

# A review of electrodynamics and its coupling with classical balance equations

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## Abstract

In geophysical and many modern applications, electrodynamics and classical thermodynamics are coupled. Generally, thermodynamic processes are modeled by equations of balance on the continuum scale. This paper demonstrates how electrodynamics can also be formulated by balance equations. The basic and auxiliary fields of electrodynamics are introduced in this context. Objective quantities in the classical limit are presented in order to study general problems w.r.t. moving observers. A short remark will be made reg. how authors use thermodynamical principles to find constitutive relations, applicable to moving observers.

## 1 Introduction

In classical physics, NEWTON's laws govern the mechanical response of individual mass points. In continuum mechanics these laws are extended to bodies modeled with continuous properties. This allows to study problems in 3D-bodies to be described by means of calculus. For example, EULER used this technique to formulate the equations of motion for rigid bodies.

In the last century, continuum *mechanics* was rigorously reexamined mathematically and extended by the principles of thermodynamics. The objective is to find the fields of mass density  $\rho$ , velocity  $\mathbf{v}$ , and temperature  $T$  for each “particle,” *i.e.*, statistical averages over sufficiently many atoms. The governing balance equations and mathematical theorems for their exploitation are presented in [1]. The authors also show the fundamental principles to formulate constitutive equations that satisfy the 2nd law of thermodynamics and basic physical reasoning. Furthermore, they formulate the principles of electrodynamics by means of balance equations and derive MAXWELL's equations by mathematical reasoning from those.

In general, the governing equations of mechanics and their associated fields are intuitively clear. In electrodynamics, however, it is often not easy to see what a field actually represents. Does it model the underlining physics? Or is it some mathematical tool? How do the governing relations change in moving matter? To resolve these questions, the modern approach by balance equations is advantageous. Following [1], a more recent treatise was presented by [2], who examines in detail, based on the COLEMAN-NOLL method, how the classical balance laws for momentum and energy change when electromagnetic fields are present. Here, the electromagnetic momentum density follows for any given class of matter under consideration.

A recent modified approach to electrodynamics using mixture balance equations was examined by [3] and communicated by W. DREYER. The mixture balances allow for an even deeper insight into the nature of the fields and how constitutive relations can be obtained. The author also examines the transformation laws of the electrodynamic fields w.r.t. LORENTZ and EUCLIDIAN transformations, as do all previously mentioned authors.

An electromagnetic momentum density with appropriate flux terms is defined in this work and the application of the entropy principle yields constitutive relations.

The goal of this review is to present the concepts of electrodynamics as demonstrated by the aforementioned authors. First, the two elementary fields of electrodynamics, *i.e.*, the magnetic flux density  $\mathbf{B}$  and the electric field  $\mathbf{E}$  are introduced by measurable forces. Second, the balance equation of the magnetic flux is analyzed, yielding two of MAXWELL's equations. Third, balance equations for mixtures are applied. This yields the balance of free electric charge as a corollary of the balance of mass. Based on the postulate of a balance law for total charge a balance of bound charges follows. The examination of the balances of total and bound charges with transport theorems and general surface flux balances yields, (a) the other two of MAXWELL's equations and (b) relations that show how the densities of bound charges and bound electric currents can be calculated by constitutive relations. Fourth, the coupling of electrodynamics and classical thermodynamics is reviewed, *i.e.*, electrodynamic production and supply in the balances of momentum and energy. As the exact coupling structure is still disputed, the common assumptions are commented on. The transformation laws for the electrodynamic fields in the classical limit are reviewed w.r.t. objective fields. Thermodynamic considerations yield constitutive relations, and, depending on the approach, extended balance equations.

Such considerations are necessary for problems of geophysics. For example, the Earth's magnetic field is generated by dynamo action, which can be modeled by coupled momentum and electrodynamic equations.

## 2 The elementary fields of electrodynamics

There are two fundamental experiments that give rise to two independent elementary electrodynamic fields. The first experiment measures the force of two charged *particles* as  $\mathbf{F}_1 = -k_e Q_1 Q_2 \frac{\mathbf{r}_{12}}{|\mathbf{r}_{12}|^3}$ , where  $k_e = \frac{c_0^2 \mu_0}{4\pi}$ . This is COULOMB's law for point charges. Here,  $\mathbf{r}_{12}$  is the vector pointing from charge  $Q_1$  to  $Q_2$ . The speed of light and the magnetic permeability in vacuum are defined by  $c_0 := 299\,792\,458$  m/s, and  $\mu_0 := 4\pi 10^{-7}$  N/A<sup>2</sup>, respectively. The electric permittivity in vacuum is related to these constants by  $\varepsilon_0 = (c_0^2 \mu_0)^{-1}$ . This phenomenon is analogous to NEWTON's law of gravity. Likewise, when there are multiple sources, it is convenient to describe the total COULOMB force upon a test charge  $Q_1$  by introducing a resultant field:  $\mathbf{F}_1 = \mathbf{E}Q_1$ . This quantity  $\mathbf{E}$  is called the electric field.

