

# A mixed time-frequency domain method to describe the dynamic behaviour of a discrete medium bounded by a linear continuum

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## Abstract

To minimize the calculation time required by numerical models that describe dynamic interactions involving nonlinear behaviour, it is useful to divide the model into two separate domains. One domain close to the interaction point, which consists of a sophisticated model capable of describing nonlinear phenomena, and another domain at a distance from the interaction point where only linear behaviour remains. The key issue in such numerical models is the coupling of the two domains. The presence of said nonlinear phenomena implies the necessity to work in the time domain rather than in the frequency domain. Nevertheless, frequency domain approaches are preferred as they allow for much faster calculations than time domain approaches. So-called hybrid models exist that attempt to maximize the use of frequency domain approaches for the modelling of nonlinear dynamic behaviour, but these models are often iterative, thereby increasing calculation times.

This contribution presents a non-iterative method to describe the non-smooth dynamic behaviour of a significantly nonlinear system coupled to a linear continuum. Although this method shows the potential to be particularly effective for applications in two- or three-dimensional media, this paper treats the coupling of a one-dimensional linear medium to illustrate this method.

## 1 General Introduction

To determine the loads on a structure due to the dynamic interaction of a structure with its environment, it is vital to correctly model the response of the environment. In this paper, we focus only on the response of the environment and include the structure as an external load applied to an environment. Near the point of interaction between structure and environment, the behaviour of the environment may be governed by nonlinear phenomena and is therefore modelled by a medium capable of describing these nonlinear phenomena, which typically occur in soil-structure and ice-structure interaction. From a numerical point of view, it is desirable to keep the domain of this nonlinear medium as small as possible to minimize the required calculation time. Therefore, the environment is divided into two separate media; a sophisticated nonlinear medium in the region of interaction with the structure, and a linear-elastic medium at such distance from the interaction point that its response is

always linear. As nonlinear phenomena are difficult to capture by continuum models [2], we use a discrete lattice model in the near-field. Applications of lattice models are for example found in the fields of fracture mechanics [5] and micromechanics [4], but also in the field of ice-structure interaction [1]. The linear-elastic far-field response is modelled by a semi-infinite continuum, thus providing the lattice model with a non-reflective boundary.

In some cases, the system of equations of motion for a coupled one-dimensional model can be analytically derived in the time domain. However, as soon as multiple dimensions are considered, a time domain solution can no longer be obtained analytically. Instead, the system of equations of motion has to be derived in the frequency domain and the integration involved with the inverse transformation from the frequency to the time domain must be performed numerically [3]. For a coupled system that is completely linear, one needs to solve the algebraic system of equations of motion in the frequency domain once and then apply numerical integration at every time step to obtain the time domain solution. This requires severely less numerical effort than to solve the system of differential equations in the time domain at every time step. Unfortunately, nonlinear phenomena cannot be described in the frequency domain and thus, coupled systems that incorporate nonlinear phenomena must generally be solved in the time domain. Nevertheless, the nonlinearities in the dynamic response of lattice models may generally be described to be significantly nonlinear, i.e. the dynamic properties change at any given moment in time, but the change in behaviour is instant. In other words, every single time a nonlinearity occurs the system changes instantly, while the system itself still behaves in a linear manner during the following time steps. Thus, for the period between any two nonlinearities, the system of equations of motion can still be solved algebraically in the frequency domain. Assuming that the featured nonlinearities are significantly nonlinear thus allows us to describe the dynamic response of a nonlinear medium to be piecewise linear. The approach described in this paper can be considered as a mixed time-frequency domain (MTFD) method as the nonlinearities are applied between time steps and thus in the time domain, while the response of the coupled model, during each piecewise linear period, is found by solving the corresponding algebraic system of equations of motion in the frequency domain. This approach is somewhat superfluous for one-dimensional models, as the coupling between one-dimensional media may often be solved analytically, but shows the potential to be particularly effective for the modelling of coupled two- or even three-dimensional media. Nevertheless, the method is here explained on the basis of the one-dimensional coupled linear-elastic system depicted in figure 1, because its system of equations of motion for the coupled linear elastic model can be solved in both the frequency domain and directly in the time domain, and therefore the correct application of the MTFD-method can be easily verified for this system.

In the following, we will first discuss the methodology of the mixed time-frequency domain approach. Consecutively, we will derive the system of equations of motion for the one-dimensional discrete-continuous linear-elastic system in the time domain for zero initial conditions, which serves as a benchmark for the MTFD-method. The derivation of the system of equations of motion for the MTFD-approach including nonzero initial conditions is then discussed in section

4. In section 5, an improved statement for the inverse Laplace transform is presented that allows for its numerical application for a system with nonzero initial conditions. Subsequently, the results of the MTFD-method are compared to the benchmark system in section 6.

