Geometrically nonlinear static theory of micropolar elastic thin shallow shells

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Abstract

Micropolar thin shallow shells are considered, the elastic deflections are comparable to their thickness and at the same time are small in relation to the basic size, at the same time as the small angles of relation of the normal to the middle surface before deformation, and their free rotations. Thus, in the deformation tensor and bending-torsion tensor takes into account not only linear but also the nonlinear terms in the gradients of displacements. The hypothesis method is developed and on this base static applied theory of micropolar elastic flexible shallow shells are constructed. Some practical problems are solved.

1 Introduction

In the monographs [1],[2] the geometrically nonlinear static and dynamic general theories of thin plates and shells are constructed on the basis of the classical elasticity theory.

In the works [3],[4] the applied static and dynamic general theories of micropolar thin plates with finite deflections are constructed on the basis of three-dimensional micropolar theory of elasticity.

In the given work as initial, the general variational principle of geometrically nonlinear three-dimensional static theory of micropolar elasticity shallow shells are developed (all basic equations and natural boundary conditions of the specified theory are followed from this principle). The kinematic and static hypotheses of the works [3],[4] are accepted in the basis of construction of the adequate general variational principle of geometrically nonlinear micropolar applied theory of elastic thin shallow shells. By verifying the resulting functional of applied theory of micropolar flexible shallow shells to it’s all functional arguments balance equations, physical relations of elasticity, geometrical relations, and also natural boundary conditions are obtained.
2 The geometrically nonlinear model of three-dimensional micropolar elastic thin shallow shells with independent field of displacements and rotations

The shallow shell with the constant thickness $2h$ is considered as three-dimensional elastic micropolar isotropic body. The shell is attributed to the system of coordinate $x_1$, $x_2$, $z$. The coordinate surface $x_1$, $x_2$ is combined with the median surface of the shell. The axes $Oz$ is directed along the normal of the median surface.

The variational functional of three-dimensional theory of micropolar elasticity with the finite displacements are resulted and looks like:

