Comparison of analytical and numerical solutions for a problem of thin body motion in gas near rigid surface

M. N. Smirnova   A. V. Zvyaguin
wonrims@inbox.ru

Abstract

The two-dimensional problem of thin body motion in gas parallel to the boundary at a distance, comparable with the length of the body, is regarded. The lift force of thin body moving parallel to rigid surface is determined and compared with the obtained numerical solution. The analytical solution is determined under the assumption of fluid being ideal and compressible. The Chaplygin-Zhukovsky hypothesis of rear-edge-limited solution is taken into consideration.

The analytical solution of a problem is first reduced to singular integral equation and then to the Fredholm equation. The generalization of Zhukovski solution was obtained, which provides the lift force dependence on the altitude of the flight. The lift force increases on decreasing altitude above the rigid surface. The screen effect becomes essential on moving wing altitude being smaller than the wing's length. The effect was detected experimentally before and gave birth to construction of a special flying vehicle named "ecranoplan".

The numerical solution is obtained using the method of boundary elements. The comparison of lift force dependence upon the altitude between the numerical and analytical solutions is done. Streamlines and the distribution of the velocity along them are shown in case of a plate moving on different altitudes and with different inclination angles.

1 Introduction

At the beginning of the XX century it was observed that the lift force of a wing moving near flat surface increases strongly in comparison with free flight. That fact was used in creation of new flying devices - screen-flights, which got the Russian name "ekranoplan". The first Soviet manned jet screen-flight SM-1 was created in collaboration with R. Alekseev in 1960 - 1961. Giant screen-flight KM was finished by 1966 and "Orlyonok" type screen-flights were built from 1974 to 1983. Designing of new flying devices continues in many countries.

The production of analytical solutions for linear problems of wings moving near flat surfaces is very important in the sense of verifying numerical modeling of the process.

L.I. Sedov obtained an analytical solution for the lift force of a wing moving near rigid surface in terms of Weierstrass functions [1] using the theory of a complex variable, but the obtained solutions incorporated free constants. Approximate analytical solution of the problem of non-steady plane moving near rigid surface was obtained by K.V. Rozjdestvensky [2]. Theoretical investigation of a wing moving near rigid surface was made by A.N. Panchenkov [3, 4]. Experimental results are shown in [5].

2 Mathematical statement of the problem

It is assumed that the wing is moving with constant velocity V in an ideal compressible fluid near a motionless surface. In a motionless two-dimensional coordinate system adiabatic gas flow is described by the continuity and Euler equations. The angle α and mass forces are considered to be negligibly small. These assumptions make the flow field to be potential. Streaming condition of the equality of normal velocity component should be satisfied on the rigid surface and on the body surface contacting fluid.

In movable coordinate system $x = x' + Vt, y = y'$ connected with the wing the gas flow can be considered stable. After transfer to another independent variable $\tilde{x} = \frac{x}{\delta}, \delta = \sqrt{1 - M^2}, M-$Mach number the equations and boundary conditions describing the gas flow take the following form:

$$\frac{\partial^2 \varphi}{\partial \tilde{x}^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0;$$
$$0 \le \tilde{x} \le \frac{L}{\delta}, y = h^\pm : \quad \frac{\partial \varphi(\tilde{x}, h^\pm)}{\partial y} = V\alpha^\pm(\delta\tilde{x});$$
$$-\infty < \tilde{x} < \infty, y = 0 : \quad \frac{\partial \varphi}{\partial y} = 0;$$
$$p = p_0 - \frac{\rho V}{\delta}\frac{\partial \varphi}{\partial \tilde{x}}.$$

Boundary conditions should be supplemented with function behavior at the infinity for the uniqueness of the solution. Thus an analytical function satisfying boundary conditions and decreasing at the infinity should be developed.