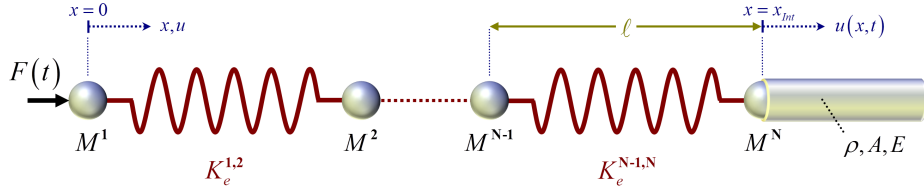


Figure 1: The one-dimensional semi-infinite discrete-continuous linear-elastic system

## 2 Methodology of the mixed time-frequency domain approach

Let us consider an arbitrary system of particles that may respond significantly nonlinear to an applied load. When the system is at rest and has zero initial conditions, it is safe to assume that this system will initially respond in a linear manner. To describe the response of said system for the time period until the occurrence of the first nonlinear event, we describe the system in the frequency domain using the Laplace transform and solve the resulting algebraic system of equations of motion yielding the frequency domain response of the system. We then obtain the time domain response for the system by straightforwardly applying the inverse Laplace transform at every time step. This method remains valid for as long as the system behaves linearly.

Now, suppose that at a given time  $t = t_0$ , a significant nonlinear event occurs. Due to the significant nonlinear event, the behaviour of the system changes, but the change is instant and the response of the system after this significant nonlinear event, i.e. for  $t > t_0$ , is once again linear. That is, for the time period until the next significant nonlinear event. As the response of the system for the time period between any two nonlinear events is always linear, we can again obtain this response by describing the system in the Laplace domain and solving the corresponding algebraic system of equations, but now for its new properties and with nonzero initial conditions. As the nonzero initial conditions represent the response of the system prior to  $t = t_0$ , we may consider the system for the time period between the first and second nonlinear event, to start anew at  $t = t_0$ . To properly consider the new situation for the system that starts at  $t = t_0$  in both the time and the Laplace domain, we reset the time domain and describe the new situation for the system to start at  $t = 0$ , but with nonzero initial conditions. The time domain response after the significant nonlinear event is then once again obtained by solving the corresponding algebraic system in the frequency domain and applying the inverse Laplace transform. This new time domain is valid until the next nonlinear event. Each time the time domain is reset, the response of the system prior to the occurrence of the nonlinear event is included in the new time domain through the nonzero initial displacement at  $t = 0$ .

Note here that, since the initial conditions are nonzero and different for each piecewise linear time period, we here apply the Laplace transform rather than the Fourier transform as the Laplace transform takes the initial conditions into account in the frequency domain, while the Fourier transform neglects the initial conditions in the frequency domain. The procedure described here has been visualized in figure 2.