$$
I = \int_{-h}^{h} \int_{S} \left\{ \sigma_{11} \left[ \gamma_{11} - \left( \frac{\partial V_1}{\partial x_1} + \frac{V_3}{R_1} + \frac{1}{2} \left( \frac{\partial V_3}{\partial x_1} \right)^2 \right) \right] + \\
+ \sigma_{22} \left[ \gamma_{22} - \left( \frac{\partial V_2}{\partial x_2} + \frac{V_3}{R_2} + \frac{1}{2} \left( \frac{\partial V_3}{\partial x_2} \right)^2 \right) \right] + \sigma_{33} \left[ \gamma_{33} - \frac{\partial V_3}{\partial z} \right] + \\
+ \sigma_{12} \left[ \gamma_{12} - \left( \frac{\partial V_2}{\partial x_1} + \frac{1}{2} \frac{\partial V_3}{\partial x_1} \frac{\partial V_3}{\partial x_2} - \omega_3 \right) \right] + \sigma_{21} \left[ \gamma_{21} - \left( \frac{\partial V_1}{\partial x_2} + \frac{1}{2} \frac{\partial V_3}{\partial x_1} \frac{\partial V_3}{\partial x_2} + \omega_3 \right) \right] + \\
+ \sigma_{13} \left[ \gamma_{13} - \left( \frac{\partial V_3}{\partial x_1} + \omega_2 \right) \right] + \sigma_{31} \left[ \gamma_{31} - \left( \frac{\partial V_1}{\partial z} - \omega_2 \right) \right] + \\
+ \sigma_{23} \left[ \gamma_{23} - \left( \frac{\partial V_2}{\partial z} + \omega_1 \right) \right] + \sigma_{32} \left[ \gamma_{32} - \left( \frac{\partial V_3}{\partial z} + \omega_1 \right) \right] + \\
+ \mu_{11} \left[ \chi_{11} - \frac{\partial \omega_1}{\partial x_1} \right] + \mu_{22} \left[ \chi_{22} - \frac{\partial \omega_2}{\partial x_2} \right] + \mu_{33} \left[ \chi_{33} - \frac{\partial \omega_3}{\partial z} \right] + \\
+ \mu_{12} \left[ \chi_{12} - \frac{\partial \omega_2}{\partial x_1} \right] + \mu_{21} \left[ \chi_{21} - \frac{\partial \omega_1}{\partial x_2} \right] + \mu_{13} \left[ \chi_{13} - \frac{\partial \omega_3}{\partial x_1} \right] + \\
+ \mu_{31} \left[ \chi_{31} - \frac{\partial \omega_1}{\partial z} \right] + \mu_{23} \left[ \chi_{23} - \frac{\partial \omega_3}{\partial x_2} \right] + \mu_{32} \left[ \chi_{32} - \frac{\partial \omega_2}{\partial z} \right]\right\} \, dx_1 dx_2 dz
\right)dz + \\
- \int_{S^+} \left[ q_1^1 V_1 + q_2^1 V_2 + q_3^1 V_3 + m_1^1 \omega_1 + m_2^1 \omega_2 + m_3^1 \omega_3 \right]_{z=h} \, dx_1 dx_2 + \\
+ \int_{S^-} \left[ q_1^1 V_1 + q_2^1 V_2 + q_3^1 V_3 + m_1^1 \omega_1 + m_2^1 \omega_2 + m_3^1 \omega_3 \right]_{z=-h} \, dx_1 dx_2 + \\
+ \int_{-h}^{h} \, dz \int_{l_1^1} d\tau \left( \sigma_{21}^0 V_1 + \sigma_{22}^0 V_2 + \sigma_{23}^0 V_3 + \mu_{21}^0 \omega_1 + \mu_{22}^0 \omega_2 + \mu_{23}^0 \omega_3 \right) \, dx_1 + \\
+ \int_{-h}^{h} \, dz \int_{l_1^"} d\tau \left[ \sigma_{31} \left( V_1 - V_1^0 \right) + \sigma_{22} \left( V_2 - V_2^0 \right) + \sigma_{23} \left( V_3 - V_3^0 \right) + \\
+ \mu_{21} \left( \omega_1 - \omega_1^0 \right) + \mu_{22} \left( \omega_2 - \omega_2^0 \right) + \mu_{23} \left( \omega_3 - \omega_3^0 \right) \right] \, dx_1 +
rotations on the other part 
elcertain part 
the top indexes zero are the set external force stresses and couple stresses on the 
are set, and on the other part displacements and rotations are set; quantities with 
the lateral surface of the shell, where on one part the external strains and moments 
are set, and on the other part displacements and rotations are set; quantities with 
the corresponding tow-dimensional theory. Also, instead of the compo-
duction the geometrically nonlinear three-dimensional theory of micropolar elas-
tics, that are statically equivalent to them: strains \((\sigma_{ij})\), \((\gamma_{ij})\), \((\chi_{ij})\), \((\chi_{kl})\), \((\chi_{1i})\), \((\chi_{1j})\), \((\chi_{1k})\), \((\chi_{1l})\), \((\Lambda_{13})\), moments 
 
\[ W = \frac{1}{2} \left( \sigma_{i1} \gamma_{i1} + \sigma_{j2} \gamma_{j2} + \sigma_{i2} \gamma_{i2} + \sigma_{21} \gamma_{21} + \sigma_{31} \gamma_{31} + \sigma_{23} \gamma_{23} + \sigma_{32} \gamma_{32} + \right. 
+ \mu_{11} \chi_{11} + \mu_{22} \chi_{22} + \mu_{33} \chi_{33} + \mu_{12} \chi_{12} + \mu_{21} \chi_{21} + \mu_{13} \chi_{13} + \mu_{31} \chi_{31} + \mu_{23} \chi_{23} + \mu_{32} \chi_{32} \right) \]

(2)

Here \(V_i\), \(V_3\) are components of the displacement vector; \(\omega_i\), \(\omega_3\) are components of the independent rotation vector; \(\sigma_{ii}, \sigma_{ij}, \sigma_{kl}, \sigma_{33}\) are components of the force stresses tensor; \(\mu_{ii}, \mu_{ij}, \mu_{kl}, \mu_{33}\) are components of the couple stresses tensor; \(\gamma_{ii}, \gamma_{ij}, \gamma_{31}, \gamma_{33}\) are components of the deformations tensor; \(\chi_{ii}, \chi_{ij}, \chi_{kl}, \chi_{33}\) are components of the bends-torsions tensor.

It is naturally, the functional (1) to call the full functional of three-dimensional micropolar theory of elasticity of shallow shells at finite deflections. On its basis the variational equation \((\delta I = 0)\) can be obtained. All general equations and natural boundary conditions of the micropolar elasticity problem at finite displacements are obtained from this equation.