3 Problem solution

The solution of the Laplas equation can be developed in the form of a real part for the analytical function of a complex variable $\varphi(\tilde{x}, y) = Re\Phi(z), z = \tilde{x} + iy$. Actually, it is necessary to develop first derivative of the analytical function, which could be denoted as: $iT(z) = \Phi'(z)$. The development of the analytical function is reduced to the following boundary problem:

$$0 \le \tilde{x} \le \frac{L}{\delta}, y = h^\pm : \quad ReT(\tilde{x} + ih) = -V\alpha^\pm(\delta\tilde{x}) = -V\tilde{\alpha}^\pm(\tilde{x});$$
$$-\infty < \tilde{x} < \infty, y = 0 : \quad ReT^+(\tilde{x}, 0^+) = 0.$$

The pressure can be developed from the following formula:

$$p(\tilde{x} + iy) = p_0 + \frac{\rho V}{\delta} ImT(\tilde{x} + iy).$$

For the uniqueness of the solution boundary conditions should be supplemented with function behavior at the infinity and at the ends of the segment $0 \le \tilde{x} \le \frac{L}{\delta}$. It is considered that the function tends to zero at the infinity and is limited at the rear edge of the wing, which is the result of the Chaplygin-Zhukovsky hypothesis [9].

The solution is developed with the help of the symmetry principle. The following non-dimensional variables are introduced $\tilde{\tilde{x}} = \frac{\tilde{x}}{\tilde{L}}, \tilde{\tilde{y}} = \frac{y}{\tilde{L}}, \tilde{\tilde{h}} = \frac{h}{\tilde{L}}, \tilde{L} = \frac{L}{\delta}$.

The development of the analytical function is reduced to the following boundary problem:

$$0 \le \tilde{\tilde{x}} \le 1, \tilde{\tilde{y}} = \tilde{\tilde{h}}^\pm : \quad ReT(\tilde{\tilde{x}} + i\tilde{\tilde{h}}) = -V\alpha^\pm(L\tilde{\tilde{x}}) = -V\tilde{\tilde{\alpha}}^\pm(\tilde{\tilde{x}});$$

$$-\infty < \tilde{\tilde{x}} < \infty, \tilde{\tilde{y}} = 0: \quad ReT^+(\tilde{\tilde{x}}, 0^+) = 0.$$

The tildes will be omitted in the following equations.

The solution of the problem can be derived in the form of Cauchy type integral [6]:

$$T(z) = X_1(z)[\frac{1}{2\pi i}\int_0^1 \frac{1}{X_1^+(t)}\frac{\mu(t)}{t+ih-z}dt] + \frac{1}{2\pi i}\int_0^1 \frac{\zeta(t)+i\eta(t)}{t+ih-z} -$$
$$- X_2(z)[\frac{1}{2\pi i}\int_0^1 \frac{1}{X_2^+(t)}\frac{\mu(t)}{t-ih-z}dt] + \frac{1}{2\pi i}\int_0^1 \frac{\zeta(t)-i\eta(t)}{t-ih-z},$$

where

$$X_1(z) = \sqrt{\frac{z-z_1}{z-z_2}}, z_1 = 1+ih, z_2 = ih;$$
$$X_2(z) = \sqrt{\frac{z-\bar{z}_1}{z-\bar{z}_2}}, \bar{z}_1 = 1-ih, \bar{z}_2 = -ih.$$

The function $T(z)$ satisfies the boundary condition $-\infty < x < \infty, y = 0, T^+(x, 0^+) = 0$. It is assumed that $\mu(t) = -V(\tilde{\tilde{\alpha}}^+ + \tilde{\tilde{\alpha}}^-), \zeta(t) = -V(\tilde{\tilde{\alpha}}^+ - \tilde{\tilde{\alpha}}^-)$. The function $T(z)$ will satisfy the boundary condition $0 \leq x \leq 1, y = h^\pm \quad ReT(x+ih) = -V\tilde{\tilde{\alpha}}^\pm(x)$, if the following singular integral equation is fulfilled:

$$\frac{1}{2\pi}\int_0^1 \frac{\eta(t)}{t-s}dt - \frac{1}{2\pi}\int_0^1 \frac{(t-s)\eta(t)}{(t-s)^2+4h^2}dt - \frac{Vh}{\pi}\int_0^1 \frac{\tilde{\tilde{\alpha}}^+(t) - \tilde{\tilde{\alpha}}^-(t)}{(t-s)^2+4h^2}dt -$$
$$- R(s)\sin\alpha(s)\frac{V}{2\pi}\int_0^1 \sqrt{\frac{t}{1-t}}\frac{(t-s)}{(t-s)^2+4h^2}(\tilde{\tilde{\alpha}}^+(t) + \tilde{\tilde{\alpha}}^-(t))dt +$$
$$+ R(s)\cos\alpha(s)\frac{Vh}{\pi}\int_0^1 \sqrt{\frac{t}{1-t}}\frac{1}{(t-s)^2+4h^2}(\tilde{\tilde{\alpha}}^+(t) + \tilde{\tilde{\alpha}}^-(t))dt = 0,$$

