Fast modeling for collective models of beam/plasma physics

Antonina N. Fedorova, Michael G. Zeitlin
zeitlin@math.ipme.ru, anton@math.ipme.ru

Abstract

We consider an application of modification of our variational-wavelet approach to some nonlinear collective models of beam/plasma physics: the Vlasov/Boltzmann-like truncation of general BBGKY hierarchy related to the modeling of the propagation of intense charged particle beams in high-intensity accelerators and transport systems. We use fast convergent multiscale variational-wavelet representations for solutions which allow to consider the polynomial and rational type of nonlinearities. The solutions are represented via the multiscale decomposition in nonlinear high-localized eigenmodes (waveletons). In contrast to different approaches we do not use perturbation technique or linearization procedures.

1 Introduction

We consider applications of numerical–analytical technique based on modification of our variational-wavelet approach to nonlinear collective models of beam/plasma physics, e.g. some forms of Vlasov/Boltzmann-like reductions of general BBGKY hierarchy (section 2). These equations are related to the modeling of propagation of intense charged particle beams in high-intensity accelerators and transport systems [1], [2]. In our approach we use fast convergent multiscale variational-wavelet representations, which allows to consider polynomial and rational type of nonlinearities [3]-[22]. The solutions are represented via the multiscale decomposition in nonlinear high-localized eigenmodes (some generalization of the so-called Gluckstern modes, in some sense), which corresponds to the full multiresolution expansion in all underlying hidden time/space or phase space scales.

In contrast with different approaches we do not use perturbation technique or linearization procedures.

In section 3 after formulation of key points we consider another variational approach based on ideas of para-products and nonlinear approximation in multiresolution approach, which provides the possibility for computations in each scale separately [23].

We consider representation (4) below, where each term corresponds to the contribution from the scale i in the full underlying multiresolution decomposition as multiscale generalization of old (nonlinear) δF approach [1], [2].

As a result, fast scalar/parallel modeling demonstrates appearance of high-localized coherent structures (waveletons) and (meta)stable pattern formation in systems with complex collective behaviour or the possibility of existence of relatively/locally stable order in the systems with full disorder.

2 Vlasov/Boltzmann–like reductions

Let M be the phase space of ensemble of N particles ($\dim M = 6N$) with coordinates $x_i = (q_i, p_i), \quad i = 1, ..., N, \quad q_i = (q_i^1, q_i^2, q_i^3) \in R^3, \quad p_i = (p_i^1, p_i^2, p_i^3) \in R^3$ with distribution function $D_N(x_1, \dots, x_N; t)$ and

$$F_N(x_1, \dots, x_N; t) = \sum_{S_N} D_N(x_1, \dots, x_N; t) \tag{1}$$

be the N-particle distribution functions (S_N is permutation group of N elements). For s=1,2 we have from general BBGKY hierarchy [22]:

$$\frac{\partial F_1(x_1; t)}{\partial t} + \frac{p_1}{m}\frac{\partial}{\partial q_1} F_1(x_1; t) = \frac{1}{\upsilon}\int dx_2 L_{12} F_2(x_1, x_2; t) \tag{2}$$

$$\frac{\partial F_2(x_1, x_2; t)}{\partial t} + \Big(\frac{p_1}{m}\frac{\partial}{\partial q_1} + \frac{p_2}{m}\frac{\partial}{\partial q_2} - L_{12}\Big) F_2(x_1, x_2; t) = \frac{1}{\upsilon}\int dx_3 (L_{13} + L_{23}) F_3(x_1, x_2; t) \tag{3}$$

where partial Liouvillean operators are described in [22]. We are interested in the cases when and where

$$F_k(x_1, \dots, x_k; t) = \prod_{i=1}^{k} F_1(x_i; t) + G_k(x_1, \dots, x_k; t),$$

where G_k are the correlation patterns, really have additional reductions as in case of the Vlasov-like systems.

Then we have in the equations (2), (3) not more than polynomial type of nonlinearities (more exactly, multilinearities), i.e. we can apply our general approach [3]–[22] based on Local Nonlinear (non-abelian) Harmonic Analysis [23].

