From localization to zoo of patterns in complex dynamics of ensembles

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Abstract

A fast and efficient numerical-analytical approach is proposed for modeling the complex collective behaviour in complex plasma physics models based on the BBGKY hierarchy of kinetic equations. Our calculations are based on variational and multiresolution approaches in the bases of polynomial tensor algebras of high-localized generalized coherent modes generated by action of the internal hidden symmetry of the underlying functional space. We construct the representation for hierarchy of reduced distribution functions via multiscale decomposition in high-localized eigenmodes. Numerical modeling shows the formation of zoo of various internal symmetry-generated structures (patterns) which describe the (meta)stable/unstable type of behaviour in non-equilibrium ensembles.

1 Introduction

The kinetic theory describes a lot of phenomena in beam/plasma physics which cannot be understood on the thermodynamic or/and fluid models level. We mean first of all local/metastable/non-gaussian fluctuations beyond the equilibrium state and collective/relaxation phenomena.

It is well-known that only kinetic approach can describe Landau damping, intra-beam scattering, while, e.g., Schottky noise and associated cooling technique depend on the understanding of spectrum of local fluctuations of the beam charge density [1], [2].

In this paper we review the applications of our numericalanalytical technique based on multiresolution (a.k.a.) wavelet analysis approach for calculations related to description of complex collective behaviour in the framework of general BBGKY hierarchy [3]–[21].

The rational type of nonlinearities allows us to use our results, which are based on the application of wavelet analysis technique and variational formulation of initial nonlinear problems. Wavelet analysis is a set of mathematical methods which give us a possibility to work with well-localized bases in functional spaces and provide maximum sparse forms for the general type of operators (differential, integral, pseudodifferential) in such bases.

It provides the best possible rates of convergence and minimal complexity of algorithms inside and as a result saves CPU time and HDD space.

In part 2 set-up for kinetic BBGKY hierarchy is described. In part 3 we present explicit analytical construction for solutions of hierarchy of equations from part 2 based on tensor algebra extension of multiresolution representation and variational formulation.

We give explicit representation for hierarchy of n-particle reduced/truncated distribution functions in the base of high-localized generalized coherent (regarding underlying affine group) states given by polynomial tensor algebra of base wavelets, which takes into account contributions from all underlying hidden multiscales from the coarsest scale of resolution to the finest one to provide full information about dynamics of complex process.

So, our approach resembles Bogolubov and related approaches but we do not use any perturbation technique (like virial expansion) or linearization procedures.

Numerical modeling shows the creation of different internal (coherent) structures from hidden localized modes, which are related to stable (equilibrium) or unstable/metastable type of behaviour and corresponding pattern (waveleton) formation.

2 Nonequilibrium dynamics: BBGKY hierarchy

Let M be the phase space of ensemble of N particles ($\dim M = 6N$) with coordinates $x_i = (q_i, p_i), \quad i = 1, ..., N, \quad q_i = (q_i^1, q_i^2, q_i^3) \in R^3, \quad p_i = (p_i^1, p_i^2, p_i^3) \in R^3, \quad q = (q_1, \dots, q_N) \in R^{3N}$. Individual and collective measures are:

$$\mu_i = dx_i = dq_i p_i, \quad \mu = \prod_{i=1}^{N} \mu_i \tag{1}$$

Distribution function $D_N(x_1, \dots, x_N; t)$ satisfies Liouville equation of motion for ensemble with Hamiltonian H_N :

$$\frac{\partial D_N}{\partial t} = \{H_N, D_N\} \tag{2}$$

and normalization constraint

$$\int D_N(x_1, \dots, x_N; t) d\mu = 1 \tag{3}$$

where Poisson brackets are:

$$\{H_N, D_N\} = \sum_{i=1}^{N} \Big(\frac{\partial H_N}{\partial q_i} \frac{\partial D_N}{\partial p_i} - \frac{\partial H_N}{\partial p_i} \frac{\partial D_N}{\partial q_i} \Big) \tag{4}$$