3 The geometrically nonlinear theory of micropolar elastic thin shallow shells with big deflections

Hypothesis of works [3], [4] are accepted in the base of the offered theory of micropolar elastic geometrically nonlinear thin shallow shells for the purpose of reduction the geometrically nonlinear three-dimensional theory of micropolar elasticity to the corresponding tow-dimensional theory. Also, instead of the components of stresses and couple stresses tensors there are entered integrated characteristics, that are statically equivalent to them: strains \((T_{ii}, S_{ij}, N_{33}, N_{31})\), moments 
\((M_{ii}, M_{ij}, l_{ii}, l_{ij}, l_{i3}, l_{33})\) and hypermoments \((\Lambda_{13})\):

\[ T_{ii} = \int_{-h}^{h} \sigma_{ii} dz, S_{ij} = \int_{-h}^{h} \sigma_{ij} dz, N_{33} = \int_{-h}^{h} \sigma_{33} dz(i \leftrightarrow 3), M_{ii} = \int_{-h}^{h} \sigma_{i3} dz \]
\[ M_{ij} = \int_{-h}^{h} \sigma_{ij} \mathrm{d}z, \quad L_{mn} = \int_{-h}^{h} \mu_{mn} \mathrm{d}z (m, n = 1, 2, 3), \quad \Lambda_{13} = \int_{-h}^{h} \mu_{13} \mathrm{d}z \tag{3} \]

The formula of the averaged functional \( I_0 \) of micropolar thin shallow shells are obtained from the formula (1) of the three-dimensional theory according to the accepted hypothesis:

\[ I_0 = \int \int \left\{ W_0 - \left( T_{11} \left[ \Gamma_{11} - \left( \frac{\partial u_1}{\partial x_1} + \frac{w}{R_1} + \frac{1}{2} \left( \frac{\partial w}{\partial x_1} \right)^2 \right) \right] + M_{11} \left[ K_{11} - \frac{\partial \psi_1}{\partial x_1} \right] + 
\right. \]

\[ \left. + T_{22} \left[ \Gamma_{22} - \left( \frac{\partial u_2}{\partial x_2} + \frac{w}{R_2} + \frac{1}{2} \left( \frac{\partial w}{\partial x_2} \right)^2 \right) \right] + M_{22} \left[ K_{22} - \frac{\partial \psi_2}{\partial x_2} \right] + 
\right. \]

\[ \left. + S_{12} \left[ \Gamma_{12} - \left( \frac{\partial u_2}{\partial x_1} + \frac{\partial w}{\partial x_1} - \Omega_2 \right) \right] + M_{12} \left[ K_{12} - \left( \frac{\partial \psi_2}{\partial x_1} - \tau \right) \right] + 
\right. \]

\[ \left. + S_{21} \left[ \Gamma_{21} - \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial w}{\partial x_2} + \Omega_3 \right) \right] + M_{21} \left[ K_{21} - \left( \frac{\partial \psi_1}{\partial x_2} + \tau \right) \right] + 
\right. \]

\[ \left. + N_{13} \left[ \Gamma_{13} - \left( \frac{\partial w}{\partial x_1} + \Omega_2 \right) \right] + N_{31} \left[ \Gamma_{31} - \left( \psi_1 - \Omega_2 \right) \right] + 
\right. \]

\[ \left. + N_{23} \left[ \Gamma_{23} - \left( \frac{\partial w}{\partial x_2} - \Omega_1 \right) \right] + N_{32} \left[ \Gamma_{32} - \left( \psi_2 + \Omega_1 \right) \right] + 
\right. \]

\[ \left. + L_{11} \left[ \kappa_{11} - \frac{\partial \Omega_1}{\partial x_1} \right] + L_{22} \left[ \kappa_{22} - \frac{\partial \Omega_2}{\partial x_2} \right] + L_{33} \left[ \kappa_{33} - \tau \right] + 
\right. \]

\[ \left. + L_{12} \left[ \kappa_{12} - \frac{\partial \Omega_2}{\partial x_1} \right] + L_{21} \left[ \kappa_{21} - \frac{\partial \Omega_1}{\partial x_2} \right] + L_{13} \left[ \kappa_{13} - \frac{\partial \Omega_3}{\partial x_1} \right] + 
\right. \]

\[ \left. + L_{23} \left[ \kappa_{23} - \frac{\partial \Omega_3}{\partial x_2} \right] + \Lambda_{13} \left[ \lambda_{13} - \frac{\partial \tau}{\partial x_1} \right] + \Lambda_{13} \left[ \lambda_{13} - \frac{\partial \tau}{\partial x_1} \right] \right\} \mathrm{d}x_1 \mathrm{d}x_2 - 
\]

