Steady streaming in a vibrating container at high Reynolds numbers

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Abstract

We consider the fluid flows in a variable vibrating container while assuming that the width of the Stokes layer is of the same order as the magnitude of the vibration. With no additional assumptions we build the asymptotic expansions of the general vibrational flow. In particular, we get an explicit form of the general equations and boundary conditions for the mean flow ('steady streaming'). We apply these results to exploring the steady 3D streaming in a round pipe due to the transverse deformation of the pipe wall spreading in the form of spiral wave.

1 The steady streaming and the Stokes drift

Let us consider a viscous incompressible and homogeneous flow confined within a container which moves itself and/or changes the shape of itself periodically but with no displacement or deformation on average. The characteristic scales of such motion are: the averaged size of the container, the magnitude of the displacement or deformation, and the correspondent frequency. We denote them as L, A and Ω, respectively. We take L as the unit of length , Ω^{-1} s the unit of time, $U = A\Omega$ as the unit of velocity, and $\rho AL\Omega^2$ as the unit of pressure (where ρ stands for the fluid density). The Navier-Stokes eqs. take the form

$$\mathbf{v}_\tau + \delta(\mathbf{v}, \nabla)\mathbf{v} = -\nabla p + \epsilon^2 \Delta \mathbf{v},\ \nabla \cdot \mathbf{v} = 0 \text{ in } D(\tau); \quad \mathbf{v}(\bar{x} + \delta\tilde{Y}, \tau) = \tilde{Y}_\tau, \bar{x} \in \bar{S}. \quad (1)$$

Here $\tau = \Omega t$, $\delta = A/L$, $\epsilon^2 = \nu/(\Omega L^2)$, where ν is the viscosity of the fluid. Further, $D(\tau)$ stands for the current liquid domain, $\tilde{Y} = \tilde{Y}(\bar{x}, \tau, \delta)$ describes the current displacement of the wall, $\bar{S}$ is the reference ('averaged') position of the wall and $\bar{D}$ denote the reference domain which is confined within $\bar{S}$. The Reynolds number is $\mathrm{Re} = (LU)/\nu = LA\Omega/\nu = \delta/\epsilon^2$.

While letting $\epsilon \to +0$ and $\delta \to +0$ in (1) we formally arrive at the vibrational limit which is linear Euler system $\mathbf{v}_{0\tau} = -\nabla p_0$, $\operatorname{div} \mathbf{v}_0 = 0$ endowed with an extra boundary condition, namely $\mathbf{v}_0 = \tilde{Y}_{0\tau}$ on $\bar{S}$. Thus, the leading approximation gives an irrotational ($\operatorname{curl} \mathbf{v}_0 = 0$) flow in the bulk of the fluid while the effect of the viscosity as well as the vorticity concentrate themselves within the Stokes layer alongside $\bar{S}$ the thickness of which is of order ϵ. However the senior approximations discover a *global* steady vortex flow which is widely known as *'steady streaming'*.

In spite of its relative weakness the steady streaming is capable of making effect on the long-term mixing processes. For clarity, let us consider the dimensionless equation of the fluid particles which is written in the form $dx/dT = \delta^{-1}\mathbf{v}(x,\tau,\delta)$, where $T = \delta^2\tau$ is the 'slow' time. Assume $\mathbf{v} = \tilde{\mathbf{v}}(x,\tau) + \delta(\bar{\mathbf{v}}(x) + \tilde{\mathbf{v}}_1(x,\tau)) + O(\delta^2)$, $\delta \to 0$ where 'tilde'-terms vanish on average. Then $x = \bar{x}(T) + \delta(\bar{x}_1(T) + \tilde{x}(\tau,T)) + O(\delta^2)$, $\delta \to 0$ where $d\bar{x}/dT = \bar{\mathbf{v}}(x) + \overline{[\tilde{\mathbf{v}},\tilde{\mathbf{w}}]}/2$ and the square brackets denote the common commutator of vector fields, and $\tilde{\mathbf{v}} = \tilde{\mathbf{w}}_\tau$. Generically, both summands in $d\bar{x}/dT$ possess vorticity[1] and the latter is known as the Stokes correction.

There is a number of different treatments of the steady streaming depending on the assumptions about dimensionless quantity $\mathrm{Re}_s = (\Omega A^2)/\nu = \delta^2/\epsilon^2$ which is widely known as the streaming Reynolds number. In fact, $\sqrt{\mathrm{Re}_s}$ is nothing more than the ratio of amplitude of the displacements of the boundary to the thickness of the Stokes layer. Assume that they are of the same order *i.e.*

$$\delta \to 0,\ \epsilon \to 0, \quad \sqrt{Re_s} = \delta/\epsilon \stackrel{\text{def}}{=} \beta \equiv \text{const} \sim 1, \quad Re = \beta\epsilon^{-1} \to \infty. \tag{2}$$

Considering of such scales goes back to Craik and Leibovich (1976), Duck and Smith (1979), Haddon and Riley, (1979), Gopinath (1993) [2]. Our approach is more formal and general. We develop asymptotic expansion of general problem (1) with the use of the Vishik-Lyusternik method. V. Levenshtam (2000) employed similar approach in the case of a constant domain and vibrating mass force. Our analysis results in the universal description of the steady streaming and Stokes drift with no use of special assumptions about the flow except for those of (2). On such base we easily treat a number of important particular cases.