In a second (seemingly independent) experiment a charge is moved at a velocity  $\mathbf{v}$ . Then near magnets or electric currents another force on the test charge can be observed:  $\mathbf{F}_1 = Q_1 \mathbf{v} \times \mathbf{B}$ , where the field  $\mathbf{B}$  is called magnetic flux density. Historically, the name magnetic induction is also used. Since  $Q_1 \mathbf{v}$  is an electric current, this phenomenon can be generalized by using the total electric current  $\mathbf{J}$  to read:  $\mathbf{F} = \mathbf{J} \times \mathbf{B}$ .

In total, we have LORENTZ' force law:

$$\mathbf{f}^L = q\mathbf{E} + \mathbf{J} \times \mathbf{v} = q\mathfrak{E} + \mathbf{j} \times \mathbf{B}, \quad \text{with} \quad \mathfrak{E} = \mathbf{E} + \mathbf{v} \times \mathbf{B}. \quad (1)$$

The symbols  $\mathbf{f}^L$ ,  $q$ , and  $\mathfrak{E}$  denote the LORENTZ force density, the total charge density, and the electromotive force density, respectively. The symbol  $\mathbf{j}$  follows from the decomposition  $\mathbf{J} = q\mathbf{v} + \mathbf{j}$ . A deeper insight into these symbols is given in Sec. 4. The theory of relativity shows that  $\mathbf{E}$  and  $\mathbf{B}$  are in fact related to components of a world tensor and therefore not independent of each other, as these experiments suggest.

### 3 Balance of the magnetic flux density

Due to FARADAY we have the postulated law:

$$\frac{d}{dt} \int_A \mathbf{B} \cdot \mathbf{n} dA = - \oint_{\partial A} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \boldsymbol{\tau} d\ell .$$

It balances magnetic flux density w.r.t. a specific surface. Furthermore, it is assumed that in any given point in space there was a time when  $\mathbf{B} = \mathbf{0}$ . This is succinctly termed as the “switching-on” argument for the field. There are two common ways to exploit these postulates. The first is to apply FARADAY’s law to a closed surface of a body  $\Omega$  with  $A \rightarrow \partial\Omega$  and  $\partial A \rightarrow \partial\partial\Omega = \emptyset$ . We immediately have:

$$\frac{d}{dt} \oint_{\partial\Omega} \mathbf{B} \cdot \mathbf{n} dA = 0 \quad \Rightarrow \quad \oint_{\partial\Omega} \mathbf{B} \cdot \mathbf{n} dA = \text{const.} \Big|_t .$$

Due to the “switching-on” argument the constant is zero everywhere. Using the generalized GAUSS theorem<sup>1</sup> yields:

$$\int_{\Omega} \nabla \cdot \mathbf{B} dV + \int_I \mathbf{e} \cdot \llbracket \mathbf{B} \rrbracket dA = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{B} = 0, \quad \mathbf{e} \cdot \llbracket \mathbf{B} \rrbracket = 0, \quad (2a)$$

for regular and singular points, respectively. The second way of exploiting FARADAY’s law is by comparison to a general surface flux balance<sup>2</sup>. Consider a material surface that may be cut by a singular surface. The general surface flux balance yields:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0}, \quad -\llbracket \mathbf{B} \rrbracket w_{\perp} + \mathbf{e} \times \llbracket \mathbf{E} \rrbracket = \mathbf{0}, \quad (2b)$$

for regular and singular points, respectively. Eqns. (2) are a direct consequence of the initially given two postulates and constitute the first set of MAXWELL’s equations. By regarding singular surfaces, as it is usually done in continuum mechanics, relations for singular points were also obtained without need for further arguments.

### 4 Balance equations for mixtures and the balances of electric charge

This section starts with an introduction to mixture theory. The concepts of mass and charge of a particle are examined. Then, the well known balance of mass for mixtures yields insight into balances of charge and the underlying quantities.

Consider a mixture of  $N$  components:  $A_1, A_2, \dots, A_N$ . In short hand notation a component is denoted by  $A_{\alpha}$ , where  $\alpha \in \{1, 2, \dots, N\}$ . Each  $A_{\alpha}$  consists of particles that carry a mass  $m_{\alpha}$  and an electric charge  $e_{\alpha}$ . The particle’s charge may only be a multiple of the elementary charge  $e_0$ , *i.e.*,  $e_{\alpha} = z_{\alpha} e_0$  where  $z_{\alpha} \in \mathbb{Z}$ . In the mixture there may occur  $N_{\text{R}}$  simultaneous reactions. For the  $i$ th reaction we have:

$$\sum_{\alpha=1}^N a_{\alpha}^i A_{\alpha} \rightleftharpoons \sum_{\alpha=1}^N b_{\alpha}^i A_{\alpha}, \quad i \in \{1, 2, \dots, N_{\text{R}}\} .$$

<sup>1</sup>For GAUSS’ generalized theorem and the general surface flux balance, see [4, P. 57, P. 346].

<sup>2</sup>See Footnote 1.