where

$$\alpha(s) = \frac{1}{2}[\arctan(\frac{2h}{1-s}) + \arctan(\frac{2h}{s})], R(s) = \sqrt[4]{\frac{(1-s)^2+4h^2}{s^2+4h^2}}.$$

4 The case of wing being a plate

If the wing has a shape of a plate $\tilde{\tilde{\alpha}}^\pm = -\gamma$ the singular integral equation takes the following form:

$$\frac{1}{\pi}\int_0^1 \frac{\eta(t)}{t-s}dt - \frac{1}{\pi}\int_0^1 \frac{(t-s)\eta(t)}{(t-s)^2+4h^2}dt + R(s)\sin\alpha(s)\frac{V\gamma}{\pi}\int_0^1 \sqrt{\frac{t}{1-t}}\frac{(t-s)}{(t-s)^2+4h^2}dt -$$
$$- R(s)\cos\alpha(s)\frac{2Vh\gamma}{\pi}\int_0^1 \sqrt{\frac{t}{1-t}}\frac{1}{(t-s)^2+4h^2}dt = 0.$$

The third and the fourth integrals can be taken analytically using the theory of residues. The following singular integral equation can be determined:

$$\frac{1}{\pi}\int_0^1 \frac{\tilde{\eta}(t)}{t-s}dt = \frac{1}{\pi}\int_0^1 \frac{(t-s)\tilde{\eta}(t)}{(t-s)^2+4h^2}dt - F(s),$$

where

$$\tilde{\eta}(s) = \frac{\eta(s)}{V\gamma}, F(s) = R(s)\sin\alpha(s) - 1, R(s) = \sqrt[4]{\frac{(1-s)^2+4h^2}{s^2+4h^2}},$$
$$\alpha(s) = \frac{1}{2}[\arctan(\frac{2h}{1-s}) + \arctan(\frac{2h}{s})].$$

5 The case of wing being convex

If the wing has a convex shape $\tilde{\tilde{\alpha}}^+(x) = -2\gamma x, \tilde{\tilde{\alpha}}^-(x) = -\gamma$. The singular integral equation takes the following form:

$$\frac{1}{\pi}\int_0^1 \frac{\eta(t)}{t-s}dt - \frac{1}{\pi}\int_0^1 \frac{(t-s)\eta(t)}{(t-s)^2+4h^2}dt + \frac{2\gamma Vh}{\pi}\int_0^1 \frac{t-\frac{1}{2}}{(t-s)^2+4h^2}dt+$$
$$+ R(s)\sin\alpha(s)\frac{V\gamma}{\pi}\int_0^1 \sqrt{\frac{t}{1-t}}\frac{(t-s)}{(t-s)^2+4h^2}(t+\frac{1}{2})dt-$$
$$- R(s)\cos\alpha(s)\frac{2\gamma Vh}{\pi}\int_0^1 \sqrt{\frac{t}{1-t}}\frac{1}{(t-s)^2+4h^2}(t+\frac{1}{2})dt = 0.$$

The third, the fourth and the fifth integrals can be taken analytically using the theory of residues. The following singular integral equation can be determined:

$$\frac{1}{\pi}\int_0^1 \frac{\tilde{\eta}(t)}{t-s}dt = \frac{1}{\pi}\int_0^1 \frac{(t-s)\tilde{\eta}(t)}{(t-s)^2+4h^2}dt - F(s),$$