2 Governing equation

We start with a number of auxiliary matters. Given a vector field $\mathbf{a}$ on $\bar{D}$ consider the orthogonal (in the natural metric of the energy) decomposition $\mathbf{a} = \mathbf{b} + \nabla\chi$ and associated projectors $\Pi : \mathbf{a} \mapsto \nabla\chi \quad \Pi' : \mathbf{a} \mapsto \mathbf{b}$, where $\mathbf{b}$ is divergent-free field being tangential to $\bar{S}$.

Let $\bar{g} = \bar{g}(\cdot)$ be the averaged value of $g = g(\cdot,\tau)$ with respect to τ. We set $\tilde{g} = g - \bar{g}$. Apparently, $\bar{\tilde{g}} = \tilde{\bar{g}} = 0$. In what follows we overline the non-oscillating terms and put tilde over the terms vanishing on average. The integration operator $\partial_\tau^{-1} : g \mapsto f,\ \partial_\tau f = g$ is well-defined in the space of time-periodic $g : \bar{g} = 0$.

Let a be a power expansion. Denote as ${}_m a$ the $m-$th order polynomial obtained by the truncating of a. Further, let $b, c, \ldots$ be polynomials perhaps with variable and vector-valued coefficients. Denote as $\mathrm{op}(b, c, \ldots)$ an algebraic expression over $b, c\ldots$, may be, involving the derivatives of the coefficients.

We seek asymptotic solution $\mathbf{v}, p$ to the system (1) subject to assumptions (2) in the form $(\mathbf{v}, p) = (\mathbf{v}^\iota, p^\iota) + (\mathbf{v}^\flat, p^\flat)$ where ι -terms are to describe the inner flow inside the bulk of the fluid while $\flat$ -terms are to describe the flow near the boundary

[1]The commutator of two irrotational generically has nonzero curl

[2]We omit the detailed references for the sake of compactness. The details can be found in the articles listed at the end of this one. They all are readily accessible via arxiv.org or www.researchgate.net

within the Stokes layer the width of which is of order ϵ. Both $\mathbf{v}^\iota, p^\iota$ and $\mathbf{v}^\flat, p^\flat$ are expanded in ϵ^k, $k = 0, 1, 2, \dots$. The coefficients $(\mathbf{v}^\iota_k, p^\iota_k)$ and $(\mathbf{v}^\flat_k, p^\flat_k)$ of both expansions have to be smooth in $x \in \bar{D}$ and 2π -periodic in τ. In addition, $\flat$ - coefficients depend on $\eta = \rho/\epsilon$, where ρ is a non-tangential coordinate nearby $\bar{S}$. These dependencies have to be such that $(\mathbf{v}^\flat_k, p^\flat_k)(\cdot, \eta) = o(\eta^{-n})$ for every n.

With this in mind we substitute the unknowns in (1) with the above expansions and arrive at the chain of equation for ι - coefficients: $\partial_\tau \mathbf{v}^\iota_k = -\nabla p^\iota_k + \mathbf{f}_k$, $\nabla \cdot \mathbf{v}^\iota_k = 0$ inside $\bar{D}$, $\mathbf{v} \cdot \mathbf{n} = \gamma_k$ on $\bar{S}$, where $\mathbf{f}_k = \mathbf{op}(\,_{k-1}\mathbf{v}^\iota)$, $\gamma_k = \mathrm{op}(\,_{k-1}\mathbf{v}^\flat,\,_{k-1}\mathbf{v}^\iota,\,_k\tilde{Y})$ and $\mathbf{n}$ stands for the *inward* normal filed on $\bar{D}$. There exists a solution provided that $\Pi' \bar{\mathbf{f}}_k = 0$ and no solutions exist otherwise; the solution (if any) is defined up to the mean field $\bar{\mathbf{v}}^\iota_k$. We have to find the mean field while reducing the solvability condition for the next approximation.

