The system of inverted pendulums

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Abstract

We present a mathematical model of the inverted pendulums. Theoretical analysis is carried out for the system of two objects. The stability problem of N inverted pendulums is analysed numerically.

1 Introduction

From the point of view of studying various physical processes the system of coupled pendulums is an instructive model. The case of small linear oscillations in a vicinity of the stable equilibrium position is described in the literature [1]. In the unstable (inverted) position there are interesting features in the behavior. For example, the upper vertical position of the pendulum might be stable when the driving frequency is fast [2],[3]. Pyotr Kapitza was the first to analyze this highly unusual phenomenon in 1951 [4].

Stability of two inverted connected pendulums was investigated in [5]. The stability problem of a finite number of inverted pendulums in a linear interaction hasn't been considered yet. It is pointed that behavior of the inverted pendulums is linked with the study of the domino effects [6] . In particular it is shown [7]-[9] that the domino-structure plays an essential role in determination of rock brittleness and instability at failure. Therefore, the inverted interacting pendulums are naturally called domino system with weak interaction. This paper presents a mathematical model of the inverted pendulums and examines its properties.

2 Construction of the Lagrangian

We determine the Lagrangian and the equation of motion of the following chain. The chain is a set of massless upright pendulums with $N+1$ mass points . A mass m is fixed to one end of a massless bar; the other end of the bar is fixed to a hinge. The $N+1$ hinges are placed in the vertical direction and divided by the constant distance a. The pendulums of equal length l are connected by a non-linear spring, and the coupling is not weak in the general case (Figure. 1.).

Let the displacement of the mass j be denoted by $\mathbf{r}_j = (x_j,\ y_j)$. Then we have the following presentation

$$(x_j,\ y_j) = (aj + l\cos\varphi_j,\ l\sin\varphi_j), \quad j = 0, 1, \ldots, N \tag{1}$$

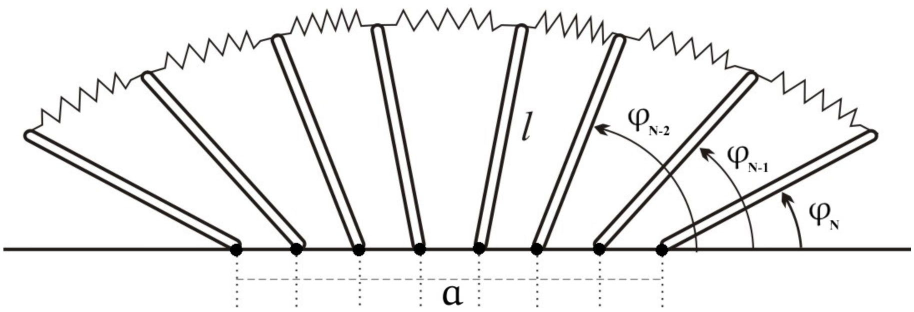

Figure 1: Interacting dominoes.

where φ_j is an angle between the bar and the horizontal axis. We introduced plane polar coordinates for the problem and adopt the set of the polar angles $\{\varphi_j\}$ as generalized coordinates of the chain.

The Lagrangian of the system is equal to $L = T - P$. The kinetic energy of the chain T is calculated in accordance with the formula

$$T = \sum_{j=0}^{N} m\frac{1}{2}(\dot{x}_j^2 + \dot{y}_j^2) = \sum_{j=0}^{N} m\frac{(l\dot{\varphi}_j)^2}{2}. \tag{2}$$

The total potential energy P of the chain is given by the expression

$$P = P_g + P_{nonlin}. \tag{3}$$

Here P_g is the potential energy corresponding to the external uniform gravity and P_{nonlin} arises from internal forces of the interacting masses. The potential energy of a mass due to uniform gravity is $P(j)_g = mgl\sin\varphi_j$ then the corresponding potential energy of the chain is

$$P_g = \sum_{j=0}^{N} P(j)_g = \sum_{j=0}^{N} mgl\sin\varphi_j \ . \tag{4}$$

The internal potential energy function P_{nonlin} depends on a potential interaction between the masses:

$$P_{nonlin} = \sum_{j=0}^{N-1} U(l_{j,j+1}) \tag{5}$$

here $l_{j,j+1}$ is the distance between neighboring masses. We use (1) for calculation of $l_{j,j+1}$:

$$l_{j,j+1} = \sqrt{[a + l(\cos\varphi_{j+1} - \cos\varphi_j)]^2 + l^2(\sin\varphi_{j+1} - \sin\varphi_j)^2}. \tag{6}$$

From (2)-(5) we have the Lagrangian of the system

$$L = T - P = \sum_{j=0}^{N} [m\frac{(l\dot{\varphi}_j)^2}{2} - mgl\sin\varphi_j] - \sum_{j=0}^{N-1} U(l_{j,j+1}) \ . \tag{7}$$