Our constructions can be applied to the following general Hamiltonians:

$$H_N = \sum_{i=1}^{N} \Big(\frac{p_i^2}{2m} + U_i(q) \Big) + \sum_{1 \le i \le j \le N} U_{ij}(q_i, q_j) \tag{5}$$

where potentials $U_i(q) = U_i(q_1, \dots, q_N)$ and $U_{ij}(q_i, q_j)$ are not more than rational functions on coordinates. Let L_s and L_{ij} be the Liouvillean operators (vector fields)

$$L_s = \sum_{j=1}^{s} \Big(\frac{p_j}{m} \frac{\partial}{\partial q_j} - \frac{\partial u_j}{\partial q} \frac{\partial}{\partial p_j} \Big) - \sum_{1 \le i \le j \le s} L_{ij} \tag{6}$$

$$L_{ij} = \frac{\partial U_{ij}}{\partial q_i}\frac{\partial}{\partial p_i} + \frac{\partial U_{ij}}{\partial q_j}\frac{\partial}{\partial p_j} \tag{7}$$

For s=N we have the following representation for Liouvillean vector field

$$L_N = \{H_N, \cdot\} \tag{8}$$

and the corresponding ensemble equation of motion:

$$\frac{\partial D_N}{\partial t} + L_N D_N = 0 \tag{9}$$

L_N is self-adjoint operator regarding standard pairing on the set of phase space functions. Let

$$F_N(x_1, \ldots, x_N; t) = \sum_{S_N} D_N(x_1, \ldots, x_N; t) \tag{10}$$

be the N-particle distribution function (S_N is permutation group of N elements). Then we have the hierarchy of reduced distribution functions (V^s is the corresponding normalized volume factor)

$$F_s(x_1, \ldots, x_s; t) = V^s \int D_N(x_1, \ldots, x_N; t) \prod_{s+1 \le i \le N} \mu_i \tag{11}$$

After standard manipulations we arrived to BBGKY hierarchy [2]:

$$\frac{\partial F_s}{\partial t} + L_s F_s = \frac{1}{\upsilon}\int d\mu_{s+1} \sum_{i=1}^{s} L_{i,s+1} F_{s+1} \tag{12}$$

It should be noted that we may apply our approach even to more general formulation than (12). Some particular case is considered in [22]. For s=1,2 we have from (12):

$$\frac{\partial F_1(x_1; t)}{\partial t} + \frac{p_1}{m}\frac{\partial}{\partial q_1} F_1(x_1; t) = \frac{1}{\upsilon}\int dx_2 L_{12} F_2(x_1, x_2; t) \tag{13}$$

$$\begin{aligned} &\frac{\partial F_2(x_1, x_2; t)}{\partial t} + \Big(\frac{p_1}{m}\frac{\partial}{\partial q_1} + \frac{p_2}{m}\frac{\partial}{\partial q_2} - L_{12}\Big) \cdot F_2(x_1, x_2; t) \\ &= \frac{1}{\upsilon}\int dx_3 (L_{13} + L_{23}) F_3(x_1, x_2; t) \end{aligned} \tag{14}$$

3 Multiscale analysis

The infinite hierarchy of distribution functions satisfying system (12) in the thermodynamical limit is:

$$F = \{F_0, F_1(x_1; t), F_2(x_1, x_2; t), \ldots, F_N(x_1, \ldots, x_N; t), \ldots\}$$

where $F_p(x_1, \ldots, x_p; t) \in H^p$, $H^0 = R$, $H^p = L^2(R^{6p})$ (or any different proper functional space), $F \in H^\infty = H^0 \oplus H^1 \oplus \cdots \oplus H^p \oplus \ldots$ with the natural Fock-space like norm (of course, we keep in mind the positivity of the full measure) introduced by us [3]–[21]:

$$(F, F) = F_0^2 + \sum_i \int F_i^2(x_1, \dots, x_i; t) \prod_{\ell=1}^{i} \mu_\ell \tag{15}$$

$$F_k(x_1, \dots, x_k; t) = \prod_{i=1}^{k} F_1(x_i; t) \tag{16}$$

First of all we consider $F = F(t)$ as function on time variable only, $F \in L^2(R)$, via multiresolution decomposition which naturally and efficiently introduces the infinite sequence of underlying hidden scales into the game [23]. Because affine group of translations and dilations is inside the approach, this method resembles the action of a microscope. We have contribution to final result from each scale of resolution from the whole infinite scale of spaces. Let the closed subspace $V_j (j \in \mathbf{Z})$ correspond to level j of resolution, or to scale j.

We consider a multiresolution analysis of $L^2(R)$ (of course, we may consider any different functional space) which is a sequence of increasing closed subspaces V_j: $...V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset ...$ satisfying the following properties: let W_j be the orthonormal complement of V_j with respect to V_{j+1}: $V_{j+1} = V_j \bigoplus W_j$ then we have the following decomposition:

$$\{F(t)\} = \bigoplus_{-\infty<j<\infty} W_j \tag{17}$$

or in case when V_0 is the coarsest scale of resolution:

$$\{F(t)\} = \overline{V_0 \bigoplus_{j=0}^{\infty} W_j}, \tag{18}$$

Subgroup of translations generates basis for fixed scale number: $\mathrm{span}_{k\in Z}\{2^{j/2}\Psi(2^j t - k)\} = W_j$. The whole basis is generated by action of the full affine group:

$$\mathrm{span}_{k\in Z, j\in Z}\{2^{j/2}\Psi(2^j t - k)\} = \mathrm{span}_{k,j\in Z}\{\Psi_{j,k}\} = \{F(t)\} \tag{19}$$

Let the sequence $\{V_j^t\}, V_j^t \subset L^2(R)$ correspond to multiresolution analysis on time axis, $\{V_j^{x_i}\}$ correspond to multiresolution analysis for coordinate x_i, then

$$V_j^{n+1} = V_j^{x_1} \otimes \dots \otimes V_j^{x_n} \otimes V_j^t \tag{20}$$

corresponds to multiresolution analysis for n-particle distribution function $F_n(x_1, \dots, x_n; t)$.

E.g., for $n = 2$:

$$V_0^2 = \{f : f(x_1, x_2) = \sum_{k_1, k_2} a_{k_1,k_2} \phi^2(x_1 - k_1, x_2 - k_2),\ a_{k_1,k_2} \in \ell^2(Z^2)\}, \tag{21}$$

where $\phi^2(x_1, x_2) = \phi^1(x_1)\phi^2(x_2) = \phi^1 \otimes \phi^2(x_1, x_2)$, and $\phi^i(x_i) \equiv \phi(x_i)$ form a multiresolution basis corresponding to $\{V_j^{x_i}\}$.

If $\{\phi^1(x_1-\ell)\}$, $\ell \in Z$ form an orthonormal set, then $\phi^2(x_1-k_1, x_2-k_2)$ form an orthonormal basis for V_0^2. Action of affine group provides us by multiresolution representation of $L^2(R^2)$. After introducing detail spaces W_j^2, we have, e.g. $V_1^2 = V_0^2 \oplus W_0^2$. Then 3-component basis for W_0^2 is generated by translations of three functions [23]:

$$\begin{aligned}
\Psi_1^2 &= \phi^1(x_1)\otimes\Psi^2(x_2),\\
\Psi_2^2 &= \Psi^1(x_1)\otimes\phi^2(x_2),\\
\Psi_3^2 &= \Psi^1(x_1)\otimes\Psi^2(x_2)
\end{aligned} \tag{22}$$

In general case we can use the rectangle lattice of scales and one-dimentional wavelet decomposition :

$$f(x_1, x_2) = \sum_{i,\ell;j,k} < f, \Psi_{i,\ell}\otimes\Psi_{j,k} > \Psi_{j,\ell}\otimes\Psi_{j,k}(x_1, x_2)$$

where the base functions $\Psi_{i,\ell}\otimes\Psi_{j,k}$ depend on two scales 2^{-i} and 2^{-j}.