Let us come through more details. Let $\mathbf{N}\tilde{\gamma}$ denote the velocity of the irrotational flow the normal component of which on $\bar{S}$ is equal to $\tilde{\gamma}$. To get the iterating started, we set $\mathbf{f}_0 = 0$, $\tilde{\gamma}_0 = \bar{\mathbf{n}} \cdot \tilde{Y}_{0\tau}$, where $\bar{\gamma}_0 = 0$ as the boundary does not vary on average, and $\tilde{Y}_0 = \tilde{Y}(x, \tau, 0)$. Then $\mathbf{v}^\iota_0 = \bar{\mathbf{v}}^\iota_0 + \mathbf{N}\tilde{\gamma}_0$; $\nabla \tilde{p}^\iota_0 = -\partial_\tau \mathbf{N}\tilde{\gamma}_0$. While calculating the first-order terms, we face the solvability condition which coincides with the steady incompressible Euler equation in $\bar{D}$. In sec. 3 we'll see that we have to seek the solution obeying both no-flux and no-slip boundary conditions. Such problem seems to be overdetermined but there always exists the trivial solution $\bar{\mathbf{v}}_0 \equiv 0$. Although another choice can be of sense too we do not explore such possibility here.

Thus the first-order term $\bar{\mathbf{v}}_1$ is the leading one for the steady streaming. Next we pass to the second order terms but the evaluating of them produce no equation for $\bar{\mathbf{v}}_1$ as the correspondent solvability condition turns out to be always satisfied. The solvability condition for the third-order terms takes the following form

$$\Delta \bar{\mathbf{v}}^\iota_1 - \nabla H_1 = \beta \bar{\boldsymbol{\omega}}^\iota_1 \times \mathbf{V};\ \nabla \cdot \bar{\mathbf{v}}^\iota_1 = 0;\ \mathbf{V} = \bar{\mathbf{v}}^\iota_1 + \beta \overline{[\boldsymbol{\xi}_\tau, \boldsymbol{\xi}]}/2,\ \boldsymbol{\xi} = \mathbf{N}(\partial_\tau^{-1}\tilde{\gamma}_0) \quad (3)$$

where $\bar{\boldsymbol{\omega}}^\iota_1 = \nabla \times \bar{\mathbf{v}}^\iota_1$.[3] Thus, $\mathbf{V}$ is the total drift velocity which is responsible for the transporting of the mean vorticity into the bulk of the fluid. The averaged commutator $\beta \overline{[\boldsymbol{\xi}_\tau, \boldsymbol{\xi}]}/2$ is nothing more then the Stokes correction.

3 Boundary condition.

Currently, our target is the deriving of the boundary conditions for the mean flow. To this end, we have to consider asymptotic expansion inside the Stokes layer. To do so we employ coordinates $x \mapsto (\rho, \theta)$ where $\rho(x) = \mathrm{dist}(x, \bar{S})$, θ is the point in $\bar{S}$ which is nearest to x. Such mapping induces the decomposition $\mathbf{h} = \mathbf{h}^{\shortparallel} + \mathbf{h}_n$, where the former summand is the tangential component and the latter one is the normal component of $\mathbf{h}$.[4] We rewrite system (1) relative to the coordinates (ρ, θ) and then separate the normal and tangential projections of it. Next we inflate the Stokes layer with the stretched normal coordinate $\eta = \rho/\epsilon$. We use slightly different expansions for the normal and tangential velocities as well as for the pressure; namely

[3]For the deriving of Eq.(3) we employ the equality $\mathbf{a} \times [\mathbf{b}, \mathbf{c}] + \mathbf{c} \times [\mathbf{a}, \mathbf{b}] + \mathbf{b} \times [\mathbf{c}, \mathbf{a}] = \nabla(\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}))$ which is valid for every three of the divergence free fields.

[4]Now we direct the normal towards the bulk of the fluid.

$(\mathbf{v}_0^\flat)_n = p_0^\flat = p_1^\flat = 0$, $(\mathbf{v}_{k+1}^\flat)_n = u_k^\flat$, $p_{k+2}^\flat = P_k^\flat$, $\mathbf{w}_k^\flat = (\mathbf{v}_k^\flat)^{\shortparallel}$ $k = 0, 1, \ldots$. Then the equations for the boundary layer corrections and boundary conditions take the form

$$\left(\partial_\tau + \beta\tilde{\gamma}_0\partial_\eta + \partial_\eta^2\right)\mathbf{w}_k^\flat{=}\mathbf{F}_k; \quad \partial_\eta u_k^\flat = S_k; \quad \partial_\eta P_k^\flat = R_k; \quad \eta > \beta\tilde{\eta}_0(\theta,\tau); \tag{4}$$

$$\mathbf{w}_k^\flat = \beta^k(\tilde{Y}_{k\tau} - \mathbf{b}_k)^{\shortparallel} - \mathbf{w}_k^\iota; \quad u_k^\iota = \beta^k(\tilde{Y}_{k\tau} - \mathbf{b}_k)_n - u_{k-1}^\flat; \quad \text{if } \eta = \beta\eta_0, \tag{5}$$

$$\tilde{\eta}_0(\theta,\tau) = \partial_\tau^{-1}\tilde{\gamma}_0(\theta,\tau) = (\tilde{Y}_0)_n|_{\rho=0}; \quad (\mathbf{w}_k^\flat, u_k^\flat, P_k^\flat) = o(\eta^{-s}), \ \eta \to \infty \ \forall s > 0; \tag{6}$$