3 Multiresolution via para-products

Our goal is the demonstration of advantages of the following formal representation

$$F = \sum_{i \in Z} \delta^i F, \tag{4}$$

for the full exact solution for the systems related to equations (2), (3). It is possible to consider the representation (4) as multiscale generalization of old (nonlinear) δF approach [1], [2]. So, in our modified version of the decomposition (4) each $\delta^i F$ term corresponds to the contribution from the scale i in the full underlying multiresolution decomposition

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \dots \tag{5}$$

of the proper function space (L^2, Hilbert, Sobolev, etc.) to which the tower F really belongs according to the properly chosen physical hypothesis. It should be noted that (4) doesn't based neither on perturbations nor on linearization procedures. Although usually physicists, who prefer computer modeling as a main tool of understanding of the physical reality, don not think about underlying functional spaces, but many concrete features of complicated complex dynamics are really related not only to concrete form/class of operators/equations but also depend on the proper choice of function spaces where operators actully act. Moreover, we have for arbitrary N in the finite N-mode approximation

$$F^N = \sum_{i=1}^{N} \delta^i F \tag{6}$$

the following more useful decompositions:

$$\{F(t)\} = \bigoplus_{-\infty<j<\infty} W_j \quad \text{or} \quad \{F(t)\} = \overline{V_0 \bigoplus_{j=0}^{\infty} W_j}, \tag{7}$$

in the case when V_0 is the coarsest scale of resolution and where $V_{j+1} = V_j \bigoplus W_j$ and the bases in the scale spaces $W_i(V_j)$ are generated from the base functions $\psi(\varphi)$ by action of the underlying affine group of the translations and dilations (the so called "wavelet microscope"). The following constructions based on the variational approach provide the best possible fast convergence properties in the sense of the combined norm:

$$\|F^{N+1} - F^N\| \leq \varepsilon \tag{8}$$

introduced and considered before in [3]–[22]. Our five basic points after the choice of the model for the functional space are as follows:

1. The ansatz-oriented choice of the (multidimensional) bases related to some polynomial tensor algebra. Some example related to the general BBGKY hierarchy is considered in [22].

2. The choice of the proper variational principle. A few projection/ Galerkin-like principles for the (weak) solution construction are considered in [3] - [21]. It should be noted the advantage of formulation related to biorthogonal (wavelet) decomposition.

3. The choice of base functions in scale spaces W_j from the whole wavelet zoo. They correspond to high-localized (nonlinear) oscillations/excitations, coherent (nonlinear) resonances, etc. Besides the fast convergence properties of the corresponding variational-wavelet expansions it should be noted the minimal complexity of all underlying calculations, especially in case of choice of wavelet packets which minimize Shannon entropy.

4. The operator representations provide the best possible sparse representations for the arbitrary (pseudo) differential/integral operators
   $df/dx, d^n f/dx^n, \int T(x,y)f(y)dy)$, etc [23].

5. (Multi)linearization. Besides variation approach we consider now a different method to deal with (polynomial) nonlinearities.

We modify the scheme of our variational approach in such a way in which we consider the different scales of the multiresolution decomposition (5) separately. For this reason we need to compute errors of approximations. The main problems come, of course, from nonlinear (polynomial) terms. We follow according to the multilinearization (in case below – bilinearization) approach of Beylkin, Meyer etc. from [23]. Let P_j be the projection operators on the subspaces V_j (5):

$$(P_j f)(x) = \sum_k < f, \varphi_{j,k} > \varphi_{j,k}(x) \tag{9}$$

and Q_j are projection operators on the subspaces W_j: $Q_j = P_{j-1} - P_j$. So, for $u \in L^2(R)$ we have $u_j = P_j u$ and $u_j \in V_j$. It is obviously that we can represent u_0^2 in the following form:

$$u_0^2 = 2\sum_{j=1}^{n}(P_j u)(Q_j u) + \sum_{j=1}^{n}(Q_j u)(Q_j u) + u_n^2 \tag{10}$$

In this formula there is no interaction between different scales. We may consider each term of (10) as a bilinear mappings:

$$M_{VW}^j : V_j \times W_j \to L^2(\mathbf{R}) = V_j \oplus_{j' \ge j} W_{j'} \tag{11}$$

$$M_{WW}^j : W_j \times W_j \to L^2(\mathbf{R}) = V_j \oplus_{j' \ge j} W_{j'} \tag{12}$$