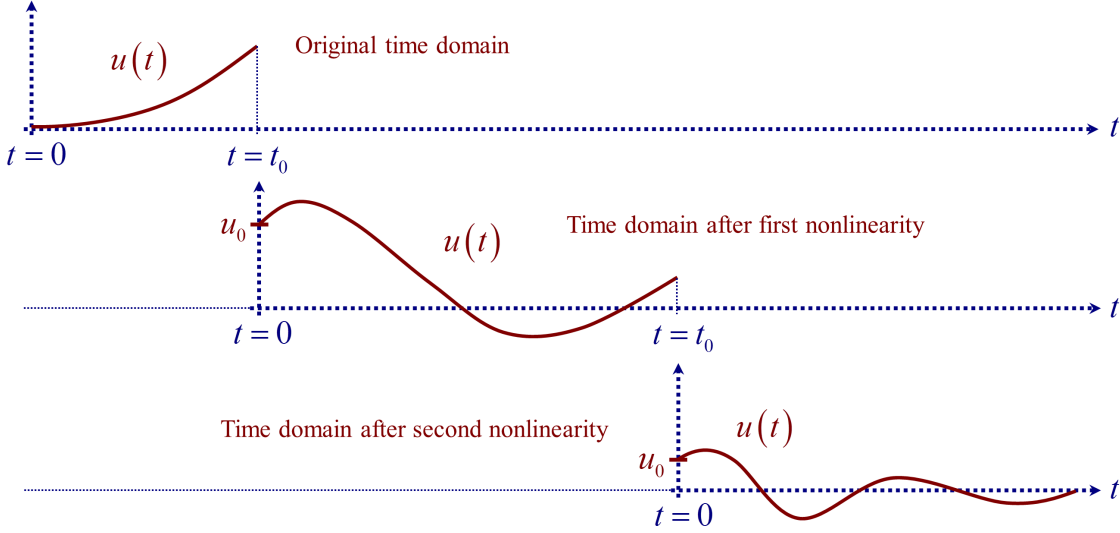


Figure 2: Resetting the time domain every time a nonlinear event occurs

### 3 Governing equations for the discrete-continuous system in the time domain

The one-dimensional semi-infinite discrete-continuous linear-elastic system, previously depicted in figure 1, is comprised of a one-dimensional linear-elastic discrete lattice composed of  $\mathbf{N}$  particles in series, and a semi-infinite linear-elastic rod. Each particle  $\mathbf{n}$  has a dimensionless mass  $M^n$  and the distance between any two adjacent nodes is  $\ell$ . Each two adjacent particles  $\mathbf{n}$  and  $\mathbf{n}+1$  are kinematically related by a spring with a dimensionless stiffness  $K_e^{n,n+1}$ . The linear-elastic rod has a density  $\rho$ , cross-section area  $A$  and Young's modulus  $E$ . The discrete lattice and the semi-infinite linear-elastic rod are connected at particle  $\mathbf{N}$  with coordinate  $x = x_{\text{int}}$ . The coupling between the one-dimensional Hooke system and the linear-elastic rod is described by the equation of motion of particle  $\mathbf{N}$ .

The dimensionless equations of motion for particles  $\mathbf{n}=1\dots\mathbf{N}-1$  respectively read:

$$M^1 \ddot{u}^1 - K_e^{1,2} e^{1,2} = F(t) \quad (1)$$

$$M^n \ddot{u}^n + K_e^{n-1,n} e^{n-1,n} - K_e^{n,n+1} e^{n,n+1} = 0 \quad (2)$$

Here,  $e^{n,n+1}$  denotes the elongation of the kinematic element between particles  $\mathbf{n}$  and  $\mathbf{n}+1$ . Furthermore, dimensionless time and space are respectively introduced

as  $\mathbf{t} \rightarrow \mathbf{t}\omega_0$  and  $\mathbf{x} \rightarrow \mathbf{x}\omega_0/\mathbf{c}$ , where  $\omega_0$  and  $\mathbf{c}$  are respectively the natural frequency of and the wave speed in the system.

To assure homogeneity between lattice and rod, their material properties are matched. The relation between the material properties of the lattice and the rod is established by matching the equation of motion for the rod with the homogeneous equation of motion for an arbitrary particle inside the lattice in the long-wave limit. Applying the Taylor expansion to the equation of motion for an arbitrary particle inside the lattice by replacing the particle displacement  $\mathbf{u}^{\mathbf{n}}$  by the continuum displacement  $\mathbf{u}(\mathbf{x}, \mathbf{t})$  and replacing the displacements of the adjacent particles,  $\mathbf{u}^{\mathbf{n}-1}$  and  $\mathbf{u}^{\mathbf{n}+1}$ , by the second order Taylor polynomial of the corresponding continuum displacement  $\mathbf{u}(\mathbf{x} \pm \ell, \mathbf{t})$  yields the equation of motion for an arbitrary particle inside the lattice in the long-wave limit [7]. Comparing the resulting expression with the equation of motion for the rod shows that the mass of a particle inside the homogeneous system is found as  $M = \rho A \ell$  and the spring stiffness of the springs inside the homogeneous system is found as  $K_e = EA/\ell$ . Therefore, the dimensionless mass of a particle  $\mathbf{n}$  and the dimensionless stiffness of the spring between particles  $\mathbf{n}$  and  $\mathbf{n}+1$  in equations (1) and (2) are generally introduced as  $M^{\mathbf{n}} \rightarrow M^{\mathbf{n}}/M$  and  $K_e^{\mathbf{n},\mathbf{n}+1} \rightarrow K_e^{\mathbf{n},\mathbf{n}+1}/K_e$ . For a homogeneous system, we thus find that  $M^{\mathbf{n}} = K_e^{\mathbf{n},\mathbf{n}+1} = 1$ .

The equation of motion of particle  $\mathbf{N}$ , also denoted as the interface equation, is obtained by combining the one-dimensional wave equation for the semi-infinite rod and the interface conditions that follow from respectively the equilibrium of forces and the displacement relation at the lattice-continuum interface. In the time domain, the dimensionless coupling statement reads:

$$\ddot{\mathbf{u}}(\mathbf{x}, \mathbf{t}) - \mathbf{u}''(\mathbf{x}, \mathbf{t}) = 0 \quad (3)$$

$$\mathbf{u}'(\mathbf{x}_{\text{Int}}, \mathbf{t}) = M^{\mathbf{N}} \ddot{\mathbf{u}}^{\mathbf{N}} + K_e^{\mathbf{N}-1, \mathbf{N}} \mathbf{e}^{\mathbf{N}-1, \mathbf{N}} \quad (4)$$

$$\mathbf{u}(\mathbf{x}_{\text{Int}}, \mathbf{t}) = \mathbf{u}^{\mathbf{N}} \quad (5)$$

