Lagrange multiplier method implementations for two-dimensional contact problems

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Abstract

Two-dimensional elastic contact problem is considered. Finite element method with bilinear shape functions is used. The Lagrange multiplier method for contact conditions implementation is used in three ways: node-to-surface method, mortar method and advanced mortar method. In the first method integration is performed with one point from master body and one point from slave body (for each finite element), in the second method integral over a segment of master body is evaluated. The third method is more like the second, except dividing each segment of master body on subsegments according to segments of slave body. Tests showed that the mortar method and the advanced mortar method are more accurate than the node-to-surface method. The advanced mortar method is able to smooth the stress field fluctuations, but only in limited number of problems. A plane problem of contact interaction of the metal rail and composite orthotropic shell in cross-cut section of the electromagnetic accelerator barrel (railgun) is considered. Parallel software package for sparse linear systems of equations solving with MPI technology is designed.

1 Introduction

The contact of one deformable body with another has affect on in almost every mechanical structure behavior. Because of the contact problems importance, a considerable effort has been made in the modeling and numerical simulations.
The problem of elastic contact has been treated numerically by many authors [1, 2, 3, 4].
Contact problems are difficult for simulation since the most surfaces of solid bodies are rough. In addition, contact problems are often beeing simulated with mismatches meshes. Also, implementation of the stress contact conditions isn't a trivial problem. There are a few methods used for implementation of contact conditions: Lagrange multiplier method[5, 6], penalty method[6], Schwarz method [7] and others.
In this paper we consider Lagrange multiplier method for contact conditions with three variants of numerical integration: node-to-surface contact method, standart mortar method and advanced mortar method[8]. Methods efficiency are compared, numerical results are shown.

2 2d contact problem

2.1 Contact theory

Description of the problem is given in [10, 6].

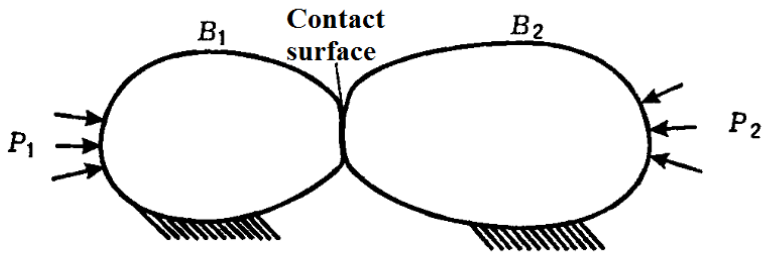


Figure 1: Contact between two bodies

Consider a 2-dimensional contact between two bodies B_1 and B_2. A contact law is characterized by a geometric condition of non-penetration:

$$g^\alpha = \min_{\boldsymbol{x}^\beta \in \Gamma^\beta} (\boldsymbol{x}^\alpha - \boldsymbol{x}^\beta) \cdot \boldsymbol{n}^\alpha \geq 0, \; \boldsymbol{x}^\alpha \in \Gamma^\alpha, \tag{1}$$

where α, β are indexes of contact bodies, Γ^α is surface of body B_α, turn to B_β, Γ^β is surface of body B_β, turn to B_α, $\boldsymbol{n}^\alpha$ is outer normal line in point x^α to contact surface for body B_α, g^α is a gap between x^α and body B_β. There is a contact between bodies if at least for one point of body B_α gap is null.
During the contact proccess distributed contact forces $\boldsymbol{t}$ are appeared. There are normal and tangent components of the contact force:

$$t_n = \boldsymbol{t} \cdot \boldsymbol{n} \leq 0, \tag{2}$$

$$t_t = \boldsymbol{t} \cdot \boldsymbol{\tau}, \tag{3}$$

where $\boldsymbol{n}$ is outer normal line to contact surface, $\boldsymbol{\tau}$ is tangent vector to contact surface.
We consider the case without fricion and sticking. In this case tangent component of contact force is equal to zero.

$$t_t = 0. \tag{4}$$

2.1.1 Basic equations

In this paper we use Cartesian coordinate system and consider a plane stress case. There are a few basic equations:
1) Cauchy tensor:

$$\{\varepsilon\} = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}\}^T = [B]\{u\}, \tag{5}$$

where $\{\varepsilon\}$ — strain tensor, $\{u\}(M) = \{u(M), v(M)\}^T$ — displacement of point M.