\[ - \int \int_{S^+} \left[ q_1 u_1 + q_1^+ h \psi_1 + q_2 u_2 + q_2^+ h \psi_2 + + q_3^+ w + 
\right. \]

\[ + m_1^+ \Omega_1 + m_2^+ \Omega_2 + m_3^+ \Omega_3 + m_4^+ h \mathrm{d}x_1 \mathrm{d}x_2 + 
\]

\[ + \int \int_{S^-} \left[ q_1 u_1 - q_1^+ h \psi_1 + q_2 u_2 - q_2^+ h \psi_2 + + q_3^+ w + 
\right. \]

\[ + m_1^+ \Omega_1 + m_2^+ \Omega_2 + m_3^+ \Omega_3 - m_4^+ h \mathrm{d}x_1 \mathrm{d}x_2 + 
\]

\[ + \int_{1'} \left( S_{21} (u_1 - u_1^0) + T_{22} (u_2 - u_2^0) + M_{21} (\psi_1 - \psi_1^0) + M_{22} (\psi_2 - \psi_2^0) + N_{23} (w - w^0) + 
\right. \]

\[ + L_{21} (\Omega_1 - \omega_1^0) + L_{22} (\Omega_2 - \omega_2^0) + L_{23} (\Omega_3 - \omega_3^0) + \Lambda_{23} (\tau - \tau^0) \right) \mathrm{d}x_1 + 
\]
\[
+ \int_{l_2} \left( T_{11}^0 u_1 + S_{12}^0 u_2 + M_{11}^0 \psi_1 + M_{12}^0 \psi_2 + N_{13}^0 w + L_{11}^0 \Omega_1 + L_{12}^0 \Omega_2 + L_{13}^0 \Omega_3 + \Lambda_{13}^0 t \right) \, dx_2 + \\
+ \int_{l_2} \left[ T_{11} \left( u_1 - u_1^0 \right) + S_{12} \left( u_2 - u_2^0 \right) + M_{11} \left( \psi_1 - \psi_1^0 \right) + M_{12} \left( \psi_2 - \psi_2^0 \right) + N_{13} \left( w - w^0 \right) + \\
+ L_{11} \left( \Omega_1 - \omega_1^0 \right) + L_{12} \left( \Omega_2 - \omega_2^0 \right) + L_{13} \left( \Omega_3 - \omega_3^0 \right) + \Lambda_{13} \left( t - t^0 \right) \right] \, dx_2 \right) 
\]

\[W_0 = \frac{1}{2} \left( T_{11} \Gamma_{11} + T_{22} \Gamma_{22} + S_{12} \Gamma_{12} + S_{21} \Gamma_{21} + M_{11} \kappa_{11} + M_{22} \kappa_{22} + M_{12} \kappa_{12} + \\
M_{21} \kappa_{21} + N_{13} \Gamma_{13} + N_{31} \Gamma_{31} + N_{23} \Gamma_{23} + N_{32} \Gamma_{32} + L_{11} \kappa_{11} + L_{22} \kappa_{22} + \\
L_{33} \kappa_{33} + L_{12} \kappa_{12} + L_{21} \kappa_{21} + L_{13} \kappa_{13} + L_{23} \kappa_{23} + \Lambda_{13} \kappa_{13} + \Lambda_{23} \right) \]

Let us notice, that in the specified theory the displacements, independent rotations, components of deformation and bend-torsion tensors are expressed by formulas:

\[
\gamma_{ii} = \Gamma_{ii} (x_1, x_2), \quad \gamma_{ij} = \Gamma_{ij} (x_1, x_2) + zK_{ij}
\]

\[
\chi_{ii} = \kappa_{ii} (x_1, x_2), \quad \chi_{33} = t (x_1, x_2), \quad \chi_{ij} = \kappa_{ij} (x_1, x_2), \quad \chi_{13} = \kappa_{13} (x_1, x_2) + zL_{13} (x_1, x_2)
\]

\[
V_i = \omega_i (x_1, x_2) + z\psi_i (x_1, x_2), \quad V_3 = w (x_1, x_2)
\]

\[
\omega_1 = \Omega_1 (x_1, x_2), \quad \omega_3 = \Omega_3 (x_1, x_2) + zt (x_1, x_2)
\]