where

$$\tilde{\eta}(s) = \frac{\eta(s)}{V\gamma},$$

$$F(s) = \frac{h}{\pi}\ln\frac{(1-s)^2+4h^2}{s^2+4h^2} + \frac{1}{\pi}(s+\frac{1}{2})[\arctan\frac{1-s}{2h} + \arctan\frac{s}{2h}]+$$
$$+ \frac{1}{2}(R(s)\sin\alpha(s) - 1) + (s+\frac{1}{2})R(s)\sin\alpha(s) - 2hR(s)\cos\alpha(s)-$$
$$- \frac{(1-s)}{R^2(s)}[\cos 2\alpha(s) + \frac{2h}{1-s}\sin 2\alpha(s)],$$

$$R(s) = \sqrt[4]{\frac{(1-s)^2+4h^2}{s^2+4h^2}}, \alpha(s) = \frac{1}{2}[\arctan(\frac{2h}{1-s}) + \arctan(\frac{2h}{s})].$$

6 Determination of the lift force

After the regularization the singular integral equation is reduced to the Fredholm equation:

$$\tilde{\eta}(x) + \int_0^1 \tilde{\eta}(t)K(x,t)dt = G(x),$$

where

$$K(x,t) = \frac{1}{4\pi}\sqrt{\frac{1-x}{x}}\frac{\sin[\alpha(t)-\delta(x,t)]}{R(t)\sqrt{[t(1-x)-x(1-t)]^2+4h^2}},$$
$$G(x) = \sqrt{\frac{1-x}{x}}\frac{1}{\pi}\int_0^1 \sqrt{\frac{s}{1-s}}\frac{F(s)}{s-x}ds, R(s) = \sqrt[4]{\frac{(1-s)^2+4h^2}{s^2+4h^2}},$$
$$\alpha(t) = \frac{1}{2}\left[\arctan(\frac{2h}{1-t}) + \arctan(\frac{2h}{t})\right], \delta(x,t) = \arctan\frac{2h}{t(1-x)-x(1-t)}.$$

In the case of a plate:

$$F = \frac{\pi\rho LV^2\gamma}{\delta}\left[1 + \frac{1}{\pi}\int_0^1 [-\tilde{\eta}(s)]ds\right].$$

The derived lift force differs from its analog for boundless surface by an extra summand which decreases with the increase of the height.

In the case of a convex wing:

$$F = \frac{\pi\rho L V^2\gamma}{\delta}[\frac{5}{4} - \frac{1}{\pi}\int_0^1 \tilde{\eta}(s)ds].$$

7 Numerical problem solution

The boundary elements method is based upon the distribution of segments along the contour of a wing with piecewise constant dipole and a vortex at the rear edge of the contour.

The velocity is given by the following formula:

$$\mathbf{u}(\mathbf{x}) = \mathbf{U} + \sum_{k=0}^{N+1} \mu_k[\mathbf{u}^{PV}(\mathbf{x}, \xi_k) - \mathbf{u}^{PV}(\mathbf{x}, \bar{\xi}_k) + \mathbf{u}^{PV}(\mathbf{x}, \bar{\xi}_{k+1}) - \mathbf{u}^{PV}(\mathbf{x}, \xi_{k+1})] +$$

$$+ k[\mathbf{u}^{PV}(\mathbf{x}, \xi_0) - \mathbf{u}^{PV}(\mathbf{x}, \bar{\xi}_0)]$$

where $\mathbf{U} = (V, 0)$ is the velocity of undisturbed gas, $\mathbf{u}^{PV} = \frac{1}{2\pi r^2}(-r_y, r_x)$ is the velocity of disturbed gas from a vortex with the center in ξ and a unit power, $\mathbf{x} = (x, y)$, $\xi = (\zeta, \eta)$, $\bar{\xi} = (\zeta, -\eta)$, $r_x = x - \zeta$, $r_y = y - \eta$, $r^2 = r_x^2 + r_y^2$, $[\xi_k, \xi_{k+1}], k = 1...N$ are the segments, which approximate the contour.

Boundary condition on the rigid surface is satisfied.

Unknown coefficients μ_k, k can be developed from the streaming conditions on the contour, fulfilled in the centers of the segments. The system of linear algebraic equations is closed with the Chaplygin-Zhukovsky hypothesis, which minimizes the module of the velocity at the rear edge of the contour: $\mu_0 - \mu_{N-1} + k = 0$.