The symbols  $a_\alpha^i, b_\alpha^i \in \mathbb{Z}$  denote the particle count of the  $i$ th reaction w.r.t. component  $A_\alpha$  before and after the reaction, respectively. The stoichiometric coefficients  $\gamma_\alpha^i$  follow from the particle counts:  $\gamma_\alpha^i = b_\alpha^i - a_\alpha^i$ . In classical physics, mass and electric charge of each reaction are conserved, *i.e.*:

$$\sum_{\alpha=1}^N m_\alpha \gamma_\alpha^i = 0 \quad \text{and} \quad \sum_{\alpha=1}^N e_\alpha \gamma_\alpha^i = 0 .$$

Here and henceforth in this review, EINSTEIN'S summation convention is only applied to tensor indices. In order to switch from point particles to the scale of continua, a particle density  $n_\alpha$  for each component is introduced. The densities of mass  $\rho_\alpha$  and of *free* electric charge  $q_\alpha^f$  for each component follow directly as  $\rho_\alpha = m_\alpha n_\alpha$  and  $q_\alpha^f = e_\alpha n_\alpha$ . The adjective "free" indicates charges as properties of the particles. *Bound* charges are examined later. As the components  $A_\alpha$  may possess different velocities  $\mathbf{v}_\alpha$ , the densities of mass flow rate  $\boldsymbol{\mu}_\alpha$  and free electric current  $\mathbf{J}_\alpha^f$  of the components are distinctively given by  $\boldsymbol{\mu}_\alpha = \rho_\alpha \mathbf{v}_\alpha$  and  $\mathbf{J}_\alpha^f = q_\alpha^f \mathbf{v}_\alpha$ . For the whole mixture there follow the densities:

$$\rho = \sum_{\alpha=1}^N \rho_\alpha , \quad q^f = \sum_{\alpha=1}^N q_\alpha^f , \quad \boldsymbol{\mu} = \sum_{\alpha=1}^N \boldsymbol{\mu}_\alpha , \quad \mathbf{J}^f = \sum_{\alpha=1}^N \mathbf{J}_\alpha^f .$$

The mixture's *barycentric* velocity  $\mathbf{v}$  is introduced analogously to the definition of the center of mass:  $\rho \mathbf{v} = \boldsymbol{\mu}$ . For each  $A_\alpha$ , the diffusive mass flow rate w.r.t. the barycentric movement of the whole mixture follow as:  $\hat{\boldsymbol{\mu}}_\alpha \equiv \boldsymbol{\mu}_\alpha - \rho_\alpha \mathbf{v}$ . Accordingly, the mass flow w.r.t. all components is conserved:  $\sum_{\alpha=1}^N \hat{\boldsymbol{\mu}}_\alpha = \mathbf{0}$ . With the obtained relations, the individual velocities can be expressed as:  $\mathbf{v}_\alpha = \mathbf{v} + \hat{\boldsymbol{\mu}}_\alpha / \rho_\alpha$ . Now, the free electric current can be denoted by:

$$\mathbf{J}^f = \sum_{\alpha=1}^N \mathbf{J}_\alpha^f = \sum_{\alpha=1}^N q_\alpha^f \mathbf{v}_\alpha = q^f \mathbf{v} + \sum_{\alpha=1}^N \frac{q_\alpha^f}{\rho_\alpha} \hat{\boldsymbol{\mu}}_\alpha = q^f \mathbf{v} + \mathbf{j}^f , \quad \mathbf{j}^f = \sum_{\alpha=1}^N \frac{e_\alpha}{m_\alpha} \hat{\boldsymbol{\mu}}_\alpha .$$

This representation shows that  $\mathbf{j}^f$  and  $q^f \mathbf{v}$  can be interpreted as free diffusive and free non-diffusive electric current, respectively. Some authors regard them as free non-convective and free convective electric currents, respectively. As all free electric currents are convective in nature, this interpretation is problematic. In order to illustrate the decomposition, consider an ordinary fixed copper wire used to conduct electric currents. In the wire, positively charged copper ions construct a fixed crystal lattice. In this environment, easily conveyed conductor electrons are nested—the electron gas. In this example we have  $N = 2$ . Then,  $A_1$  and  $A_2$  represent the copper ions and the electron gas, respectively (say). The lattice does not move, *i.e.*,  $\mathbf{v}_1 = \mathbf{0}$ . As  $m_1 \gg m_2$ ,  $\mathbf{v} \approx \mathbf{0}$  is a good approximation. We immediately see from the last equation, that in this process:  $\mathbf{J}^f \approx \mathbf{j}^f = \hat{\boldsymbol{\mu}}_2 e_2 / m_2$ . The technical current is due to mass diffusion. A simple stationary model for this diffusion process is given by OHM'S law:  $\mathbf{j}^f = \sigma \boldsymbol{\mathcal{E}}$ .