$$\mathbf{b}_k = \mathbf{op}(\,_{k-1}\mathbf{v}^\iota, \,_{k-1}\mathbf{w}^\flat, \,_{k-1}u^\flat, \,_k\tilde{Y}); \quad S_k = \mathrm{op}(\,_k\mathbf{v}^\iota, \,_kp^\iota, \,_{k-1}u^\flat, \,_{k-1}P^\flat, \,_k\mathbf{w}^\flat); \tag{7}$$

$$\mathbf{F}_k = \mathbf{op}(\,_{k-1}u^\flat, \,_{k-1}\mathbf{w}^\flat, \,_{k-1}P^\flat, \,_{k-1}\mathbf{v}^\iota, \,_{k-1}p^\iota); \quad R_k = \mathrm{op}(\,_k\mathbf{v}^\iota, \,_kp^\iota, \,_ku^\flat, \,_{k-1}P^\flat, \,_k\mathbf{w}^\flat). \tag{8}$$

Here all the ι—terms or their derivatives are restricted onto $\bar{S}$ i.e. to $\rho = \eta = 0$. Note also that $\mathbf{F}_k$, R_k and S_k possesses the decay rate (6) provided that the $\flat$—terms they involve possesses the same decay rate.

Substituting η with $s = \eta - \beta\eta_0 > 0$ transforms the first equation of (4) into the canonical heat equation in a semi-plane $\{(s,\tau) : s > 0\}$ and the first boundary condition (5) then moves to the line $s = 0$. Note that the appearing of the advection in the first equation of (4) is the direct consequence of assumption (2).

The oscillating projection of the first equation in (5) gives boundary condition to the first equation in (4) while the oscillating projection of the second gives the boundary condition to the normal velocity of the inner flow. The averaging of the both gives the boundary values for $\bar{\mathbf{v}}_k^\iota$ i.e. for the mean flow. The calculating of them involves $\bar{\mathbf{w}}_k^\flat$ and $\bar{u}_k^\flat$ which one can get via the averaging of the first and second equations of (4) with account of the decay condition. We emphasize that the averaging is always performed relative to the moving frame *i.e.* we *first* write all the things using s-variable first and *next* perform the averaging.

System (4-8) possesses triangular structure which allows us to find $\mathbf{w}_k^\flat$ first, $u_k^\flat$ next and $P_k^\flat$ finally. To get the process started we set $\mathbf{b_0} = 0$, $\mathbf{F}_0 = 0$. Then $\bar{\mathbf{w}}_0^\iota = 0$ and $\bar{\gamma}_0 = 0$ as stated in Sec. 2.

Now we formulate the boundary conditions for $\bar{\mathbf{v}}_1^\iota$. We denote as $\hat{h}_m$, $m \in \mathbb{Z}$ the Fourier coefficients for periodic function h. Also, $\hat{Y}_m$ stands for the Fourier coefficient of $\tilde{Y}_0 = \tilde{Y}(x,\tau,0)$. On $\bar{S}$, we define a vector field $\mathbf{q} = \tilde{Y}_0 - \xi$, $\mathbf{q} \parallel \bar{S}$. Thus,

$$\begin{aligned}
&\beta^{-1}\bar{\mathbf{w}}_1^\iota\big|_{\bar{S}} = \overline{(\nabla^{\shortparallel}\cdot\mathbf{q})\mathbf{q}_\tau}/2 - \sum |m|\left(\nabla^{\shortparallel}|\hat{\mathbf{q}}_m|^2 + 4(\nabla^{\shortparallel}\cdot\hat{\mathbf{q}}_m)\hat{\mathbf{q}}_{-m}\right)/4 - \overline{[\tilde{Y}^{\shortparallel}_{0\tau}, \tilde{Y}^{\shortparallel}_0]}/2 - \\
&-\frac{1}{2}\sum |m|\left(\nabla^{\shortparallel}\cdot(\hat{Y}_m\times\nabla\rho)\right)(\nabla\rho\times\hat{\mathbf{q}}_{-m}) - 2\overline{(\mathrm{curl}\,(\xi\times\nabla\rho)\cdot\nabla\rho - \tilde{\eta}_0\Delta\rho)\mathbf{q}_\tau} - \\
&-\overline{\tilde{\eta}_0\nabla^{\shortparallel}\tilde{\eta}_{0\tau}} - 2\overline{\tilde{\eta}_{0\tau}(\tilde{Y}^{\shortparallel}_0,\nabla)\nabla\rho} - \overline{(\nabla^{\shortparallel}\cdot(\tilde{Y}_0\times\nabla\rho))(\nabla\rho\times\xi_\tau)} - \nabla^{\shortparallel}\overline{(\xi^{\shortparallel}_\tau\cdot\tilde{Y}^{\shortparallel}_0)} - \qquad (9)\\
&-\beta\overline{(\tilde{Y}^{\shortparallel}_0\cdot\nabla\tilde{\eta}_0)\,\mathbf{W}_s|_{s=0}}; \quad \mathbf{W}_\tau = \mathbf{W}_{ss},\ s > 0,\ \mathbf{W}(0,\tau) = \mathbf{q}_\tau,\ \mathbf{W}(\infty,\tau) = 0.\\
&\beta^{-1}\bar{u}_1^\iota\big|_{\bar{S}} = \beta^{-1}\bar{\gamma}_1^\iota = \overline{[\xi,\xi_\tau]}_n/2. \qquad (10)
\end{aligned}$$