3 Background equations

Lagrange's equations corresponding to the Lagrangian (7) are the following ones:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}_j} - \frac{\partial L}{\partial \varphi_j} = \frac{d}{dt}\frac{\partial T}{\partial \dot{\varphi}_j} + \frac{\partial U}{\partial \varphi_j} \Leftrightarrow$$

$$\Leftrightarrow ml^2\ddot{\varphi}_j = mgl\cos\varphi_j + \frac{\partial U(l_{j-1,j})}{\partial \varphi_j} + \frac{\partial U(l_{j,j+1})}{\partial \varphi_j}. \tag{8}$$

The two last derivatives are equal to

$$\frac{\partial U(l_{j-1,j})}{\partial \varphi_j} + \frac{\partial U(l_{j,j+1})}{\partial \varphi_j} = \frac{\partial U(l_{j-1,j})}{\partial l_{j-1,j}}\frac{\partial l_{j-1,j}}{\partial \varphi_j} + \frac{\partial U(l_{j,j+1})}{\partial l_{j,j+1}}\frac{\partial l_{j,j+1}}{\partial \varphi_j}. \tag{9}$$

It is clear that:

$$\frac{\partial l_{j,j+1}}{\partial \varphi_j} = \frac{1}{2l_{j,j+1}}\frac{\partial l^2_{j,j+1}}{\partial \varphi_j}, \quad \frac{\partial l_{j-1,j}}{\partial \varphi_j} = \frac{1}{2l_{j-1,j}}\frac{\partial l^2_{j-1,j}}{\partial \varphi_j}. \tag{10}$$

The distance $l_{j,j+1}$ is expressed in the terms of the quantities

$$\varphi_{+,j} = \frac{\varphi_{j+1} + \varphi_j}{2}, \ \varphi_{-,j} = \frac{\varphi_{j+1} - \varphi_j}{2}.$$

For this purpose we use the formulae

$$\cos\varphi_{j+1} - \cos\varphi_j = -2\sin\frac{\varphi_{j+1} + \varphi_j}{2}\sin\frac{\varphi_{j+1} - \varphi_j}{2},$$

$$\sin\varphi_{j+1} - \sin\varphi_j = 2\cos\frac{\varphi_{j+1} + \varphi_j}{2}\sin\frac{\varphi_{j+1} - \varphi_j}{2}$$

and rewrite (6) in the form

$$l_{j,j+1} = \sqrt{a^2 - 4al\sin\varphi_{+,j}\sin\varphi_{-,j} + 4l^2\sin\varphi_{-,j}}. \tag{11}$$

Since

$$\frac{\partial \varphi_{+,j}}{\partial \varphi_j} = \frac{1}{2}, \ \frac{\partial \varphi_{-,j}}{\partial \varphi_j} = -\frac{1}{2}$$

we substitute (11) into (10), and it results in

$$\frac{\partial l^2_{j-1,j}}{\partial \varphi_k} = -2al\cos\varphi_{+,j-1}\sin\varphi_{-,j-1} - 2al\sin\varphi_{+,j-1}\cos\varphi_{-,j-1} + 4l^2\sin\varphi_{-,j-1}\cos\varphi_{-,j-1},$$

$$\frac{\partial l^2_{j,j+1}}{\partial \varphi_k} = -2al\cos\varphi_{+,j}\sin\varphi_{-,j} + 2al\sin\varphi_{+,j}\cos\varphi_{-,j} - 4l^2\sin\varphi_{-,j}\cos\varphi_{-,j}. \tag{12}$$

If we consider the weak coupling between the pendulums then the potential $U(l_{j,j+1})$ can be written in the traditional form

$$U(l_{j,j+1}) \equiv \frac{k}{2}(\varepsilon_j)^2, \quad \varepsilon_j \equiv \frac{l_{j,j+1} - a}{a}, \tag{13}$$

where ε_j is the relative deformation. The particles displacements are not small in the general case, i.e. $\varphi_k \sim 1$. But the relative deformation ε_j is supposed to be a first order linear with respect to $\varphi_{-,k}$ then from (11), (13) one has

$$|\varepsilon_j| \approx \frac{2l}{a}|\varphi_{-,j}| \sin \varphi_{+,j} \ll 1. \tag{14}$$