\[
\omega_i, \ u_1, \ u_2, \ w \text{ are displacements of the points of the shallow shell's median surface along the axes } x_1, \ x_2, \ z; \ \psi_i, \ \psi_2 \text{ are full angels of the rotation; } \Omega_1, \ \Omega_2, \ \Omega_3 \text{ are free rotations of the initially normal element of the shallow shell's median surface round the lines } x_1, \ x_2, \ z; \ t \text{ is intensity of the full rotation along the } z; \ \Gamma_{ii} \text{ are elongation deformations along the } x_1, \ x_2; \ \Gamma_{ij}, \ \Gamma_{13}, \ \Gamma_{3i} \text{ are shears in the corresponding planes; } K_{ii} \text{ are flexures of the shallow shell's median surface caused by the stresses; } K_{ij} \text{ are torsions of the shallow shell's median surface caused by the stresses; } K_{13} \text{ are flexures of the shallow shell's median surface caused by the couple stresses; } K_{ij} \text{ are torsions of the shallow shell's median surface caused by the couple stresses; } L_{13} \text{ are hyper shears of the shallow shell's median surface caused by the couple stresses.}
\]

By the verifying \( I_0 \) to it's all functional arguments, the general equations and the natural boundary conditions of the micropolar elastic geometrically nonlinear thin shallow shells with the independent fields of displacements and rotations are obtained from the variational equation \( \delta I_0 = 0 \). These equations and boundary conditions are follows as:

**Balance equations**

\[
\frac{\partial T_{ii}}{\partial x_i} + \frac{\partial S_{ji}}{\partial x_j} = - \left( p_i^+ - p_i^- \right), \quad \frac{\partial M_{ii}}{\partial x_i} + \frac{\partial M_{ji}}{\partial x_j} - N_{3i} = -h \left( p_i^+ + p_i^- \right)
\]

\[
\frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} + \frac{\partial}{\partial x_1} \left[ T_{11} \frac{\partial w}{\partial x_1} + \frac{1}{2} \left( S_{12} + S_{21} \right) \frac{\partial^2 w}{\partial x_2^2} \right] +
\]
metrically nonlinear micropolar elastic thin shallow shells with independent fields \( \alpha \) geometrically nonlinear Timoshenko type classical model are also obtained, if to put micropolar shallow shells are obtained, if the nonlinear members to ignore, and the full account of shear deformations.

Geometrically nonlinear static theory of micropolar elastic thin shallow shells

Boundary conditions (on \( x_i = \text{const} \))

The obtained system of the equations (7)-(9) and boundary conditions (10) are the mathematical static model of the geometrically nonlinear micropolar elasticity thin shallow shells with independent fields of displacements and rotations with the full account of shear deformations.

Let us notice, that from constructed model the geometrically linear model of micropolar shallow shells are obtained, if the nonlinear members to ignore, and the geometrically nonlinear Timoshenko type classical model are also obtained, if to put \( \alpha = 0 \).

It is necessary to have in view, that the mathematical dynamic model of geometrically nonlinear micropolar elastic thin shallow shells with independent fields
of displacements and rotations are obtained, if the corresponding inertial members
\[ 2\rho h \frac{\partial^2 u_i}{\partial t^2}, \quad 2\rho h \frac{\partial^2 w}{\partial t^2}, \quad \frac{2\rho h}{3} \frac{\partial^2 \psi_i}{\partial t^2}, \quad 2Jh \frac{\partial^2 \Omega_i}{\partial t^2}, \quad 2Jh \frac{\partial^3 \Omega_i}{\partial t^3} \] to add.

Let us notice, that the formulated above variational problem corresponds to
the most general variational principle of the micropolar elastic thin shallow shells.
Therefore, from the last result the Lagrange and Kastiliano type principles of mi-
cropolar elastic thin shallow shells will follow as special cases. The direct methods
of their approach decision can be made to each of obtained variational equations (in
particular Ritz and Galerkin methods). Using these methods, the boundary prob-
lem of the theory of flexible micropolar elastic thin shallow shells can be reduce to
the decision of the nonlinear algebraic equations system.

As special case, the variational principle of geometrically nonlinear classical the-
ory of elastic thin shallow shells [3],[4] will follow from the variational principle of
geometrically nonlinear micropolar theory of elastic thin shallow shells.

Acknowledgements

- This work was supported by State Committee of Science MES RA, in frame of the
research project SCS 13-2C154.

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