For numerical purposes we need formula (10) with a finite number of scales, but when we consider limit $j \to \infty$ we have

$$u^2 = \sum_{j \in \mathbf{Z}} (2P_j u + Q_j u)(Q_j u), \tag{13}$$

which is the very useful para-product of Bony, Coifman and Meyer [23]. Now we need to expand (10) into the wavelet bases. To expand each term in (10) we need to consider the integrals of the products of the basis functions corresponding to decomposition (7), e.g.

$$M_{WWW}^{j,j'}(k, k', \ell) = \int_{-\infty}^{\infty} \psi_k^j(x)\psi_{k'}^j(x)\psi_\ell^{j'}(x)dx, \tag{14}$$

where $j' > j$ and

$$\psi_k^j(x) = 2^{-j/2}\psi(2^{-j}x - k) \tag{15}$$

are the basis functions proper for (7). For compactly supported wavelets

$$M^{j,j'}_{WWW}(k, k', \ell) \equiv 0 \quad \text{for} \quad |k - k'| > k_0, \tag{16}$$

where k_0 depends on the overlap of the supports of the basis functions and

$$|M^r_{WWW}(k - k', 2^r k - \ell)| \leq C \cdot 2^{-r\lambda M} \tag{17}$$

Let us define j_0 as the distance between scales such that for a given ε all the coefficients in (17) with labels $r = j - j'$, $r > j_0$ have absolute values less than ε. For the

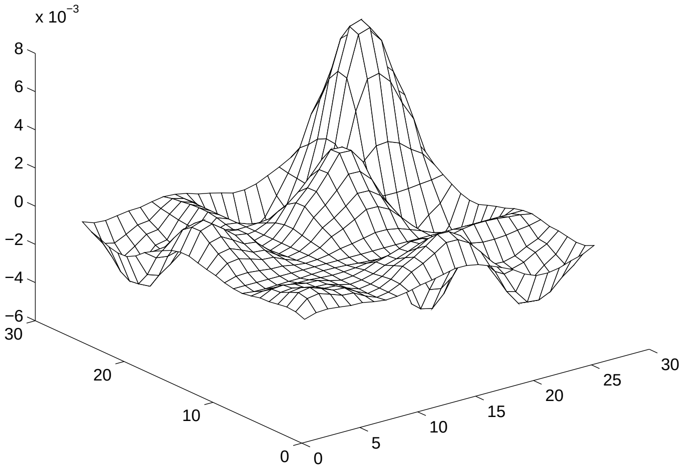

Figure 1: $N = 1$ coarse grain contribution to (6).

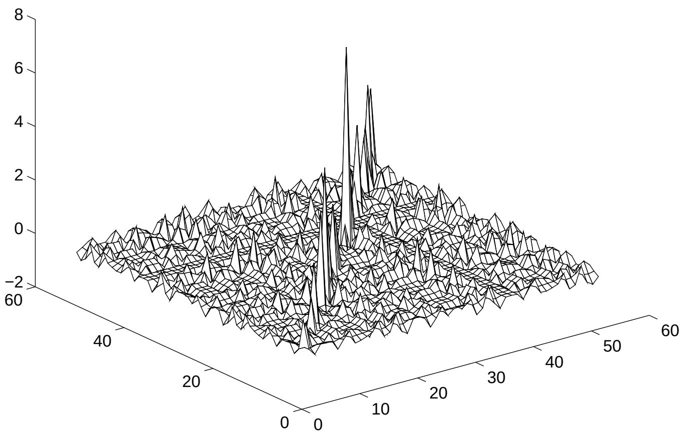

Figure 2: Localized metastable pattern.

purposes of computing with accuracy ε we replace the mappings in (11), (12) by

$$M^j_{VW} : V_j \times W_j \to V_j \oplus_{j \leq j' \leq j_0} W_{j'} \tag{18}$$

$$M^j_{WW} : W_j \times W_j \to V_j \oplus_{j\leq j'\leq j_0} W_{j'} \tag{19}$$

Since

$$V_j \oplus_{j\leq j'\leq j_0} W_{j'} = V_{j_0-1},\ V_j \subset V_{j_0-1},\ W_j \subset V_{j_0-1}$$

we may consider bilinear mappings (18), (19) on $V_{j_0-1} \times V_{j_0-1}$. For the evaluation of (18), (19) as mappings $V_{j_0-1} \times V_{j_0-1} \to V_{j_0-1}$ we need significantly fewer coefficients than for mappings (18), (19). It is enough to consider only coefficients