Forces that keep the pendulums together are assumed to be a first order linear with respect to the relative deformation ε_j. It allows us to replace $\sin \varphi_{-,j} \to 0$ and $\cos \varphi_{-,j} \to 1$ in (12) and remain the linear term with respect to $\varphi_{-,j}$ in $l_{j,j+1}$:

$$\frac{\partial l^2_{j-1,j}}{\partial \varphi_j} \approx -2al \sin \varphi_{+,j-1}, \quad l_{j,j-1} \approx a - 2\varphi_{-,j-1} l \sin \varphi_{+,j-1},$$

$$\frac{\partial l^2_{j,j+1}}{\partial \varphi_j} \approx 2al \sin \varphi_{+,j}, \quad l_{j,j+1} \approx a - 2\varphi_{-,j} l \sin \varphi_{+,j}.$$

From here and (10) we obtain

$$\frac{\partial l_{j,j+1}}{\partial \varphi_j} = l \sin \varphi_{+,j}, \frac{\partial l_{j-1,j}}{\partial \varphi_j} = -l \sin \varphi_{+,j-1}, \tag{15}$$

including the leading order with respect to $|\varphi_{-,k}| \ll 1$. Substitution of the expression (15) into (9) results in

$$\frac{\partial U(l_{j-1,j})}{\partial \varphi_j} + \frac{\partial U(l_{j,j+1})}{\partial \varphi_j} = l\frac{\partial U(l_{j-1,j})}{\partial l_{j-1,j}} \sin \varphi_{+,j-1} - l\frac{\partial U(l_{j,j+1})}{\partial l_{j,j+1}} \sin \varphi_{+,j}. \tag{16}$$

Combination of (16) and (8) allows us to obtain the corresponding equations of a discrete chain:

$$ml^2\ddot{\varphi}_0 - k\left(\frac{l}{a}\right)^2 (\varphi_1 - \varphi_0) \sin^2 \varphi_{+,0} + mgl \cos \varphi_0 = 0,$$

$$ml^2\ddot{\varphi}_j - k\left(\frac{l}{a}\right)^2 [(\varphi_{j+1} - \varphi_j) \sin^2 \varphi_{+,j} - (\varphi_j - \varphi_{j-1}) \sin^2 \varphi_{+,j-1}] + mgl \cos \varphi_j = 0, \tag{17}$$

$$ml^2\ddot{\varphi}_N + k\left(\frac{l}{a}\right)^2 (\varphi_N - \varphi_{N-1}) \sin^2 \varphi_{+,N-1} + mgl \cos \varphi_N = 0, \quad j = 1, \dots, N-1.$$

Introducing the pendulum time $t_0 = \sqrt{l/g}$ of the chain, we can go to $t \to t\sqrt{l/g}$ and write the motion equation (17) in the following form

$$\ddot{\varphi}_0 - \frac{1}{p^2}(\varphi_1 - \varphi_0) \sin^2 \varphi_{+,0} + \cos \varphi_0 = 0, \quad p^2 = \frac{mgl}{k\left(\frac{l}{a}\right)^2},$$

$$\ddot{\varphi}_j - \frac{1}{p^2}[(\varphi_{j+1} - \varphi_j) \sin^2 \varphi_{+,j} - (\varphi_j - \varphi_{j-1}) \sin^2 \varphi_{+,j-1}] + \cos \varphi_j = 0, \tag{18}$$

$$\ddot{\varphi}_N + \frac{1}{p^2}(\varphi_N - \varphi_{N-1}) \sin^2 \varphi_{+,N-1} + \cos \varphi_N = 0, \quad j = 1, \dots, N-1.$$

4 Analysis of the two pendulum model

The system (18) for the two pendulums can be written as

$$\ddot{\varphi}_0 - \frac{1}{p^2}(\varphi_1 - \varphi_0)\sin^2\varphi_{+,0} + \cos\varphi_0 = 0, \quad \ddot{\varphi}_1 + \frac{1}{p^2}(\varphi_1 - \varphi_0)\sin^2\varphi_{+,0} + \cos\varphi_1 = 0. \quad (19)$$

We use functions $\varphi_{+,0}$, $\varphi_{-,0}$ (12) in (19):

$$\ddot{\varphi}_{+,0} + \cos\varphi_{+,0}\cos\varphi_{-,0} = 0, \quad \ddot{\varphi}_{-,0} + \frac{2\varphi_{-,0}}{p^2}\sin^2\varphi_{+,0} - \sin\varphi_{+,0}\sin\varphi_{-,0} = 0. \quad (20)$$

Since there is the condition (14) we can replace $\cos\varphi_{-,0} \to 1$ with accuracy of $(\varphi_{-,0})^2\cos\varphi_{+,0}$ and $\sin\varphi_{-,0} \to \varphi_{-,0}$ with accuracy of $(\varphi_{-,0})^3\sin\varphi_{+,0}$ in (20). The system (20) is reduced to

$$\ddot{\varphi}_{+,0} + \cos\varphi_{+,0} = 0, \quad \ddot{\varphi}_{-,0} + \left[\frac{2}{p^2}\sin^2\varphi_{+,0} - \sin\varphi_{+,0}\right]\varphi_{-,0} = 0.$$