Consider the material balance of mass for a mixture component  $A_\alpha$  for a domain cut by a singular surface  $I$ :

$$\frac{d}{dt} \left[ \int_{\Omega} \rho_\alpha dV + \int_{\Omega \cap I} \rho_{I\alpha} dA \right] = \int_{\Omega} r_\alpha dV + \int_{\Omega \cap I} r_{I\alpha} dA - \oint_{\partial \Omega} \rho_\alpha (\mathbf{v}_\alpha - \mathbf{v}) \cdot \mathbf{n} dA - \oint_{\partial \Omega \cap I} \rho_{I\alpha} (\mathbf{v}_{I\alpha} - \mathbf{w}) \cdot \boldsymbol{\nu} d\ell .$$

All quantities with the index " $I$ " are fields on the singular surface. The mass transition rate densities are denoted by  $r_\alpha$ . Recalling that  $e_\alpha$  and  $m_\alpha$  are constants for each component,

the mass balance can be multiplied by  $e_\alpha/m_\alpha$  and, due to the previous definitions, we immediately find:

$$\frac{d}{dt} \left[ \int_{\Omega} q_\alpha^f dV + \int_{\Omega \cap I} q_{I\alpha}^f dA \right] = \int_{\Omega} r_\alpha \frac{e_\alpha}{m_\alpha} dV + \int_{\Omega \cap I} r_{I\alpha} \frac{e_\alpha}{m_\alpha} dA - \oint_{\partial\Omega} q_\alpha^f (\mathbf{v}_\alpha - \mathbf{v}) \cdot \mathbf{n} dA - \oint_{\partial\Omega \cap I} q_{I\alpha}^f (\mathbf{v}_{I\alpha} - \mathbf{w}) \cdot \boldsymbol{\nu} dl .$$

Noting that  $\sum_{\alpha=1}^N r_\alpha = 0$ , because total mass is conserved, we find after summation:

$$\frac{d}{dt} \left[ \int_{\Omega} q^c dV + \int_{\Omega \cap I} q_I^c dA \right] = - \oint_{\partial\Omega} (\mathbf{J}^c - q^c \mathbf{v}) \cdot \mathbf{n} dA - \oint_{\partial\Omega \cap I} (\mathbf{J}_I^c - q_I^c \mathbf{w}) \cdot \boldsymbol{\nu} dl , \quad (3)$$

if we set in this equation  $q^c = q^f$ ,  $\mathbf{J}^c = \mathbf{J}^f$ , etc. This is the balance of free electric charge. It followed as a direct corollary of the balance of mass. Observation shows that there are other non-free *bound* electric charges and currents due to phenomena referred to as *polarization* and *magnetization*.<sup>3</sup> Without considering the specific physical origins of these charges and currents, it is generally assumed that the densities of total charge and current can be decomposed:  $q = q^f + q^r$ ,  $\mathbf{J} = \mathbf{J}^f + \mathbf{J}^r$ . Here, the index “r” indicates bound charges and currents independent of the origin. Furthermore, it is postulated that total charge can also be balanced. It is given by Eq. (3) when all “c”-indices are removed, *i.e.*, total densities are implied. Due to the decomposition of  $q$  and  $\mathbf{J}$ , we may subtract the balance of free charge from the balance of total charge and obtain the balance for bound charges. It is also given by Eq. (3) when all “c”-indices are exchanged by “r”. In order to solve the balances of total and bound charge, two potentials are introduced: the potential of total charges  $\mathbf{D}$  and the potential of bound charges  $\mathbf{P}$  by

$$\oint_{\partial\Omega} \mathbf{D} \cdot \mathbf{n} dA \equiv \int_{\Omega} q dV + \int_{\Omega \cap I} q_I dA \quad \text{and} \quad - \oint_{\partial\Omega} \mathbf{P} \cdot \mathbf{n} dA \equiv \int_{\Omega} q^r dV + \int_{\Omega \cap I} q_I^r dA .$$

The signs follow the common convention. The potential  $\mathbf{P}$  is also referred to as *polarization*. Application of GAUSS’s generalized theorem and localization yield:

$$\nabla \cdot \mathbf{D} = q , \quad -\nabla \cdot \mathbf{P} = q^r , \quad \text{and} \quad \llbracket \mathbf{D} \rrbracket \cdot \mathbf{e} = q_I , \quad -\llbracket \mathbf{P} \rrbracket \cdot \mathbf{e} = q_I^r , \quad (4)$$

for regular and singular points, respectively. Expressions for free charges follow by subtracting these relations:  $\nabla \cdot (\mathbf{D} + \mathbf{P}) = q^f$ ,  $\llbracket \mathbf{D} + \mathbf{P} \rrbracket \cdot \mathbf{e} = q_I^f$ . This leads to the interpretation of  $\mathfrak{D} = \mathbf{D} + \mathbf{P}$  as charge potential of free charges. The symbol  $\mathfrak{D}$  was introduced by [1]. In literature,  $\mathfrak{D}$  is most often denoted by  $\mathbf{D}$ . This mix-up will be discussed later. Now, the two potentials are used to formulate:

$$\begin{aligned} \frac{d}{dt} \int_A \mathbf{D} \cdot \mathbf{n} dA &= - \int_A (\mathbf{J} - q\mathbf{v}) \cdot \mathbf{n} dA - \oint_{A \cap I} (\mathbf{J}_I - q_I \mathbf{w}) \cdot \boldsymbol{\nu} dl + \oint_{\partial A} (\mathbf{H} + \mathbf{D} \times \mathbf{v}) \cdot \boldsymbol{\tau} dl , \\ - \frac{d}{dt} \int_A \mathbf{P} \cdot \mathbf{n} dA &= - \int_A (\mathbf{J}^r - q^r \mathbf{v}) \cdot \mathbf{n} dA - \oint_{A \cap I} (\mathbf{J}_I^r - q_I^r \mathbf{w}) \cdot \boldsymbol{\nu} dl + \oint_{\partial A} (\mathbf{M} - \mathbf{P} \times \mathbf{v}) \cdot \boldsymbol{\tau} dl . \end{aligned}$$