Solving the one-dimensional wave equation (3) in the Laplace domain accounting for the appropriate behaviour of the linear-elastic rod for  $\mathbf{x} \rightarrow \infty$ , and inserting the resulting Laplace domain displacement along the semi-infinite rod into the interface conditions (4), as well as taking into account (5), yields the interface equation in the Laplace domain. Subsequently applying the inverse Laplace transform to the resulting Laplace domain expression then yields the interface equation, i.e. the equation of motion for particle  $\mathbf{N}$ , in the time domain as:

$$M^{\mathbf{N}} \ddot{\mathbf{u}}^{\mathbf{N}} + K_e^{\mathbf{N}-1, \mathbf{N}} \mathbf{e}^{\mathbf{N}-1, \mathbf{N}} + \dot{\mathbf{u}}^{\mathbf{N}} = 0 \quad (6)$$

By choosing the point of coupling between lattice and rod at a particle and keeping the distance between the particles equal to  $\ell$ , it follows from the systems' geometry that particle  $\mathbf{N}$  only represents half the length  $\ell$ . To maintain a homogeneous distribution of mass and spring stiffness along the coupled system, it then follows

that  $M^N = \frac{1}{2}$  and  $K_e^{N-1,N} = 1$ . Note here that different combinations of mass and stiffness may be chosen at the discrete-continuous interface as testified by Metrikine et al. [8], where it is shown that there will be no reflection at the lattice-rod interface in the long-wave limit as long as the interface mass and stiffness are related as:

$$K_e^{N-1,N} = \frac{2}{1 + 2M^N} \quad (7)$$

Together, equations (1), (2) and (6) give the full system of dimensionless equations of motion for the one-dimensional coupled system in the time domain.

## 4 Laplace domain equations for the coupled 1D-system for nonzero initial conditions

In the previous section, we have derived the equations of motion for the one-dimensional coupled linear-elastic system in the time domain assuming zero initial conditions. For the application of the MTFD-method however, we require the corresponding system of equations to account for nonzero initial conditions in the Laplace domain. Applying the unilateral Laplace transform to equations (3) to (5) with respect to time, which accounts for possible nonzero initial conditions, yields the dimensionless coupling statement in the Laplace domain as:

$$s^2\tilde{u}(x, s) - \tilde{u}''(x, s) = s\mathbf{u}_0(x) + \mathbf{v}_0(x) \quad (8)$$

$$\tilde{u}'(x_{\text{Int}}, s) + M^N(s\mathbf{u}_0^N + \mathbf{v}_0^N) = M^N s^2\tilde{u}^N + K_e^{N-1,N}\tilde{e}^{N-1,N} \quad (9)$$

$$\tilde{u}(x_{\text{Int}}, s) = \tilde{u}^N \quad (10)$$

Here, the tilde denotes a variable in the Laplace domain, while  $s$  is the complex Laplace parameter. Furthermore,  $\mathbf{u}_0(x)$  and  $\mathbf{v}_0(x)$  are respectively the initial displacement and initial velocity along the linear-elastic rod, while  $\mathbf{u}_0^N$  and  $\mathbf{v}_0^N$  denote the initial displacement and initial velocity of the interface particle. Accounting for the appropriate behaviour of the rod for  $x \rightarrow \infty$  and noting that  $\text{Re}(s) > 0$ , the general solution to equations (8) reads:

$$\tilde{u}(x, s) = C_1 e^{-sx} + \tilde{u}_p(x, s) \quad (11)$$

The first right-hand-side term is the solution to the homogeneous equation, where  $C_1$  is derived by considering the boundary condition at  $x_{\text{Int}}$ . Furthermore,  $\tilde{u}_p(x, s)$  denotes the yet unknown particular solution. For zero initial conditions, equation (8) reduces to a homogeneous equation, so that the particular solution is exclusively related to the nonzero initial conditions. Applying the differentiation to space to equation (11) and rearranging yields:

$$\tilde{u}'(\chi, s) = -s\tilde{u}(\chi, s) + s\tilde{u}_p(\chi, s) + \tilde{u}'_p(\chi, s) \quad (12)$$

Substituting equation (12) into equation (9), as well as taking equation (10) into account, yields the interface equation in the Laplace domain as:

$$M^N s^2 \tilde{u}^N + K_e^{N-1, N} \tilde{e}^{N-1, N} + s\tilde{u}^N = M^N (s\mathbf{u}_0^N + \mathbf{v}_0^N) + \mathbf{f}_0(s) \quad (13)$$

$$\text{where : } \mathbf{f}_0(s) = s\tilde{u}_p(\chi_{\text{Int}}, s) + \tilde{u}'_p(\chi_{\text{Int}}, s) \quad (14)$$

Note here that all terms related to the nonzero initial conditions are collected at the right-hand-side of equation (13).