Equations (3) and boundary conditions (9-10) gives us the total drift velocity $\mathbf{V}$. Note that $\mathbf{V}$ is always tangential to $\bar{S}$.

4 Examples

1. The tangential and torsional vibrations are natural provided that a subgroup of the motions acts on $\bar{S}$. For such vibrations, the Stokes term is always equal to zero.

For instance, consider the tangential vibrations of a circular pipe. Then $\bar{D} = \{r < 1\}$, and $\rho = 1 - r$ (relative to proper cylindrical coordinates). Every tangential motion of the boundary can be written as $Y_0 = \tilde{\kappa}_0(\tau)\mathbf{e}_\theta + \tilde{\kappa}_1(\tau)\mathbf{e}_z = \mathbf{q}.$ where $\tilde{\kappa}_0(\tau)$ and $\tilde{\kappa}_1(\tau)$ are given scalar periodic functions. Apparently, $\nabla^{\shortparallel}\cdot\mathbf{q} = 0$, $\nabla^{\shortparallel}|\hat{\mathbf{q}}_k|^2 = 0$ and $\nabla^{\shortparallel}\cdot(\hat{\mathbf{q}}_k \times \nabla\rho) = 0$ once again. Thus, *translational-rotational tangential vibrations of the circular pipe produce no steady streaming in the leading approximation.*

Consider now *the torsional oscillations of a ball submerged in unbounded fluid.* Then the averaged flow domain is $\bar{D} = \{r > 1\}$, $r = |x|$, $\bar{S} = \{r = 1\}$, $\rho = r - 1$, $\nabla\rho = \theta = x/r$. Let $\tilde{Y}_0 = \mu(\tau)\mathbf{k} \times \theta$, where $\mathbf{k} \equiv$ const, $|\mathbf{k}| = 1$, μ is scalar 2π-periodic function vanishing on average. Then $\hat{\mathbf{q}}_m = \hat{\mu}_m\mathbf{k} \times \theta$ and no one term on the righthand side of (9) contributes in $\bar{\mathbf{w}}_1^1$ except for $-\frac{1}{4}\sum|m|\nabla^{\shortparallel}|\hat{\mathbf{q}}_m|^2$ and for $-\frac{1}{2}\sum|m|\left(\nabla^{\shortparallel}\cdot(\hat{Y}_m \times \nabla\rho)\right)(\nabla\rho \times \hat{\mathbf{q}}_{-m})$. In the end, we get $\bar{\mathbf{w}}_1^1|_{\bar{S}} = -(\kappa\beta/4)\sin 2\psi\mathbf{e}$, $\kappa = \sum|m||\hat{\mu}_m|^2$, $\cos\psi = \mathbf{k}\cdot\theta$ here ψ is the latitude on S, $\psi = \pi/2$ on the equator and $\mathbf{e}$ is the unit vector associated with ψ; on the equator, $\mathbf{e}$ is directed as $\mathbf{k}$. Thus, *the steady streaming driven by the torsional oscillations of a submerged ball moves the fluid from the poles towards the equator* that is in agreement with the experiments of Hollerbach *et al* (2009), see Morgulis (2010) for more detailed analysis.

Consider *the translational vibrations of a submerged ball.* Then $\tilde{Y}_0 = \mu(\tau)\mathbf{k}$ (with the same $\bar{D}$, $\bar{S}$, μ and $\mathbf{k}$ as in the case of the torsional vibration). Such vibrations are not tangential but produce no Stokes correction again, *i.e.* $\overline{[\xi_\tau, \xi]} = 0$ everywhere in the flow domain. As a result, $\bar{u}_1^1 = 0$ on $\bar{S}$, see (10). Furthermore, $\mathbf{q} = 3\mu(\theta \times (\mathbf{k} \times \theta))/2$. No one term on the righthand side of (9) contributes in $\bar{\mathbf{w}}_1^1$ except for $-\frac{1}{4}\sum|m|\left(\nabla^{\shortparallel}|\hat{\mathbf{q}}_m|^2 + 4(\nabla^{\shortparallel}\cdot\hat{\mathbf{q}}_m)\hat{\mathbf{q}}_{-m}\right)$, and $\bar{\mathbf{w}}_1^1 = (45\kappa\beta/16)\sin 2\psi\mathbf{e}$. This time, *the steady streaming driven by the translational oscillations of a submerged ball moves the fluid from the equator towards the poles.*