These equations solve the balances of total and bound charge formally, as  $A \rightarrow \partial\Omega$  yields  $\partial A = \partial\partial\Omega = \emptyset$ . New degrees of freedom were introduced in form of two potentials: the current potential w.r.t. the total current  $\mathbf{H}$ , and the current potential w.r.t. the bound

<sup>3</sup>A phenomenological physical introduction of these potentials is given in [4, Sec. 13.6–13.7].

current  $\mathbf{M}$ . The latter one is usually referred to as *magnetization*. Simple subtraction leads to another form, the balance of free electric charges:

$$\frac{d}{dt} \int_A \mathfrak{D} \cdot \mathbf{n} dA = - \int_A (\mathbf{J}^f - q^f \mathbf{v}) \cdot \mathbf{n} dA - \oint_{A \cap I} (\mathbf{J}_I^f - q_I^f \mathbf{w}) \cdot \boldsymbol{\nu} d\ell + \oint_{\partial A} (\mathfrak{H} + \mathfrak{D} \times \mathbf{v}) \cdot \boldsymbol{\tau} d\ell.$$

Here,  $\mathfrak{H} = \mathbf{H} - \mathbf{M}$  denotes the current potential w.r.t. the free electric current. In literature,  $\mathfrak{H}$  is mostly denoted by  $\mathbf{H}$ . By application of a general surface flux balance, the local form of the total and bound charge balances are obtained for regular and singular forms:

$$\begin{aligned} \frac{\partial \mathbf{D}}{\partial t} - \nabla \times \mathbf{H} &= -\mathbf{J}, & -[\mathbf{D}]w_\perp - \mathbf{e} \times [\mathbf{H}] &= -\mathbf{J}_I, \\ -\frac{\partial \mathbf{P}}{\partial t} - \nabla \times \mathbf{M} &= -\mathbf{J}^r, & [\mathbf{P}]w_\perp - \mathbf{e} \times [\mathbf{M}] &= -\mathbf{J}_I^r. \end{aligned} \quad (5)$$

In summary, MAXWELL's equations are shown in Tab. 1. The first row presents the balances of magnetic flux and can always be found in this form when MAXWELL's equations are presented in textbooks or other sources. It is usually accompanied by the groups of the balance of total or free charge. The formulation of total charge is only practical for use in media without polarization and magnetization. Of course, in that case the balances of total and free charge are identical. Therefore, most authors only regard the balance of free charges and put  $\mathfrak{D} \rightarrow \mathbf{D}$ ,  $\mathfrak{H} \rightarrow \mathbf{H}$ . The original distinction is still beneficial when the laws of transformation are considered.

Table 1: MAXWELL's equations in different formulations w.r.t. the charges.

| balance of:   | regular points   | singular points  |
|---------------|--|--|
| magnetic flux | $\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0}$<br>$\nabla \cdot \mathbf{B} = 0$            | $-[\mathbf{B}]w_\perp + \mathbf{e} \times [\mathbf{E}] = \mathbf{0}$<br>$\mathbf{e} \cdot [\mathbf{B}] = 0$                |
| total charge  | $\frac{\partial \mathbf{D}}{\partial t} - \nabla \times \mathbf{H} = -\mathbf{J}$<br>$\nabla \cdot \mathbf{D} = q$           | $-[\mathbf{D}]w_\perp - \mathbf{e} \times [\mathbf{H}] = -\mathbf{J}_I$<br>$\mathbf{e} \cdot [\mathbf{D}] = q_I$           |
| free charge   | $\frac{\partial \mathfrak{D}}{\partial t} - \nabla \times \mathfrak{H} = -\mathbf{J}^f$<br>$\nabla \cdot \mathfrak{D} = q^f$ | $-[\mathfrak{D}]w_\perp - \mathbf{e} \times [\mathfrak{H}] = -\mathbf{J}_I^f$<br>$\mathbf{e} \cdot [\mathfrak{D}] = q_I^f$ |
| bound charge  | $-\frac{\partial \mathbf{P}}{\partial t} - \nabla \times \mathbf{M} = -\mathbf{J}^r$<br>$-\nabla \cdot \mathbf{P} = q^r$     | $[\mathbf{P}]w_\perp - \mathbf{e} \times [\mathbf{M}] = -\mathbf{J}_I^r$<br>$-\mathbf{e} \cdot [\mathbf{P}] = q_I^r$       |