The particular solution to equation (8) is found using a Green's function approach. To this purpose, we replace the right-hand-side of equation (8) by the Dirac delta function. Applying the Fourier transform with respect to space to the resulting equation of motion then yields the corresponding Greens displacement in the frequency-wavenumber domain. Applying the inverse Fourier transform using contour integration by means of the residue theorem yields the Green's displacement of the linear-elastic rod in the Laplace domain as  $\tilde{g}_u(\chi, s) = \frac{1}{2s} e^{-s|\chi|}$ . Consequently, we find the particular solution at  $\chi = \chi_{\text{Int}}$  as:

$$\tilde{u}_p(\chi_{\text{Int}}, s) = \frac{1}{2s} \int_0^\infty e^{-s|\chi_{\text{Int}} - \xi|} (s\mathbf{u}_0(\xi) + \mathbf{v}_0(\xi)) d\xi \quad (15)$$

Using Leibniz' rule for differentiation under the integral sign to obtain the spatial derivative of the particular solution and subsequently substituting equation (15) and its spatial derivative into equation (14) then yields the expression  $\mathbf{f}_0(s)$  as:

$$\mathbf{f}_0(s) = \int_{\chi_{\text{Int}}}^\infty e^{-s(\xi - \chi_{\text{Int}})} (s\mathbf{u}_0(\xi) + \mathbf{v}_0(\xi)) d\xi \quad (16)$$

Equation (16) denotes the contribution of the rod's initial conditions to the interface equation. If we only allow loads to be applied inside the lattice, it is evident that at any time moment the response of the rod is due to the input at its interface with the lattice, and thus, we can express the displacement and velocity along the rod in terms of the response of the interface particle  $\mathbf{N}$ . The relation between the time domain response of the rod and that of the interface particle can be derived by considering the boundary value problem for the rod assuming zero initial conditions. In the Laplace domain, the relation between the displacement along the rod and that of the interface particle is found as  $\tilde{u}(\chi, s) = \tilde{u}^N e^{-s(\chi - \chi_{\text{Int}})}$ . Applying the inverse Laplace transform then yields the corresponding relation in the time domain as  $\mathbf{u}(\chi, t) = \mathbf{u}^N(t - (\chi - \chi_{\text{Int}}))H(t - (\chi - \chi_{\text{Int}}))$ . Deriving the velocity relation

accordingly and substituting both displacement and velocity relations into equation (16), as well as replacing the variable of integration  $\xi$  by the variable of integration  $\tau = \xi - x_{Int}$  then yields the expression  $f_0(s)$  as:

$$f_0(s) = \int_0^{t_0} e^{-s\tau} (su_0^N(t_0 - \tau) + v_0^N(t_0 - \tau)) d\tau \quad (17)$$

The remaining expression for  $f_0(s)$  can now be interpreted as a convolution integral over the time domain prior to the time moment  $t_0$  at which the system was reinitiated and new nonzero initial conditions were specified. Consequently, equations (13) and (17) together describe the Laplace domain interface equation exclusively in terms of the discrete lattice. The Laplace domain system of equations of motion for the one-dimensional coupled linear-elastic system that accounts for nonzero initial conditions is completed by including the Laplace domain equations of motion for the particles  $n=1\dots N-1$  that read:

$$M^1 s^2 \tilde{u}^1 - K_e^{1,2} \tilde{e}^{1,2} = \tilde{F}(s) + M^1 (su_0^1 + v_0^1) \quad (18)$$

$$M^n s^2 \tilde{u}^n + K_e^{n-1,n} \tilde{e}^{n-1,n} - K_e^{n,n+1} \tilde{e}^{n,n+1} = M^n (su_0^n + v_0^n) \quad (19)$$

Here,  $\tilde{F}(s)$  is the Laplace domain expression for the external force applied at particle 1. Note here that, once the time domain is reinitiated at a time  $t_0$  this must also be accounted for in the Laplace domain expression of the applied load.

## 5 Transformation to the time domain accounting for nonzero initial conditions

Solving the algebraic system of equations of motion given by equations (13) and (17) to (19) yields the Laplace domain displacements of all particles in the one-dimensional lattice. The time domain displacements of all particles in the lattice are then obtained by applying the inverse Laplace transform to the corresponding Laplace domain displacements. The inverse Laplace transform of these Laplace domain displacements can not be derived analytically and must therefore be obtained numerically. As a consequence, the semi-infinite domain of integration of the inverse Laplace transform has to be truncated. This truncation leads to several problems, in particular for nonzero initial conditions. If we describe the inverse Laplace transform in terms of frequency, rather than in terms of the complex Laplace operator  $s$ , we may find the time domain displacement of a particle  $n$  in the lattice as:

$$u^n = \frac{e^{\sigma t}}{\pi} \int_0^{\omega_{tr}} \text{Re} \{ \tilde{u}^n e^{i\omega t} \} d\omega \quad (20)$$