2. The normal vibrations are such that $\tilde{Y}_0^{\shortparallel} = 0$. Then $\tilde{Y}_0 = \tilde{\eta}_0\nabla\rho$, $\mathbf{q} = -\xi^{\shortparallel}$. For instance, consider *the normal vibrating of the lid of a liquid half-space.* Let $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ be the unit vectors of Cartesian coordinates system $Oxyz$ relative to which $\bar{D} = \{z > 0\}$, $\bar{S} = \{z = 0\}$ and $\rho = z$. Let $\tilde{\eta}_0 = \tilde{\eta}_0(x, \tau)$. Then $\xi = \xi\mathbf{e}_x + \eta\mathbf{e}_z$, $\tilde{\eta}_0 = \eta|_{z=0}$ and $\mathbf{q} = -\xi\mathbf{e}_x$. Further, $\mathrm{curl}\,(\xi \times \nabla\rho)\cdot\nabla\rho = -\xi_x$, and $\Delta\rho = 0$. Then

$$\bar{\mathbf{w}}_1^1|_{\bar{S}} = w\mathbf{e}_x,\ w = \beta\overline{\tilde{\eta}_{0x}\tilde{\eta}_{0\tau}} - (3\beta/2)(\overline{\xi_x\xi_\tau} + (\partial_x/2)\textstyle\sum|k||\hat{\xi}_k|^2)|_{z=0};$$
$$\tfrac{\beta}{2}\overline{[\xi_\tau, \xi]} = \beta(\bar{\psi}_z\mathbf{e}_x - \bar{\psi}_x\mathbf{e}_z);\quad \bar{u}_1^i = \beta\bar{\psi}_x|_{z=0},\quad \bar{\psi} = \overline{\eta\xi_\tau};.$$

Consider now normal vibration being produced with *a planar traveling wave, i.e.* set $\tilde{\eta}_0(x, \tau) = f(\alpha x - \tau)$, where $f(\sigma) = \sum\hat{f}_k e^{ik\sigma}$ is prescribed. Then

$$\eta = \textstyle\sum\hat{f}_k e^{-\alpha|k|z + ik\sigma};\ \xi = -i\sum\hat{f}_k \mathrm{sgn}\,k e^{-\alpha|k|z + ik\sigma};\ \sigma = \alpha x - \tau;$$
$$\overline{\tilde{\eta}_{0x}\tilde{\eta}_{0\tau}} = \overline{\xi_x\xi_\tau}|_{z=0} = -\alpha\overline{f'^2};\ \bar{\psi} = -\textstyle\sum|\hat{f}_k|^2|k|e^{-2|k|\alpha z};$$
$$\bar{u}_1^1 = 0,\ \bar{\mathbf{v}}_1^1 = w\mathbf{e}_x,\ w = \alpha\beta\overline{f'^2}/2 \equiv \text{const},$$
$$\tfrac{\beta}{2}\overline{[\xi_\tau, \xi]} = \beta\bar{\psi}'(z)\mathbf{e}_x = 2\beta\alpha\textstyle\sum|\hat{f}_k|^2k^2e^{-2|k|\alpha z}.$$

Here the mean field $\bar{\mathbf{v}}_1^1$ is constant. The total drift velocity is

$$\mathbf{V} = W(z)\mathbf{e}_x,\ \text{where } W(z) = \alpha\beta\overline{f'^2}/2 + 2\beta\alpha\left(\sum|\hat{f}_k|^2k^2e^{-2|k|\alpha z}\right).$$

Thus *the steady streaming being induced by the spreading of the normal displacements in the form of a planar wave traveling along the lid of a liquid semi-space moves* *the drifting particles in the direction of the wave propagation (as $W(z) > 0$ for every $z > 0$). The streamlines are everywhere parallel to the direction of the wave propagation. The velocity magnitude depends on the depth only and attains a nonzero limit far down from the lid as the Stokes correction becomes negligible.*

Finally, consider *the normal vibrations of the wall of a circular pipe being produced with a spiral traveling wave.* Again, $\bar{D} = \{r < 1\}$, $\bar{S} = \{r = 1\}$ and $\rho = 1 - r$ relative to the cylindrical coordinates r, θ, z, and

$$\tilde{\eta}_0(\theta, z, \tau) = f(\alpha z + n\theta - \tau),\ n \in \mathbb{N}, \quad f = f(\sigma) = \sum \hat{f}_k e^{ik\sigma}.$$

We treat this case by analogy with the previous one. As usual, $I_p(s)$, $p \in \mathbb{N}$, stands for modified Bessel function of first kind (bounded for $s \to +0$) of order p. For the sake of convenience, define

$$\chi_{k,n,\alpha}(s) = \frac{d}{2ds}\left(\frac{I_{n|k|}(s)}{I'_{n|k|}(\alpha n|k|)}\right)^2;\ \mu_{k,n,\alpha} = \frac{I_{|k|n}(\alpha|k|)}{I'_{|k|n}(\alpha|k|)}, \quad k \in \mathbb{Z};$$