Then, after constructing such multidimension bases we can apply some of our variational procedures introduced in [3]-[21]. As a result the solution of equations (12) has the following multiscale/multiresolution decomposition via nonlinear high-localized eigenmodes

$$\begin{aligned}
F(t, x_1, x_2, \dots) &= \sum_{(i,j)\in Z^2} a_{ij} U^i\otimes V^j(t, x_1, x_2, \dots)\\
V^j(t) &= V_N^{j,slow}(t) + \sum_{l\geq N} V_l^j(\omega_l t), \quad \omega_l \sim 2^l\\
U^i(x_s) &= U_M^{i,slow}(x_s) + \sum_{m\geq M} U_m^i(k_m^s x_s), \quad k_m^s \sim 2^m,
\end{aligned} \tag{23}$$

which corresponds to the full multiresolution expansion in all underlying time/space scales.

Formal representation (23) provide us with expansion into the slow part $\Psi_{N,M}^{slow}$ (coarse graining) and fast oscillating parts (fine scales) for arbitrary N, M.

So, we can move from coarse scales of resolution to the finest one for obtaining more detailed information about our complex dynamical process.

The first terms in the RHS of formulas (23) correspond on the global level of function space decomposition to resolution space and the second ones to detail space. In this way we collect contributions to the exact solution from each scale of resolution or each hidden time/space scale or from each nonlinear hidden eigenmode [3]–[21].

It should be noted that such representations provide the best possible localization properties in the corresponding (phase)space/time coordinates. In contrast with different approaches representation (23) do not use perturbation technique or linearization procedures.

Numerical calculations are based on compactly supported wavelets and related wavelet families [24] and on evaluation of the accuracy regarding norm (15):

$$\|F^{N+1} - F^N\| \leq \varepsilon \tag{24}$$

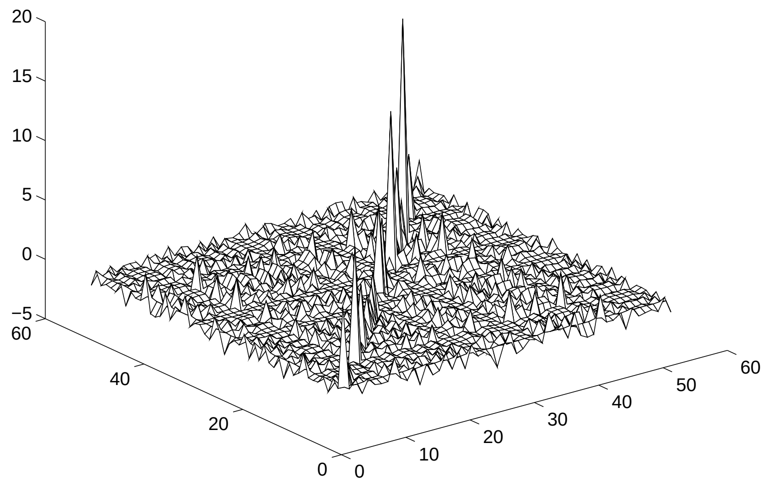

Figure 1: 6-eigenmodes representation for waveletons.

Fig. 1 demonstrates waveleton (high-localized and metastable) pattern generated at level 6 of the scale/eigenmodes decomposition for solutions of hierarchies like (12).

So, finally, using multiresolution decomposition constructed by properly generated action of hidden symmetry of the underlying functional spaces, we provide the best possible (phase) space/time localization properties and as a result the construction of high-localized metastable waveleton structures in spatially-extended stochastic systems with collective behaviour. Fig. 1 represents some possible image for the energy confinement state in the plasma fusion model [20].

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