Here,  $\omega_{tr}$  is the truncation frequency. The corresponding velocities and accelerations



may then be obtained by taking the first and second time derivatives of equation (20), which is equivalent to respectively applying the inverse Laplace transform to  $\tilde{v}^n = s\mathbf{u}^n$  and  $\tilde{a}^n = s^2\mathbf{u}^n$ . Now, we may only apply the inverse Laplace transform numerically, if their integrands are sufficiently convergent within the truncated domain of integration. Solving equation (19) for  $\tilde{u}^n$  shows that, taking nonzero initial conditions into account, the Laplace domain displacement has a convergence  $\omega^{-1}$  for  $\omega \rightarrow \infty$ . Consequently, it follows that the Laplace domain velocity and acceleration, i.e.  $\tilde{v}^n$  and  $\tilde{a}^n$ , are both non-convergent. As shown by figure 3, the lack of convergence is due to the nonzero initial conditions; Figure 3a shows the convergent Laplace domain displacement, velocity and acceleration of a particle  $\mathbf{n}$  for zero initial conditions, while figure 3b gives the response for nonzero initial conditions.

As an alternative, and in correspondence with the Laplace transform, we may respectively describe the velocity and acceleration in the Laplace domain as  $\tilde{v}^n = s\mathbf{u}^n - \mathbf{u}_0^n$  and  $\tilde{a}^n = s^2\mathbf{u}^n - s\mathbf{u}_0^n - \mathbf{v}_0^n$ . Substituting  $\tilde{u}^n$ , found from solving equation (19), into these expressions shows that the Laplace domain velocity and acceleration now both have the same convergence for  $\omega \rightarrow \infty$  as the Laplace domain displacement. Although the given Laplace domain expressions for displacement, velocity and acceleration are now convergent, these expressions still do not lead to a proper numerical application of the inverse Laplace transform. To illustrate this, let us consider the inverse Laplace transform and note that by applying the causality principle, the integrand of the inverse Laplace transform may alternatively be described as either one of  $\text{Re}\{\tilde{u}^n e^{i\omega t}\} = 2\text{Re}\{\tilde{u}^n\} \cos(\omega t) = -2\text{Im}\{\tilde{u}^n\} \sin(\omega t)$ . These integrands may however only be interchanged if, and only if, the limit  $\omega \rightarrow \infty$  is taken into account. For either integrand, numerically applying the integration over its truncated domain yields an incorrect behaviour of the time domain displacement near the nonzero initial conditions as depicted in figure 4. Due to the finite domain of integration, the integrand with the term  $\sin(\omega t)$  must always give a zero displacement at time  $t = 0$  of the time domain reinitiated at  $t = t_0$ . Additionally, taking the time derivative of the integrand with the term  $\cos(\omega t)$  shows that  $\dot{u}_0 = \sigma u_0$ , where  $\sigma$  is the small positive real value of the complex Laplace parameter. Consequently,

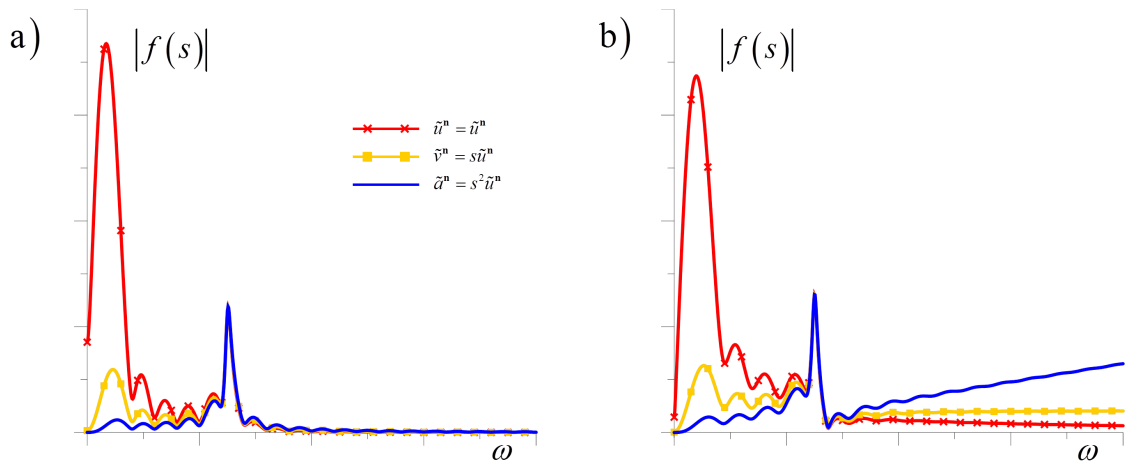


Figure 3: Absolute Laplace domain displacement, velocity and acceleration for: a) zero initial conditions; b) nonzero initial conditions.