The tangential and normal components of the mean velocity on $\bar{S}$, the mean velocity itself, the Stokes correction, and the drift velocity field have the forms

$$\bar{\mathbf{w}}_1^\iota = \beta C_{n,\alpha}\left(n\mathbf{e}_\theta + \alpha\mathbf{e}_z\right); \bar{u}_1^\iota = 0;\ \bar{\mathbf{v}}_1^\iota = \beta C_{n,\alpha}(nr\mathbf{e}_\theta + \alpha\mathbf{e}_z); \tag{11}$$

$$C_{n,\alpha} = \sum k^2|\hat{f}_k|^2\left(\tfrac{3}{2}(1 + \tfrac{n^2}{\alpha^2})\mu^2_{k,n,\alpha} - \tfrac{2\mu_{k,n,\alpha}}{|k|\alpha} - 1\right); \tag{12}$$

$$(\beta/2)\overline{[\boldsymbol{\xi}_\tau, \boldsymbol{\xi}]} = \beta(\nu_{n,\alpha}(r)n\mathbf{e}_\theta + w_{n,\alpha}(r)\alpha\mathbf{e}_z); \tag{13}$$

$$\nu_{n,\alpha} = \tfrac{1}{r}\sum k^2|\hat{f}_k|^2\left(\chi'_{k,n,\alpha}(\alpha|k|r) - \tfrac{\chi_{k,n,\alpha}(\alpha|k|r)}{\alpha|k|r}\right); \tag{14}$$

$$w_{n,\alpha} = \sum k^2|\hat{f}_k|^2\left(\chi'_{k,n,\alpha}(\alpha|k|r) + \tfrac{\chi_{k,n,\alpha}(\alpha|k|r)}{\alpha|k|r}\right); \tag{15}$$

$$\mathbf{V} = (\beta/2)[\boldsymbol{\xi}_\tau, \boldsymbol{\xi}] + \bar{\mathbf{v}}_1^\iota = \beta\left(\nu^d_{n,\alpha}n\mathbf{e}_\theta + w^d_{n,\alpha}\alpha\mathbf{e}_z\right); \tag{16}$$

$$\nu^d_{n,\alpha} = rC_{n,\alpha} + \nu_{n,\alpha};\ w^d_{n,\alpha} = C_{n,\alpha} + w_{n,\alpha} \tag{17}$$

Thus the steady streaming being induced with the spreading of the normal displacements in the form of a spiral wave traveling along the wall of a circular pipe gives rise to a translational-rotational motion of the drifting particles. The axial and rotational velocities depend only on the distance from the pipe axis and the streamlines are helixes.

The Stokes corrections to the axial and rotational velocities are always positive *i.e.* the directions of the correspondent rotating and translating of the drifting particles are the same as those of the deforming wave. Indeed, the modified Bessel equation implies that

$$\left(I'_p I_p\right)' + s^{-1}I'_p I = I'^2_p + (1 + s^{-2}p^2)I_p^2;$$
$$\left(I'_p I_p\right)' - s^{-1}I'_p I = \left(I'_p - s^{-1}I_p\right)^2 + (1 + (p^2 - 1)s^{-2})I_p^2.$$

Then every summand in (14-15) is positive. Despite of this observation, the components of total drift velocity (17) can change their signs, *i.e.* the directions of the translating or revolving of different layers of the fluid can be opposite one to another. Indeed, consider the long-wave limit *i.e.* let $\alpha \to 0$. For simplicity, let f be a trigonometric polynomial of degree N. Then

$$C_{n,\alpha} = \sum k^2|\hat{f}_k|^2 \left(1/2 - 2/(|k|n) + O(\alpha)\right), \ \alpha \to 0;$$
$$v^d_{n,\alpha}\big|_{r=1} = \sum k^2|\hat{f}_k|^2 \left(5/2 - 4/(|k|n) + O(\alpha)\right), \ \alpha \to 0;$$
$$w^d_{n,\alpha}\big|_{r=1} = \sum k^2|\hat{f}_k|^2 \left(5/2 - 2/(|k|n) + O(\alpha)\right), \ \alpha \to 0;$$

Thus $C_{n,\alpha}$ can be negative, that is, the rotational and axial components of mean velocity can be directed oppositely to those of the deforming wave provided that $n = 1, 2, 3$ while α is small enough. If $n = 1$ in addition then $v^d_{n,\alpha}$ can be negative for $r = 1$, that is, the rotating of the drifting material particle can be opposite to that of the deforming wave near the wall[5] of the pipe. Finally, $w^d_{1,\alpha}$ is positive in the long-wave limit, that is, the axial drift always follows the wave near the pipe wall.