The lift force can be derived from the following equation:

$$\mathbf{F} = \frac{1}{2}\rho_0 \sum_{k=0}^{N-1} [(\xi_{k+1} - \xi_k)\mathbf{n}_k \hat{\mathbf{u}}_k^2],$$

where $\hat{\mathbf{u}}_k = \mathbf{u}(\hat{\mathbf{x}}_k), \hat{\mathbf{x}}_k = \frac{1}{2}(\xi_k + \xi_{k+1})$.

8 Results and discussions

The dependence of the reduced lift force $\frac{F\delta}{\pi\rho_0 L V^2\gamma}$ of a plate and a convex contour upon the altitude $H = \tilde{\tilde{h}} = \frac{h\delta}{L}$, which was developed analytically, is shown on Fig. 1. ρ_0−the density of undisturbed gas, L−the length of a contour.

It can be seen that the lift force decreases with the increase of the height above the rigid surface until the force reaches its magnitude in a boundless medium. The lift force of a convex contour is bigger than that of a plate for the same altitudes.

On Fig. 2. the dependence of the reduced lift force $\frac{F\delta}{\pi\rho_0 L V^2\gamma}$ of a plate upon the altitude $H = \tilde{\tilde{h}} = \frac{h\delta}{L}$, which was obtained analytically, is compared with the numerical solution for the Math number $M = 0.2$.

Streamlines and the distribution of velocity along them are shown on Fig. 3-9. in case of gas flows around a plate with an inclination angle α on the altitude $H = \frac{h}{L}$. The velocity module is presented on the colored scale, the measure unit is the velocity of undisturbed

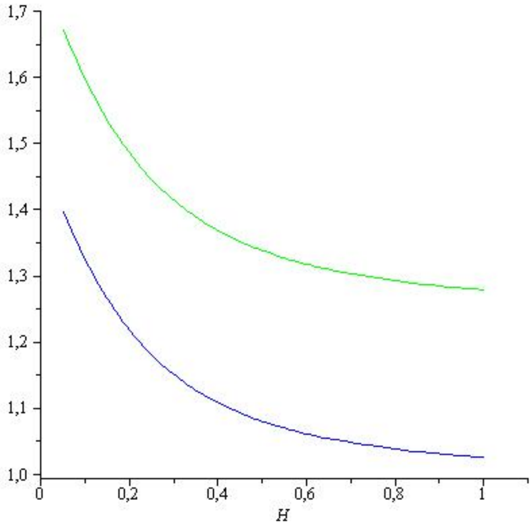

Figure 1: The dependence of the reduced lift force $\frac{F\delta}{\pi\rho_0 LV^2\gamma}$ of a plate (blue curve) and a convex contour (green curve) upon the altitude $H = \tilde{\tilde{h}} = \frac{h\delta}{L}$.

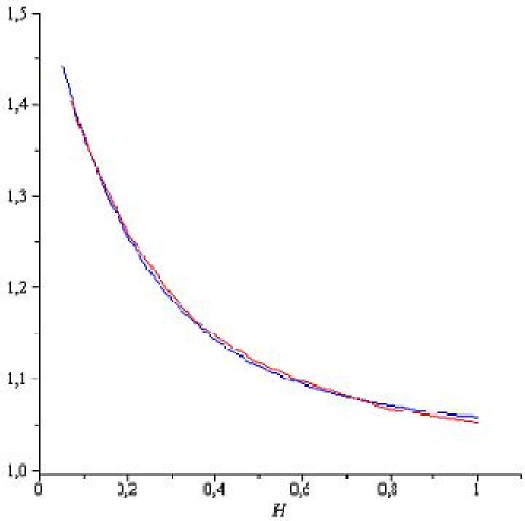

Figure 2: The dependence of the reduced lift force $\frac{F\delta}{\pi\rho_0 LV^2\gamma}$ of a plate upon the altitude $H = \tilde{\tilde{h}} = \frac{h\delta}{L}$ for $M = 0.2$, obtained analytically (blue curve) and numerically (red curve).

gas. It can be seen that the lift force of a plate with an inclination angle less than 30^0 decreases with the increase of the altitude above the surface. There is no such an effect for inclination angles more than 30^0 .