## 5 The coupling of electrodynamics and classical thermodynamics

### 5.1 The classical approach

In Sec. 1 the LORENTZ force density was introduced. In *inertial* (LORENTZ) frames, a simple identity can be found. This is due to the fact that in inertial frames the fields  $\mathbf{D}$ ,  $\mathbf{H}$  are directly proportional to the fields  $\mathbf{E}$ ,  $\mathbf{B}$ . Here, the simple relations  $\mathbf{D} = \varepsilon_0 \mathbf{E}$  and  $\mathbf{H} = 1/\mu_0 \mathbf{B}$  hold. They are referred to as MAXWELL-LORENTZ aether relations. First,

calculate the sum of (a) the vector product of  $\mathbf{B}$  with Eq. (5)<sub>1</sub> and (b) the dot product of Eq. (4)<sub>1</sub> by  $-\mathbf{E}$ . Application of vector calculus identities and the aether relations yield after some rearrangement:

$$\frac{\partial}{\partial t}(\mathbf{D} \times \mathbf{B}) - \nabla \cdot \left[ -\frac{1}{2}(\varepsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0} \mathbf{B}^2) \mathbf{1} + \varepsilon_0 \mathbf{E} \otimes \mathbf{E} + \frac{1}{\mu_0} \mathbf{B} \otimes \mathbf{B} \right] = -q\mathbf{E} - \mathbf{J} \times \mathbf{B} .$$

Examination of this identity suggests the interpretation of  $\mathbf{D} \times \mathbf{B}$  as density of electromagnetic linear momentum  $\mathbf{g}$  and the expression in the divergence as the MAXWELL stress tensor  $\mathbf{m}$ . We then have  $\frac{\partial \mathbf{g}}{\partial t} - \nabla \cdot \mathbf{m} = -\mathbf{f}^L$ . This representation follows arguments of POYNTING. According to [1, p. 686], there is general agreement that this definition of the LORENTZ force holds in media where  $\mathbf{P} = \mathbf{M} = \mathbf{0}$ , *i.e.*, where  $\mathbf{f}^L = q^f \mathbf{E} + \mathbf{J}^f \times \mathbf{B}$ . The extension to the cases where magnetization and polarization do not vanish is, in fact, based on a postulate. There is some ongoing discussion in literature as to whether this postulate holds. Other propositions were given by MINKOWSKI ( $\mathfrak{D} \times \mathbf{B}$ ) and ABRAHAM ( $\mu_0 \varepsilon_0 \mathbf{E} \times \mathfrak{H}$ ). A definition of an electromagnetic stress tensor is generally problematic, since this suggests the existence of a electromagnetic traction  $\mathbf{t}^L$ , *i.e.*:

$$\mathbf{F}^L = \int_{\Omega} \mathbf{f}^L dV = - \int_{\Omega} \frac{\partial \mathbf{g}}{\partial t} dV + \int_{\partial \Omega} \mathbf{n} \cdot \mathbf{m} dA = - \int_{\Omega} \frac{\partial \mathbf{g}}{\partial t} dV + \int_{\partial \Omega} \mathbf{t}^L dA .$$

This is obviously not a physics-based approach, in contrast to the definition of the CAUCHY stress tensor, which relies on the existence of a mechanical traction vector. One can hope that there will be some *experimentum crucis* in the future in order to settle this problem for matter on the continuum scale. Most publications on this issue argue with (in part faulty) *gedanken* experiments and promote one particular momentum density. In this section, POYNTING's version is *assumed* to hold for demonstration. The next step in coupling mechanics and electrodynamics is based on the idea that total linear momentum and total energy are conserved when all supplies are deactivated. In other words: there is no production w.r.t. total densities of momentum and energy. Even without electrodynamics, there are production terms in the balance of kinetic and internal energy. After summation, these production terms cancel out and total energy lacks production. In the presence of electromagnetic fields, charged matter may experience a (production) force  $\mathbf{k}$  due to the electromagnetic field. The classical balance of mechanical momentum then reads in local form for regular points:

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot [\rho \mathbf{v} \otimes \mathbf{v} - \boldsymbol{\sigma}] = \rho \mathbf{f} + \mathbf{k} , \quad \text{and} \quad \frac{\partial \mathbf{g}}{\partial t} - \nabla \cdot \mathbf{m} = -\mathbf{f}^L$$

can be regarded as (the correct) balance of electromagnetic momentum. The total balance of momentum must not contain a production term, *i.e.*:

$$\frac{\partial}{\partial t}(\rho \mathbf{v} + \mathbf{g}) + \nabla \cdot [\rho \mathbf{v} \otimes \mathbf{v} - \boldsymbol{\sigma} - \mathbf{m}] = \rho \mathbf{f} + \underbrace{\mathbf{k} - \mathbf{f}^L}_{=0} \quad \Rightarrow \quad \mathbf{k} \equiv \mathbf{f}^L !$$