Consider now the vicinity of the pipe axis. It is convenient to watch the behaviour of angular velocity. Then

$$\lim_{r\to+0} \tfrac{v_{n,\alpha}(r)}{r} = \alpha^2 \sum_{0<n|k|\le 2} \tfrac{k^2|\hat{f}_k|^2}{4^{|kn|} I'^2_{n|k|}(\alpha|k|)}; \qquad n = 1, 2;$$

$$\lim_{r\to+0} \tfrac{v_{n,\alpha}}{r} = 0, \ n > 2; \quad w_{1,\alpha}|_{r=0} = \tfrac{5|\hat{f}_1|^2}{2I'^2_1(\alpha)}, \ w_{1,\alpha}|_{r=0} = 0, \ n > 1.$$

We conclude that in the long-wave limit, $w^d_{n,\alpha}$, $n = 1, 2, 3$ can be negative on the axis but always positive near the wall. Consequently, the axial drift of the material particles can change its direction in the bulk of the fluid; namely, the axial drift always follows the wave near the pipe wall but the counter drifting can occur near the axis. However if $n = N = 1$ then the axial drift follows the wave on the axis too. (We remind that N is the degree of polynomial f).

Consider now the long-wave limit for the angular velocity $\Gamma_{n,\alpha}(r) = C_{n,\alpha} + r^{-1}v_{n,\alpha}$. If $n > 3$ then the drifting particles and the deforming wave revolve themselves in the same direction. Otherwise the rotating keeps the same direction near the pipe wall while the counter revolving can occur near the axis. If $n = 1$ there is an extra possibility: the counter revolving occurs both near the wall and near the axis and so happens inevitably provided that $n = N = 1$, see Fig. 1.

On Fig. 1, the profiles of the angular velocity of the total drift vs the distance from the pipe axis are presented for the case of the vibrations being produced by single harmonic, that is, for $f(\sigma) = \cos\sigma$. The right hand panel displays the plots for different azimuthal wave numbers while the wave length is fixed. The left hand panel displays plots for different wave lengths while azimuthal wave number is fixed. It is worth to note that there are such the values of the wave length that the angular velocity turns out to be much smaller near the pipe wall than near the axis. While looking at the right hand panel, it can be seen that the doubling of the azimuthal wave number is able to re-direct the rotating of all of the drifting particles. Further increase of the wave number from doubled to tripled changes the direction of the rotation again; however, the changing happens not everywhere but near the axis only.

[5]"Near the wall" means that the distance from the wall is small enough but still much greater when the width of the boundary layer (which is of order ϵ)

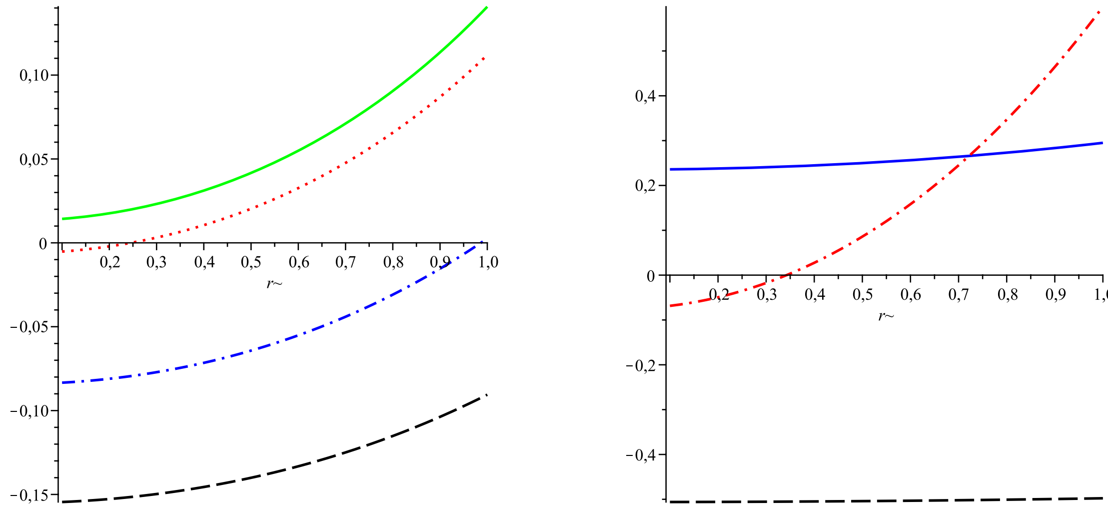

Figure 1: The left hand panel displays $\Gamma_{n,\alpha}$ vs r where $n = 1$ while $\alpha = 1.13$ (solid line), $\alpha = 1.1$ (dotted line), $\alpha = 0.99$ (dash-dotted line) and $\alpha = 0.9$ (dashed line). The right hand panel displays $\Gamma_{n,\alpha}$ vs r where $\alpha = 0.5$ while $n = 1$ (dashed line), $n = 3$ (dash-dotted line) and $n = 2$ (solid line).

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