9 Conclusion

A linear problem of thin wing motion near a rigid surface is reduced to the Fredholm equation, which is developed as a system of linear equations. So the problem is solved almost analytically.

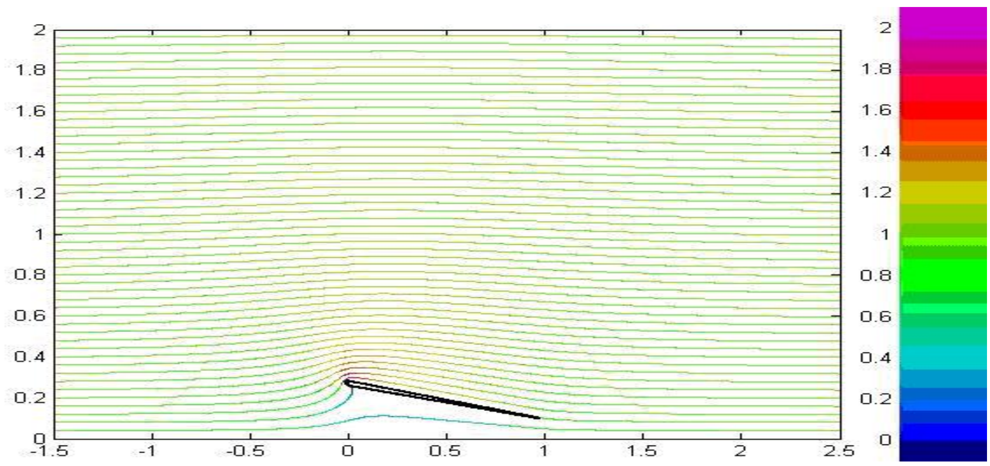

Figure 3: Streamlines and the distribution of velocity along them in case of gas flows around a plate with an inclination angle α on the altitude $H = \frac{h}{L}$: $\alpha = 10^0, H = 0.1$

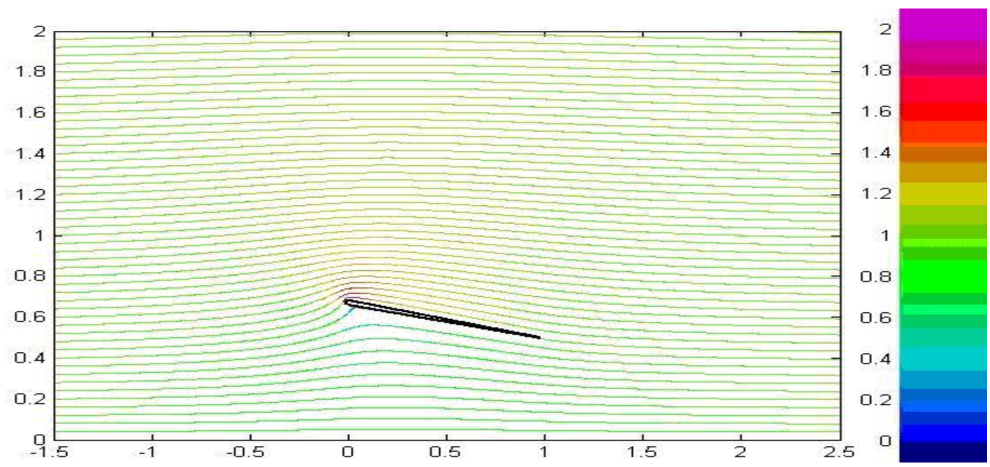

Figure 4: $\alpha = 10^0, H = 0.5$

It is shown that the lift force increases with the decrease of the wing distance from the surface. It can be seen that rigid surface affects the lift force only on the altitude, which is smaller than the length of a wing. If the altitude above the surface surpasses the wing span the screen effect practically disappears and the lift force tends to its value in an unbounded space determined by the classical Zhukovsky solution.

The dependence of the lift force of a plate upon the altitude, which was obtained analytically, is compared with the numerical solution. The results coincide.

The obtained solution evidently shows, that the increase of lift force near the screen in the orders of magnitude allows developing flying vehicles carrying much more cargo at lower fuel consumption.