$$M(k, k', \ell) = 2^{-j/2} \int_{\infty}^{\infty} \varphi(x-k)\varphi(x-k')\varphi(x-\ell)dx, \tag{20}$$

where $\varphi(x)$ is scale function. Also we have

$$M(k, k', \ell) = 2^{-j/2} M_0(k-\ell, k'-\ell), \tag{21}$$

where

$$M_0(p, q) = \int \varphi(x-p)\varphi(x-q)\varphi(x)dx \tag{22}$$

$M_0(p, q)$ satisfy the standard system of linear equations and after its solution we can recover all bilinear quantities (14). Then we may apply some variational approach from [3]-[21] but, in contrast with previous attempts, at each scale separately. Finally, after the application of points 1-5 above, we arrive to the explicit numerical-analytical realization of representations (4) or (6). Fig. 1 demonstrates the coarse grain level contribution to the full solution (6) while Fig. 2 presents our final goal: the localized non-gaussian (meta)stable pattern as the solution of the system like (2),(3). We evaluate the accuracy of calculations according to norm considered for the whole kinetic hierarchy in companion paper [22]. Various images for different types of possible patterns are parametrized by details of the underlying (multi)linear algebra related to aspects of multiresolution decomposition as well as by features related to the internal structure of the underlying functional spaces. Both structures have direct relation to the underlying physics of ensembles.

References

[1] *A. H. Boozer*, Rev. Mod. Phys., 76, 1071 (2004); A. Chao, Handbook of Accelerator Physics, World Scientific, 1999.

[2] *R. C. Davidson and H. Qin*, Physics of Intense Charged Particle Beams in High Energy Accelerators (World Scientific, Singapore, 2001); R. Balescu, Equilibrium and Nonequilibrium Statistical Mechanics, (Wiley, New York, 1975).

[3] A.N. Fedorova and M.G. Zeitlin, Quasiclassical Calculations for Wigner Functions via Multiresolution, Localized Coherent Structures and Patterns Formation in Collective Models of Beam Motion, in *Quantum Aspects of Beam Physics*, Ed. P. Chen (World Scientific, Singapore, 2002) pp. 527–538, 539–550; arXiv: physics/0101006; physics/0101007.

[4] A.N. Fedorova and M.G. Zeitlin, BBGKY Dynamics: from Localization to Pattern Formation, in *Progress in Nonequilibrium Green's Functions II*, Ed. M. Bonitz, (World Scientific, 2003) pp. 481–492; arXiv: physics/0212066.

[5] A.N. Fedorova and M.G. Zeitlin, Pattern Formation in Wigner-like Equations via Multiresolution, in *Quantum Aspects of Beam Physics*, Eds. Pisin Chen, K. Reil (World Scientific, 2004) pp. 22-35; Preprint SLAC-R-630; arXiv: quant-ph/0306197.

[6] A.N. Fedorova and M.G. Zeitlin, Localization and pattern formation in Wigner representation via multiresolution, *Nuclear Inst. and Methods in Physics Research, A*, **502A/2-3**, pp. 657 - 659, 2003; arXiv: quant-ph/0212166.

[7] A.N. Fedorova and M.G. Zeitlin, Fast Calculations in Nonlinear Collective Models of Beam/Plasma Physics, *Nuclear Inst. and Methods in Physics Research, A*, **502/2-3**, pp. 660 - 662, 2003; arXiv: physics/0212115.

[8] A.N. Fedorova and M.G. Zeitlin, Classical and quantum ensembles via multiresolution: I-BBGKY hierarchy; Classical and quantum ensembles via multiresolution. II. Wigner ensembles; *Nucl. Instr. Methods Physics Res.*, **534A** (2004)309-313; 314-318; arXiv: quant-ph/0406009; quant-ph/0406010.

[9] A.N. Fedorova and M.G. Zeitlin, Localization and Pattern Formation in Quantum Physics. I. Phenomena of Localization, in *The Nature of Light: What is a Photon? SPIE*, vol.**5866**, pp. 245-256, 2005; arXiv: quant-ph/0505114;

[10] A.N. Fedorova and M.G. Zeitlin, Localization and Pattern Formation in Quantum Physics. II. Waveletons in Quantum Ensembles, in *The Nature of Light: What is a Photon?SPIE*, vol. **5866**, pp. 257-268, 2005; arXiv: quant-ph/0505115.

