3(x) = c_0 + c_1x_1 + c_2x_1^2 + c_3x_1^3.$$

Here a_i, b_i, c_i are coefficients, which are expressed with the help of nodal displacements and rotations. Nodal displacements are denoted as follows:

$$w(0) = \delta_1,\ w'(0) = \delta_2,\ \psi(0) = \delta_3,\ \psi'(0) = \delta_4,\ \Omega_3(0) = \delta_5,\ \Omega_3'(0) = \delta_6,$$
$$w(a) = \delta_7,\ w'(a) = \delta_8,\ \psi(a) = \delta_9,\ \psi'(a) = \delta_{10},\ \Omega_3(a) = \delta_{11},\ \Omega_3'(a) = \delta_{12}. \tag{2.2}$$

As we can see above mentioned finite element has twelve degrees of independence. Substituting (2.1) into (2.2), coefficients a_i, b_i, c_i will be expressed with the help of nodal displacements and rotations δ_k. Substituting a_i, b_i, c_i into (2.1), we obtain following approximations for displacements and rotations.

$$w(x_1) = \sum_{i=1,2,7,8} \delta_i N_i(x_1),$$
$$\psi(x_1) = \sum_{i=3,4,9,10} \delta_i N_i(x_1), \tag{2.3}$$
$$\Omega_3(x_1) = \sum_{i=5,6,11,12} \delta_i N_i(x_1).$$

Here $N_i(x)$ are form functions of the element:

$$N_1 = N_3 = N_5 = 1 - \frac{3}{a^2}x_1^2 + \frac{2}{a^3}x_1^3,\ N_2 = N_4 = N_6 = x_1 - \frac{2}{a}x_1^2 + \frac{1}{a^2}x_1^3,$$

$$N_7 = N_9 = N_{11} = \frac{3}{a^2}x_1^2 - \frac{2}{a^3}x_1^3,\ N_8 = N_{10} = N_{12} = -\frac{1}{a}x_1^2 + \frac{1}{a^2}x_1^3. \tag{2.4}$$

Substituting (2.3) into functional (1.6), after integration we obtain function of twelve independent variables $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \delta_{10}, \delta_{11}, \delta_{12}$. The minimization of functional (1.6) reduces to the determination of the minimum of function of twelve independent variables:

$$\frac{\partial U}{\partial \delta_k} = 0 \quad (k = 1, 2, 3, \ldots, 12).$$

Calculating corresponding partial derivatives, we obtain system of linear algebraic equations:

$$[K] \cdot \{\delta\} = \{P\}. \tag{2.5}$$

Here K is stiffness matrix of element with size 12×12, which is the most important concept of the finite element method; $\{\delta\}^T = \{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \delta_{10}, \delta_{11}, \delta_{12}\}$ is vector of nodal displacements and rotations;

$$\{P\}^T = \left\{ \int_0^a 2qN_1 dx_1,\ \int_0^a 2qN_2 dx_1,\ \int_0^a 2q_1 hN_3 dx_1,\ \int_0^a 2q_1 hN_4 dx_1, \right.$$

$$\int_0^a 2m_2 N_5 dx_1,\ \int_0^a 2m_2 N_6 dx_1,\ \int_0^a 2qN_7 dx_1,\ \int_0^a 2qN_8 dx_1,$$

$$\left. \int_0^a 2q_1 hN_9 dx_1,\ \int_0^a 2q_1 hN_{10} dx_1,\ \int_0^a 2m_2 N_{11} dx_1,\ \int_0^a 2m_2 N_{12} dx_1 \right\}$$

is vector concentrated nodal forces and moments.

Expressions for the elements of the stiffness matrix of a finite element are introduced below:

$$K_{11} = -K_{17} = K_{77} = \frac{12h(\alpha + \mu)}{5a},\ K_{12} = K_{18} = -K_{27} = -K_{78} = \frac{h(\alpha + \mu)}{5},$$

$$K_{13} = K_{19} = -K_{37} = -K_{79} = h(\alpha - \mu),$$

$$K_{14} = -K_{1,10} = -K_{23} = K_{29} = K_{38} = -K_{47} = K_{7,10} = -K_{89} = \frac{ha(\alpha - \mu)}{5},$$

$$K_{15} = K_{1,11} = -K_{57} = -K_{7,11} = 2h\alpha,$$

$$K_{16} = -K_{1,12} = -K_{25} = K_{2,11} = K_{58} = -K_{67} = K_{7,12} = -K_{8.11} = \frac{2}{5}ha\alpha,$$

$$K_{22} = K_{88} = \frac{4ah(\alpha + \mu)}{15},\ K_{24} = K_{26} = K_{8,10} = K_{8,12} = 0,\ K_{28} = -\frac{ha(\alpha + \mu)}{15},$$

$$K_{2,10} = -K_{48} = -\frac{a^2h(\alpha - \mu)}{30},\ K_{2,12} = -K_{68} = -\frac{1}{15}a^2h\alpha,$$

$$K_{33}=K_{99}=\frac{26a^2(\alpha+\mu)+28h^2E}{35a}h,\;\; K_{34}=-K_{9,10}=\frac{11a^2(\alpha+\mu)+7h^2E}{105}h,$$

$$K_{35}=K_{9,11}=\frac{52}{35}ah\alpha,\;\; K_{36}=K_{45}=-K_{9,12}=-K_{10,11}=\frac{22}{105}a^2h\alpha,$$

$$K_{39}=\frac{9a^2(\alpha+\mu)-28h^2E}{35a}h,\;\; K_{3,10}=-K_{49}=-\frac{13a^2(\alpha+\mu)-14h^2E}{210}h,$$