Acknowledgements

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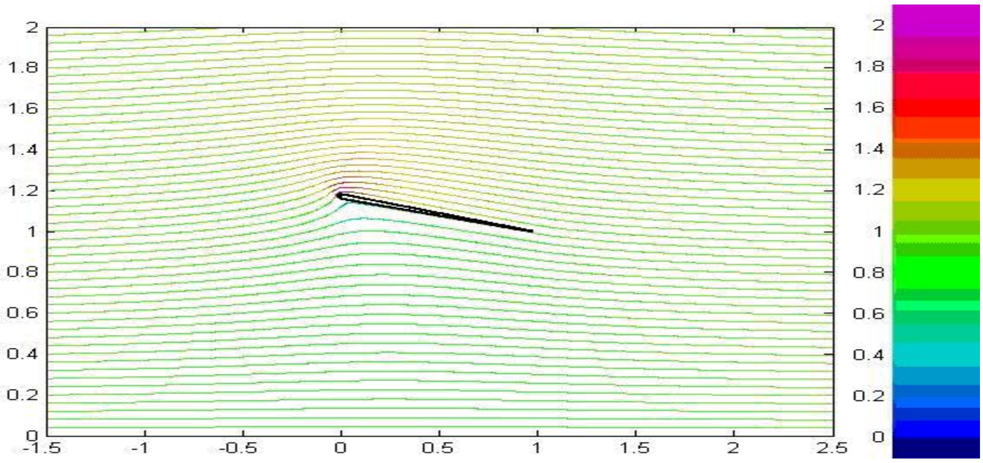