The production of linear momentum is identified. By using POYNTING's version we have explicitly:

$$\frac{\partial}{\partial t}(\rho \mathbf{v} + \mathbf{D} \times \mathbf{B}) + \nabla \cdot [\rho \mathbf{v} \otimes \mathbf{v} - \boldsymbol{\sigma} + \frac{1}{2}(\varepsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0} \mathbf{B}^2) \mathbf{1} - \varepsilon_0 \mathbf{E} \otimes \mathbf{E} - \frac{1}{\mu_0} \mathbf{B} \otimes \mathbf{B}] = \rho \mathbf{f} .$$

The coupling of energy is formulated analogously. When  $\mathbf{P} = \mathbf{M} = \mathbf{0}$ , there is agreement that  $r^J = \mathbf{J} \cdot \mathbf{E}$  is JOULE's heating term. Another identity is constructed by subtraction of

(a) the dot product of  $\mathbf{H}$  with Eq.(2b)<sub>1</sub> by (b) the dot product of  $\mathbf{E}$  with Eq. (5)<sub>1</sub>. Using the aether relations and vector identities yields after rearrangement:

$$\frac{\partial}{\partial t}[\frac{1}{2}(\varepsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0} \mathbf{B}^2)] + \nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{J} \cdot \mathbf{E} .$$

Again, one may interpret terms of this *identity* in terms of an electromagnetic energy density,  $u^{\text{EM}} = \frac{1}{2}(\varepsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0} \mathbf{B}^2)$ , and flux,  $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ . In this definition,  $\mathbf{S}$  is referred to as POYNTING vector. It is again a (disputed) postulate to assume that the previously identified symbols solve  $\frac{\partial u^{\text{EM}}}{\partial t} + \nabla \cdot \mathbf{S} = -r^{\text{J}}$  in the case where magnetization and polarization do not vanish. The classical energy balance is seen to possess some production  $\varsigma$  in the presence of electromagnetic field. It reads in local form:

$$\frac{\partial}{\partial t} \rho(\frac{1}{2} \mathbf{v}^2 + u) + \nabla \cdot [\rho(\frac{1}{2} \mathbf{v}^2 + u) \mathbf{v} + \mathbf{q} - \boldsymbol{\sigma} \cdot \mathbf{v}] = \rho \mathbf{f} \cdot \mathbf{v} + r + \varsigma .$$

If we postulate that the total energy is conserved, the production terms should vanish. We find by adding the (correct) electromagnetic energy balance:

$$\frac{\partial}{\partial t} [\rho(\frac{1}{2} \mathbf{v}^2 + u) + u^{\text{EM}}] + \nabla \cdot [\rho(\frac{1}{2} \mathbf{v}^2 + u) \mathbf{v} + \mathbf{q} - \boldsymbol{\sigma} \cdot \mathbf{v} + \mathbf{S}] = \rho \mathbf{f} \cdot \mathbf{v} + r + \underbrace{\varsigma - r^{\text{J}}}_{=0} \Rightarrow \varsigma \equiv r^{\text{J}} ,$$

hence, the production of energy is identified. Assuming that  $r^{\text{J}} = \mathbf{J} \cdot \mathbf{E}$  holds in general, the balance of total energy reads explicitly:

$$\frac{\partial}{\partial t} [\rho(\frac{1}{2} \mathbf{v}^2 + u) + \frac{1}{2}(\varepsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0} \mathbf{B}^2)] + \nabla \cdot [\rho(\frac{1}{2} \mathbf{v}^2 + u) \mathbf{v} + \mathbf{q} - \boldsymbol{\sigma} \cdot \mathbf{v} + \mathbf{E} \times \mathbf{H}] = \rho \mathbf{f} \cdot \mathbf{v} + r .$$

The relations in this sections were obtained by assuming (a) an inertial frame, where the aether relations hold and (b) POYNTING's form of electromagnetic momentum and energy densities. The problem of providing a mathematical description of volume supplies or productions by divergence terms are discussed in [5]. The authors advise against the introduction of a MAXWELL stress tensor due to lack of physical arguments. In [1] it is argued that there are *laws of interaction* that cannot be stated in a general manner but rather on a per media basis. If this statement holds, it renders the idea of a general total balance of momentum and energy impossible. In this respect, the thermodynamical formulation of [2] avoids a general interaction law. Independent of this specific point, all promising thermodynamical formulations are based upon objective fields, which will be discussed now.

## 5.2 Objective fields and equations in the classical limit

The general transformation of the electromagnetic fields are governed by the four-dimensional LORENTZ transformation. For most practical engineering purposes however, the EUCLIDIAN transformation  $x'_i = R'_{ij} x_j + b'_i$  is more suitable. As in mechanics, objective quantities, *i.e.*, those that transform as tensors w.r.t. this specific transformation, are desirable in the formulation. Also, constitutive relations must not depend on the observer. Therefore, these relations may only be objective tensors. In this regard, it can be seen that MAXWELL's stress tensor as defined above is neither objective nor symmetric in every frame. However, the classical balance of momentum is not objective, either. Also, in certain cases, CAUCHY's stress is not symmetric. Therefore, this argument cannot be used to dismiss POYNTING's choice *per se*. Further treatments start by introducing objective fields, see [2]. In the classical limit ( $|\mathbf{v}| \ll c$ ), the following quantities are objective:  $\mathbf{B}$ ,