[11] A.N. Fedorova and M.G. Zeitlin, Pattern Formation in Quantum Ensembles, *Intl. J. Mod. Physics* **B20**(2006)1570-1592; arXiv: 0711.0724.

[12] A.N. Fedorova and M.G. Zeitlin, Patterns in Wigner-Weyl approach, Fusion modeling in plasma physics: Vlasov-like systems, *Proceedings in Applied Mathematics and Mechanics (PAMM)*, Volume **6**, Issue 1, p. 625, p. 627, Wiley InterScience, 2006; arXiv:1012.4971, arXiv:1012.4965.

[13] A.N. Fedorova and M.G. Zeitlin, Localization and Fusion Modeling in Plasma Physics. Part I: Math Framework for Non-Equilibrium Hierarchies, pp.61-86, in *Current Trends in International Fusion Research*, Ed. E. Panarella, R. Raman, National Research Council (NRC) Research Press, Ottawa, Ontario, Canada, 2009; arXiv: physics/0603167.

[14] A.N. Fedorova and M.G. Zeitlin, Localization and Fusion Modeling in Plasma Physics. Part II: Vlasov-like Systems. Important Reductions, pp.87-100, in *Current Trends in International Fusion Research*, Ed. E. Panarella, R. Raman, National Research Council (NRC) Research Press, Ottawa, Ontario, Canada, 2009; arXiv: physics/0603169.

[15] A.N. Fedorova and M.G. Zeitlin, Fusion State in Plasma as a Waveleton (Localized (Meta)-Stable Pattern), p. 272, in *AIP Conference Proceedings*, Volume **1154**, Issue 1, *Current Trends in International Fusion Research*, Ed. E. Panarella, R. Raman, AIP, 2009, doi:10.1063/1.3204585.

[16] A.N. Fedorova and M.G. Zeitlin, Exact Multiscale Representations for (Non)-Equilibrium Dynamics of Plasma, p.291, in *AIP Conference Proceedings*, Volume **1154**, Issue 1, *Current Trends in International Fusion Research*, Ed. E. Panarella, R. Raman, AIP, 2009, doi:10.1063/1.3204592.

[17] A.N. Fedorova and M.G. Zeitlin, Fusion Modeling in Vlasov-Like Models, *Journal Plasma Fusion Res. Series*, Vol. **8**, pp. 126-131, 2009.

[18] A.N. Fedorova and M.G. Zeitlin, Quantum points/patterns, Part 2: from quantum points to quantum patterns via multiresolution, SPIE, vol. 8121, 81210K (2011), doi:10.1117/12.893537; arXiv:1109.5042.

[19] A.N. Fedorova and M.G. Zeitlin, Quantum points/patterns, Part 1. From geometrical points to quantum points in sheaf framework , SPIE, vol. 8121, 81210J (2011), doi:10.1117/12.893502; arXiv:1109.5035.

[20] A.N. Fedorova and M.G. Zeitlin, En route to fusion: confinement state as a waveleton, Journal of Physics CS, vol. 490, 012202, 2014, doi:10.1088/1742-6596/490/1/012202.

[21] *A. N. Fedorova and M. G. Zeitlin*, a list of papers/preprints (including links to arXiv) can be found on web sites below.

[22] *A. N. Fedorova and M. G. Zeitlin*, From localization to zoo of patterns in complex dynamics of ensembles, This Volume.

[23] Y. Meyer, Wavelets and Operators (Cambridge Univ. Press, 1990); G. Beylkin, R. Coifman, V. Rokhlin, Fast Wavelet transform and Numerical Algorithms, Comm. Pure Applid Math, 44, 141-183, 1991.

[24] F. Auger e.a., *Time-Frequency Toolbox* (CNRS, 1996); D. Donoho, *WaveLab* (Stanford, 2000).

Antonina N. Fedorova, Michael G. Zeitlin
Mathematical Methods in Mechanics Group, IPME RAS, V.O. Bolshoj pr., 61, 199178, St. Petersburg, Russia
http://www.ipme.ru/zeitlin.html, http://mp.ipme.ru/zeitlin.html