$$K_{3,11}=K_{59}=\frac{18}{35}ah\alpha,\;\; K_{3,12}=-K_{4,11}=K_{5,10}=-K_{69}=-\frac{13}{105}a^2h\alpha,$$

$$K_{44}=K_{10,10}=\frac{6a^3(\alpha+\mu)+28ah^2E}{315}h,\;\; K_{46}=K_{10,12}=\frac{4}{105}a^3h\alpha,$$

$$K_{4,10}=-\frac{9a^3(\alpha+\mu)-14ah^2E}{630}h,\;\; K_{4,12}=K_{6,10}=-\frac{1}{35}a^3h\alpha,$$

$$K_{55}=K_{11,11}=\frac{84B+104a^2\alpha}{35a}h,\;\; K_{56}=-K_{11,12}=\frac{21B+44a^2\alpha}{105}h,$$

$$K_{5,11}=-\frac{84B-36a^2\alpha}{35l}h,\;\; K_{5,12}=\frac{21B-26a^2\alpha}{105}h,\;\; K_{66}=K_{12,12}=\frac{28B+8a^2\alpha}{105}ah,$$

$$K_{6,11}=-\frac{21B-26a^2\alpha}{105}h,\;\; K_{6,12}=-\frac{7B+6a^2\alpha}{105}ah.$$

4 Examples

As a first example we'll consider problem of normal to axis x_1 evenly loaded bar, when its edges are hinged-supported.

We have following boundary condition for hinged-supported bar:

$$w=0,\;\; M_{11}=0,\;\; L_{13}=0,\;\; \text{on}\;\; x_1=0;a. \tag{3.1}$$

Conditions (3.1) are equivalent to followings with consideration of (1.3), (1.4):

$$w=0,\;\; \frac{\partial\psi}{\partial x_1}=0,\;\; \frac{\partial\Omega_3}{\partial x_1}=0,\;\; \text{on}\;\; x_1=0;a. \tag{3.2}$$

We obtain following expression for functional (1.6) with consideration of (3.1):

$$U=\int_0^a (W-2q_1h\psi-2qw-2m_2\Omega_3)dx_1.$$

Also we assume $q_1=0$, $q\neq 0$, $m_2=0$. At first, we consider the bar as a whole one finite element. We calculate the concentrated nodal forces and moments, which are equivalent to distributed load $2q=const:$

$$\{P'\}^T=\left\{qa,\frac{qa^2}{6},0,0,0,0,qa,\frac{qa^2}{6},0,0,0,0\right\}.$$

Considering, that reaction of bar supports on the edges is $-qa$, definitively, loads in the end nodes of bar will be as follows:

$$\{P\}^T=\left\{0,\frac{qa^2}{6},0,0,0,0,0,\frac{qa^2}{6},0,0,0,0\right\}.$$

The system of equations for this problem is the following:

$$[K] \cdot \{\delta\} = \{P\}.$$

It is easy to see that the first and seventh lines of stiffness matrix are proportional, i.e. $det[K] = 0$. Thus, the rank of matrix of the system is equal to ten, but the rank of the augmented matrix is also equal to ten. Quantity of unknowns is equal to twelve, so the system has infinite number of solutions. To obtain the unique solution, it is necessary to introduce two arbitrary values for two unknowns. These values are taken from the boundary condition of hinged-supported micropolar bars, this means $\delta_1 = w(0) = 0$, $\delta_7 = w(a) = 0$.

Taking into consideration (3.2), we obtain $\delta_4 = \psi'(0) = 0$, $\delta_{10} = \psi'(a) = 0$, $\delta_6 = \Omega_3'(0) = 0$, $\delta_{12} = \Omega_3'(a) = 0$. As a result, solution of the system (2.5) gives values for nodal generalized displacements δ_2, δ_3, δ_5, δ_8, δ_9, δ_{11}.

To improve accuracy of solutions it is necessary to divide the bar into several finite elements. We consider the case, when the bar is divided into two finite elements. Numerical results (maximum deflection) of the calculation are given for the case, when the physical constants have following values [14]: $\alpha = 1.6MPa$, $\mu = 2MPa$, $\lambda = 3MPa$, $B = 6KN$. Geometrical dimensions of the bar are the followings: $a = 8mm$, $h = 0,2mm$. We also introduce the result for classical theory of elastic thin bar when it is bent.

The case, when the bar is considered as a whole finite element:

1) $w_{\max} = 5,6433 \cdot 10^{-7}m$.

The case when the bar is divided into two finite elements:

2) $w_{\max} = 5,6475 \cdot 10^{-7}m$.

Exact solution gives:

3) $w_{\max} = 5,6498 \cdot 10^{-7}m$.

Value in the case of the classical theory:

4) $w_{\max} = 1,9331 \cdot 10^{-4}m$.

As we can see the micropolarity of the bar material increases its rigidity. The same qualitative results are valid for maximum bending stresses.

Analogical result of the problem is introduced, when the bar is loaded with evenly distributed normal load and its both edges are rigidly fixed.

1) When the bar is considered as a whole finite element: $w_{\max} = 5,6069 \cdot 10^{-7}m$.

2) When the bar is divided into two finite elements: $w_{\max} = 5,6224 \cdot 10^{-7}m$.

3)The exact value of the maximum deflection: $w_{\max} = 5,6246 \cdot 10^{-7}m$.

4)Classical value: $w_{\max} = 0,3962 \cdot 10^{-5}m$.

We also consider the case, when one edge of the bar is rigidly fixed and the other one is free, but is loaded by a concentrated force:

1) When the bar is considered as a whole one element: $w_{max} = 0,0002845m$.

2) Exact solution: $w_{\max} = 0,0002847m$.

3) Classical solution: $w_{\max} = 0,308192m$.

In the above discussed cases micropolarity of the bar material increases its rigidity and strength.

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