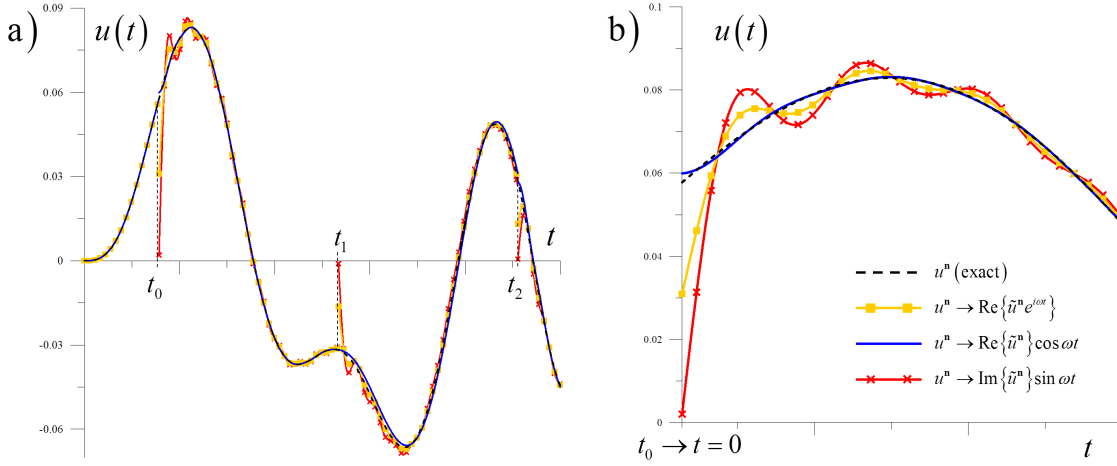


Figure 4: Displacement for a truncated domain of the inverse Laplace transform: a) in the original time domain; b) close-up in the time domain reinitiated at  $t_0$ .

the corresponding time domain displacement has a slope that is significantly smaller than the slope of the exact solution.

To improve the behaviour of the inverse Laplace transform with a truncated domain of integration for a system that, in the time domain, has nonzero initial conditions, we extract the initial conditions from the corresponding Laplace domain expression and separately include the contribution of the initial conditions in the time domain. As a consequence, the remaining Laplace domain expression, henceforth denoted as the improved Laplace domain expression, can be considered as a Laplace domain expression for a system with zero initial conditions, thereby improving its behaviour in the time domain. To obtain expressions for the displacement, velocity and acceleration that appreciate the rules of differentiation, we do not only extract the initial displacement from the Laplace domain displacement, but we also extract the contributions of the initial velocity and initial acceleration. Consequently, the time domain expressions for the displacement, velocity and acceleration of an arbitrary particle  $\mathbf{n}$  in the lattice now become:

$$\mathbf{u}^{\mathbf{n}} = \frac{e^{\sigma t}}{\pi} \int_0^{\omega_{tr}} \operatorname{Re} \{ \tilde{\mathbf{u}}_{\text{imp}}^{\mathbf{n}} e^{i\omega t} \} d\omega + \mathbf{u}_0^{\mathbf{n}} + \mathbf{v}_0^{\mathbf{n}} t + \mathbf{a}_0^{\mathbf{n}} \frac{t^2}{2} \quad (21)$$

$$\dot{\mathbf{u}}^{\mathbf{n}} = \frac{e^{\sigma t}}{\pi} \int_0^{\omega_{tr}} \operatorname{Re} \{ s \tilde{\mathbf{u}}_{\text{imp}}^{\mathbf{n}} e^{i\omega t} \} d\omega + \mathbf{v}_0^{\mathbf{n}} + \mathbf{a}_0^{\mathbf{n}} t \quad (22)$$

$$\ddot{\mathbf{u}}^{\mathbf{n}} = \frac{e^{\sigma t}}{\pi} \int_0^{\omega_{tr}} \operatorname{Re} \{ s^2 \tilde{\mathbf{u}}_{\text{imp}}^{\mathbf{n}} e^{i\omega t} \} d\omega + \mathbf{a}_0^{\mathbf{n}} \quad (23)$$

Here, the improved Laplace domain displacement is found as:

$$\tilde{\mathbf{u}}_{\text{imp}}^{\mathbf{n}} = \tilde{\mathbf{u}}^{\mathbf{n}} - \frac{\mathbf{u}_0^{\mathbf{n}}}{s} - \frac{\mathbf{v}_0^{\mathbf{n}}}{s^2} - \frac{\mathbf{a}_0^{\mathbf{n}}}{s^3} \quad (24)$$

Substituting  $\tilde{\mathbf{u}}^{\mathbf{n}}$ , found from solving equation (19), into equation (24), shows that the improved Laplace domain displacement now has a convergence  $\omega^{-3}$  for  $\omega \rightarrow \infty$ .