The first equation coincides with the equation of the nonlinear pendulum. It has a stationary solution $\varphi_{+,0} = \pi/2$. In this case $\varphi_{-,0}$ satisfies the equation

$$\ddot{\varphi}_{-,0} + \left[\frac{2}{p^2} - 1\right]\varphi_{-,0} = 0.$$

Behavior of $\varphi_{-,0}$ over time depends on the value of the parameter p . It is clearly that the condition

$$p_*^2 = \frac{mgl_*}{k_*(l_*/a_*)^2} = 2 \quad (21)$$

determines the critical value $p_* = \sqrt{2}$ separating different regions of behavior. If there is the condition $p < \sqrt{2}$ then the solution $\varphi_{-,0}$ is oscillatory one. In the case of the enequality $p > \sqrt{2}$ function $\varphi_{-,0}$ contains exponentially growing contributions.

The system (18) was investigated numerically using an implicit Runge-Kutta method with the help of the package "Mathematica 9.0". We supposed that the initial speed was equal to zero. It is shown that decrease of the parameter p^2 results in increase of the oscillation frequency. In particular, the corresponding elliptical trajectories for $p^2 = 2$ and $p^2 = 1.5$ are shown in (Figure. 2). They indicates the

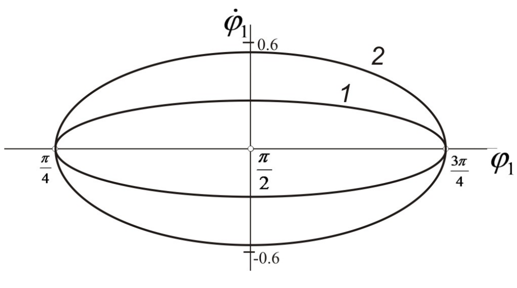

Figure 2: Phase portraits for $N = 2$ at $p^2 = 2$ (curve 1) and $p^2 = 1.5$ (curve 2).

existence of the stability regions in a vicinity of $(0, \pi/2)$. The phase portrait of the second pendulum is identical to the first one on condition that $\dot{\varphi}_1 = -\dot{\varphi}_2$.

Numerical investigation of the system (18) at $p^2 > 2$ shows that the elliptical trajectories disappear and an unstable trajectories appear. This confirms the analytical formula (21) for the critical value of the parameter p_*.

5 Investigation of stability for N pendulums

Model of two pendulums showed that the quantity p^2 is a control parameter of the system. In particular, the value of $p^2 = 2$ is the highest value of the parameter p for which the stable equilibrium is observed for $N = 2$. This allows us to formulate a heuristic idea of separating stable trajectories for $N > 2$.

We begin to calculate the trajectories of the system (18) starting from $p^2 = 2$ for different initial angles at zero initial velocity. Reducing p^2 we control the appearance of elliptic trajectories in the phase plane by means changing the initial angle and analyzing the phase trajectories of each pendulum. If there is an elliptic point on the plane than the position of a pendulam is a stable one. This procedure allows us to calculate the critical value p_* of the system. The software package Mathematica 9.0 was used to construct the solution for $N = 4;\ 6;\ 8;\ 10$ pendulums.

It is shown that the critical value p_* decreases monotonically with respect to N. Analytical investigation of stability of the system (18) was carried out on the information about zeroes of the matrix determinant in a small vicinity of the equilibrium. The asymptotic formula is obtained in case of small p_* and written in the form

$$\frac{1}{p_*^2} = \frac{2\cos\frac{\pi}{N+1}\cos^2\frac{\pi}{2(N+1)}}{\sin\frac{\pi}{2(N+1)}} - 1. \qquad (22)$$

But we expand (22) for the finite p_* as well and the calculated values are presented in the Table 1.

Table 1: Numerical and asymptotic values of with respect to N.

N	Asymptotic values p_*^2	Numerical values p_*^2
2	2.0000	2.000
4	0.2676	0.560
6	0.1493	0.220
8	0.1053	0.120
10	0.0819	0.085

6 Discussion

We constructed the 1D model of the inverted interacting pendulums. It was found that in the case of two inverted connected pendulums the upper vertical position of the system has a critical behavior with respect to the interaction parameter p. The relative angle between two pendulums contains wavy terms when $p < \sqrt{2}$. If the parameter $p > \sqrt{2}$ the solution behaves monotonically.

For a finite number N of inverted pendulums critical behavior of the parameter p is defined by the value of N. The asymptotic formula for the critical value of p with respect to N is proposed in the work. It is shown that the critical value leads to its decrease. Hence the kinematic characteristic N influences on physics behaviour effectively.

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