$\mathfrak{E} = \mathbf{E} + \mathbf{v} \times \mathbf{B}$ ,  $\mathbf{D}$ ,  $\mathfrak{D}$ ,  $\mathbf{P}$ ,  $\mathbf{j}$ ,  $\mathbf{j}^f$ ,  $\mathbf{j}^r = \mathbf{J}^r - q^r \mathbf{v}$ ,  $q$ ,  $q^f$ ,  $q^r$ ,  $\mathfrak{H} = \mathbf{H} - \mathbf{v} \times \mathbf{D}$ ,  $\mathfrak{M} = \mathbf{M} + \mathbf{v} \times \mathbf{P}$ ,  $\mathfrak{H}^f = \mathfrak{H} - \mathfrak{M} = \mathfrak{H} - \mathbf{v} \times \mathfrak{D}$ . The symbol  $\mathfrak{M}$  is referred to as LORENTZ magnetization. With the flux derivative, defined as  $\mathbf{A}^* = \dot{\mathbf{A}} + \mathbf{A} \nabla \cdot \mathbf{v} - \mathbf{A} \cdot \nabla \mathbf{v}$ , MAXWELL's equations can be reformulated in an objective manner:

$$\begin{aligned} \mathbf{B}^* + \nabla \times \mathfrak{E} &= \mathbf{0}, & \nabla \cdot \mathbf{B} &= 0, & \mathbf{D}^* - \nabla \times \mathfrak{H} &= -\mathbf{j}, & \nabla \cdot \mathbf{D} &= q, \\ \mathfrak{D}^* - \nabla \times \mathfrak{H}^f &= -\mathbf{j}^f, & \nabla \cdot \mathfrak{D} &= q^f, & -\mathbf{P}^* - \nabla \times \mathfrak{M} &= -\mathbf{j}^r, & -\nabla \cdot \mathbf{P} &= q^r. \end{aligned}$$

Using these fields, there are different methods to obtain constitutive relations. If a general form of total momentum and energy is rejected due to the reasons stated above, the proper equations have to be derived for each class of media individually.

### 5.3 Thermodynamical approaches

A deep analysis of thermodynamical methods for problems involving electromagnetic fields would go beyond the scope of this review. Thus, only two methods and their specific differences are reviewed shortly:

**Kovetz-Coleman-Noll** [2] avoids specifying a general body force due to the electromagnetic field. It is postulated (a) a general energy flux,  $\mathbf{q} + \mathfrak{E} \times \mathfrak{H}$ , and, (b) that the inequality flux remains  $\mathbf{q}$ . Whilst the first points seem reasonable, the validity of the second point is hard to follow. These postulates can be referred to as KOVETZ' hypotheses. Subsequent application of the COLEMAN-NOLL methods together with the assumption of a media's constitutive dependencies, *e.g.*,  $\mathcal{F}(\rho, T, \nabla T, \mathbf{v}, \mathfrak{E}, \mathbf{B}, \dots)$ , leads to the form of the *total* stress tensor  $\mathbf{T}$ , momentum density, *etc.* Mechanical and electromagnetic effects cannot be decomposed clearly in the representation. The author discusses several important classes of media, for example, elastic materials, viscous and purely conduction fluids. The latter is of importance in context with the analysis of the Earth's magnetic field and the so-called magnetohydrodynamic approximation.

**Dreyer-Guhlke** [3] This approach relies on POYNTING's choice for momentum and energy to hold. However, the entropy flux is regarded as a general constitutive field. Subsequent analysis is based upon the principle of objectivity and on an axiomatic exploitation of the entropy principle stated according to [6]. The formalism considers general mixtures. Temperature and chemical potentials are defined such that they are continuous on singular surfaces in equilibrium. Furthermore, thermodynamics on singular surfaces is analyzed in detail.

Both approaches are immaculate in their thermodynamical reasoning. Different results seem to stem from their corresponding initial postulates. Both methods should be further examined eventually yielding consensus in the scientific community.

## References

- [1] C. A. Truesdell and R. Toupin. "The classical field theories". In: *Handbuch der Physik, Vol. III/1*. With an appendix on tensor fields by J. L. Ericksen. Berlin: Springer, 1960, 226–793; appendix, pp. 794–858.
- [2] A. Kovetz. *Electromagnetic theory*. Oxford University Press, 2000.
- [3] C. Guhlke. "Theorie der elektrochemischen Grenzfläche". PhD thesis. Technische Universität Berlin, 2015.
- [4] W. H. Müller. *An expedition to continuum theory*. Solid mechanics and its applications series. Berlin: Springer, 2014.

- [5] C. Rinaldi and H. Brenner. “Body versus surface forces in continuum mechanics: Is the Maxwell stress tensor a physically objective Cauchy stress?” In: *Physical Review E* 65.3 (2002), p. 036615.
- [6] D. Bothe and W. Dreyer. “Continuum thermodynamics of chemically reacting fluid mixtures”. In: *Acta Mechanica* (2014), pp. 1–49.

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