Figure 5 shows the displacement of particle  $\mathbf{n}$  obtained using the improved statement for the inverse Laplace transform given by equation (21).

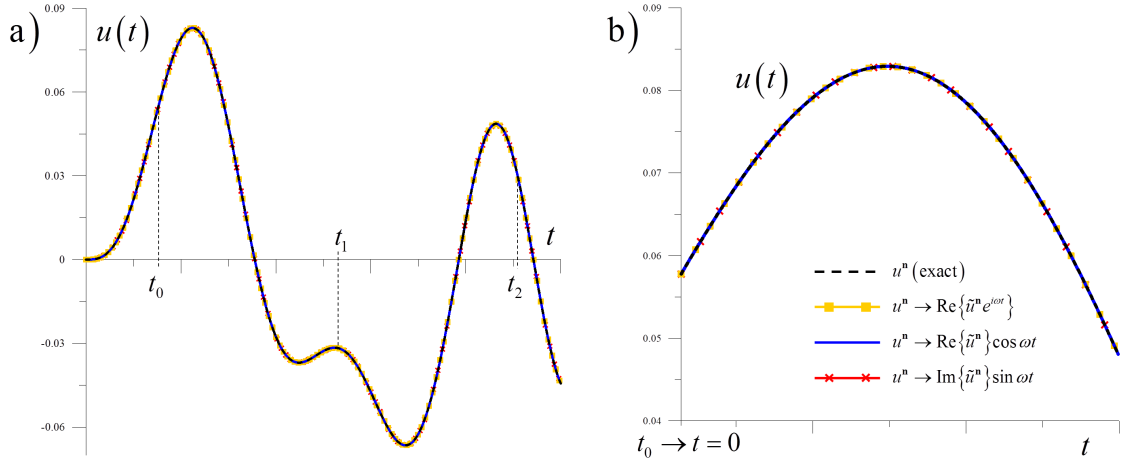


Figure 5: Displacement obtained using the improved statement: a) in the original time domain; b) close-up in the time domain reinitiated at  $t_0$ .

## 6 The mixed time-frequency domain method applied to the coupled 1D-system

Figure 6 shows the longitudinal displacements along the one-dimensional discrete-continuous linear-elastic system due to an applied single-sinus pulse load at 6 consecutive time moments. The discrete lattice consists of 80 particles at an inter-particle distance of  $\ell = 0,2\text{m}$  and the linear-elastic rod has a density  $\rho = 1960\text{kg/m}^3$ , a cross-section area  $A = 1\text{m}^2$  and a Young's modulus  $E = 19,6\text{MPa}$ .

The continuous line shows the displacements that result from applying the MTFD-method, while the dashed line shows the resulting displacements from applying a Runge-Kutta scheme to solve the system directly in the time domain, here

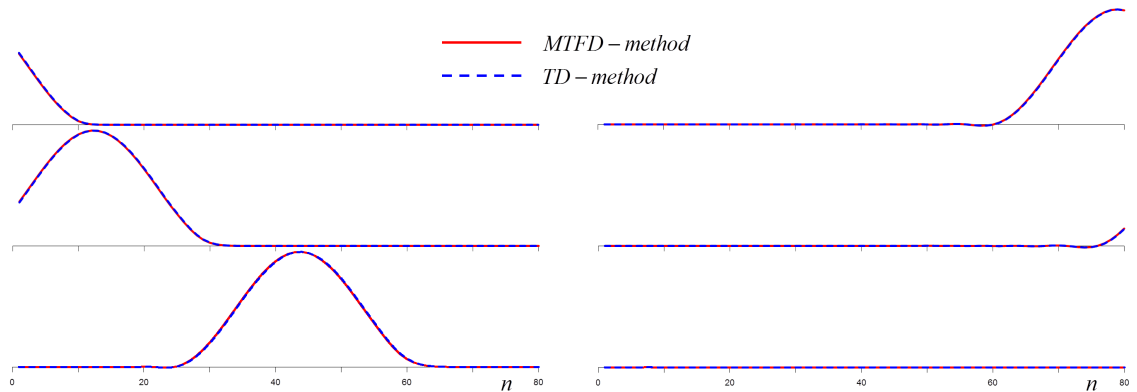


Figure 6: Longitudinal displacement along the one-dimensional discrete-continuous linear-elastic system at successive time moments due to a single-sinus pulse.

denoted as the TD-method. The simulation consists of a 1000 time steps and for the application of the MTFD-method, the system was reinitiated after every 100 time steps. Figure 6 verifies that the MTFD-method yields the proper response of the system. Although the MTFD-method is here applied to a rather simplistic linear-elastic coupled system, and thereby hardly yields any computational profit, its real gain is found in the application of the MTFD-method to a coupled system consisting of a nonlinear lattice bounded by a viscoelastic environment, especially, for multi-dimensional systems.

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