Figure 5: $\alpha = 10^0, H = 1$

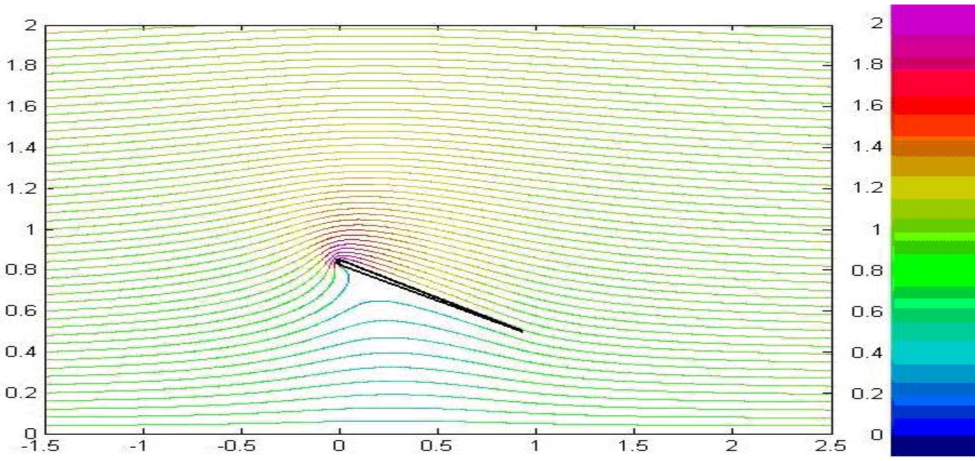

Figure 6: $\alpha = 20^0, H = 0.5$

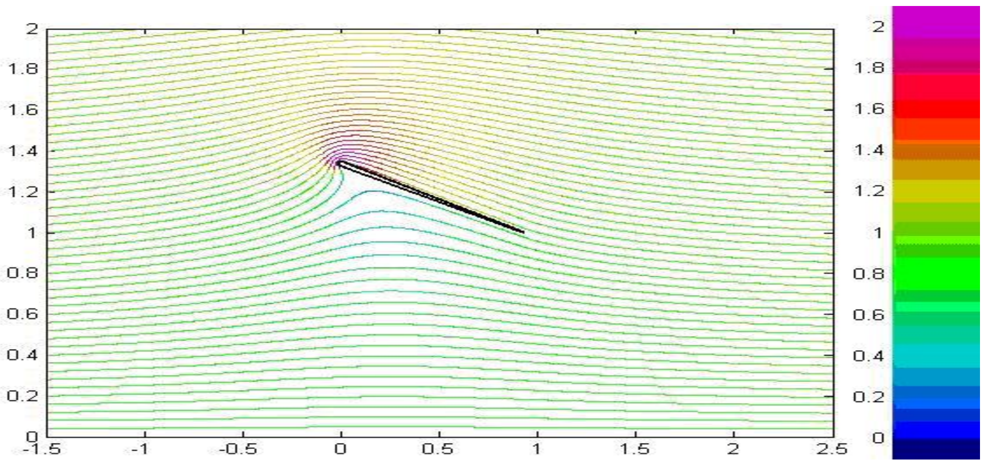

Figure 7: $\alpha = 20^0, H = 1$

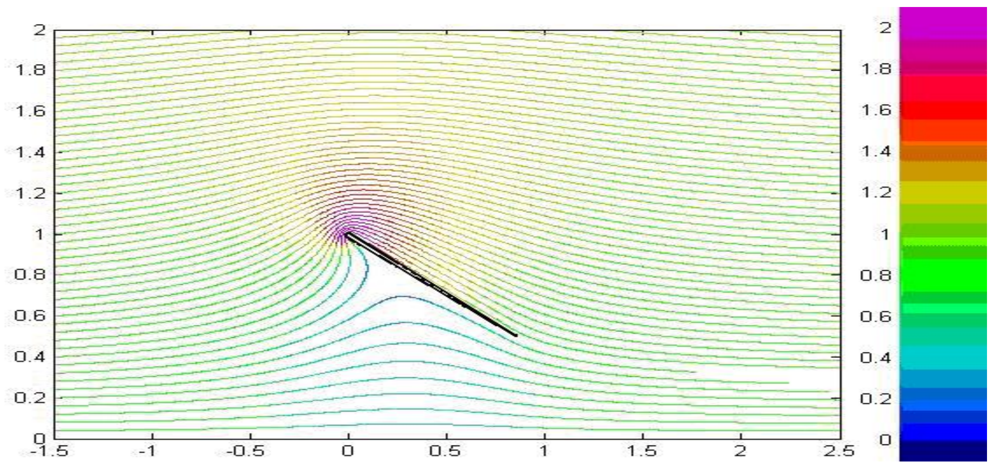

Figure 8: $\alpha = 30^0, H = 0.5$

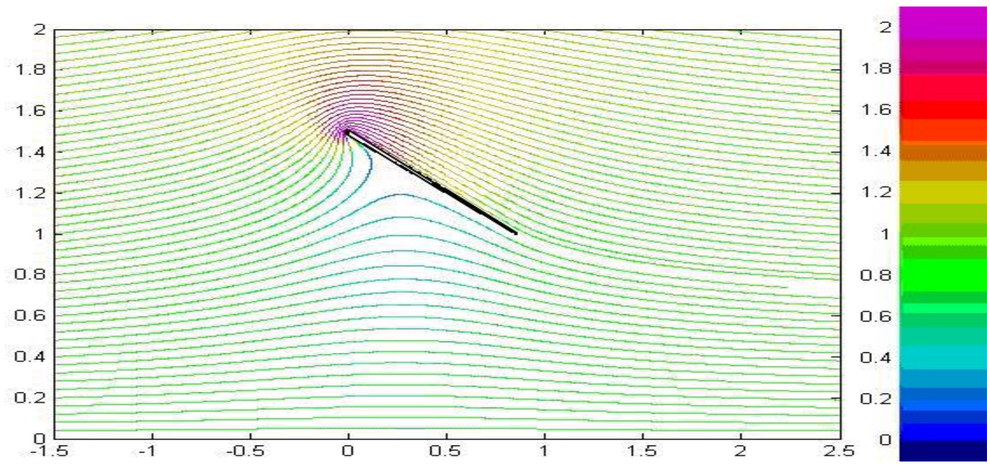

Figure 9: $\alpha = 30^0, H = 1$

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M. N. Smirnova, A. V. Zvyaguin, Department of Mechanics and Mathematics, Moscow M.V. Lomonosov State University, Russia, Moscow, 119992, Leninskie Gory 1.
M. N. Smirnova, St. Petersburg State Polytechnical University, Institute of Military Engineering and Safety Research, Russia, St. Petersburg, Polytechnicheskaya 29.