$$B = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}.$$

2) Hooke's law:

$$\{\sigma\} = \{\sigma_x, \sigma_y, \tau_{xy}\}^T = [H]\{\varepsilon - \varepsilon_0\}, \tag{6}$$

where $\{\sigma\}$ is a stress tensor, $\{\varepsilon\}$ is a strain tensor, $\{\varepsilon_0\}$ is a start strain tensor, (consider as zero-vector), $[H]$ is a elasticity tensor. For the case of plane strain, elasticity tensor for isotropic material becomes:

$$H = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & 0 \\ \dfrac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \dfrac{(1-2\nu)}{2(1-\nu)} \end{bmatrix},$$

where E and ν — Young's modulus and Poisson's ratio.
3) Equilibrium equations (without outer forces):

$$[B]^T\{\sigma\} = \{f\}, \tag{7}$$

4) Boundary conditions:

$$\begin{aligned} &u(M) = \tilde{u}(M), M \in \Gamma, \\ &v(M) = \tilde{v}(M), M \in \Gamma, \\ &\sigma_x(N)n_x(N) = \tilde{p}_x(N), N \in \Gamma, \\ &\sigma_y(N)n_y(N) = \tilde{p}_y(N), N \in \Gamma, \\ &\tau_{xy}(N)n_y(N) = \tilde{p}_{xy}(N), N \in \Gamma, \end{aligned}$$

where Γ is a surface of the body.
5) Contact conditions:

$$u_n^{(1)}|_{\Gamma_C} = u_n^{(2)}|_{\Gamma_C}, \tag{8}$$

$$\sigma_n^{(1)}|_{\Gamma_C} = \sigma_n^{(2)}|_{\Gamma_C}, \tag{9}$$

where $u_n^{(i)}$ and $\sigma_n^{(i)}$ are normal component of displacement and bf stress for body i, $i = 1, 2$, Γ_C is a contact surface.

2.2 Numerical method

The popular displacement finite element method is largely used for scientific computing in engineering. Without going into all the details, we present here just the algorithm for contact modeling. After finite element discretization in the context of small displacements, the global set of equilibrium equations of two contacting elastic bodies can be written as

$$[K]\{u\} = \{f\} + R, \tag{10}$$

where

$$[K] = \sum_e [k]^e, \ \{f\} = \sum_e \{f\}^e + G,$$
$$[k]^e = \int_{S^e} [B_N]^T [H][B_N] dS, \ \{f\}^e = \int_{\Gamma^e} [N^e]^T \{p\} d\Gamma,$$

$[K]$ — global stiffness matrix, $[k]^e$ — local stiffness matrix, $\{f\}$ — global right-side vector, $\{f\}^e$ — local right-side vector, G — vector of node forces, $[B_N] = [B][N]^T$ — derivative matrix for shape functions, S^e — square of element, Γ^e — load bound of body, R — contact reaction vector.

2.2.1 Contact conditions

In this paper we use a Lagrange multiplier method for contacn condotions [10]. According to the Lagrange multiplier method, we add a potential of contact forces to the potential energy of the whole system. Adding element can be written as:

$$W_C = -\int_{\Gamma_C} \Lambda \cdot (x^{(1)} - x^{(2)}) d\Gamma, \tag{11}$$

where Γ_C is a contact surface between B_1 and B_2, Λ is a Lagrange multiplier function, $x^{(i)} = X^{(i)} + u^{(i)}$ are deformed positions of the congruent points for bodies B_1 and B_2, $X^{(i)}$ and $u^{(i)}$ are start positions of the congruent points for bodies B_1 and B_2. Consider energy minimization method:

$$\delta\Pi = \int_B \delta\{\varepsilon\}^T [H] \{\varepsilon\} dS - \int_\Gamma \delta\{u\}^T \{t\} d\Gamma - \int_{\Gamma_C} \delta\Lambda \cdot (u^{(1)} - u^{(2)}) d\Gamma - \\ - \int_{\Gamma_C} \Lambda \cdot (\delta u^{(1)} - \delta u^{(2)}) d\Gamma = 0. \tag{12}$$

Components u and Λ ate independent, so we can write system of equations as:

$$\begin{cases} \int_S \delta\{\varepsilon\}^T [H] \{\varepsilon\} dS - \int_\Gamma \delta\{u\}^T \{t\} d\Gamma - \int_{\Gamma_C} \Lambda \cdot (\delta u^{(1)} - \delta u^{(2)}) d\Gamma = 0, \\ \int_{\Gamma_C} \delta\Lambda \cdot ((X^{(1)} + u^{(1)}) - (X^{(2)} + u^{(2)})) d\Gamma = 0. \end{cases} \tag{13}$$

Calculation of integral (11) consist of next steps. Let one discretizate the intefral (11). One of two bodies is called «master», another one - «slave» [8]. The main points are chosen from master body, after that we find congruent points from slave body. Then (11) can be written as:

$$W_C = -\int_{\Gamma_C} \lambda^T \cdot (x^{(m)} - x^{(s)}) d\Gamma, \tag{14}$$

where $x^{(m)}$ and $x^{(s)}$ are deformed positions for congruent points from master body and slave body, λ is a vector of Lagrange multipliers, Γ_C is contact surface on master body.
Common case for interpolation is:

$$x^{(m)} = N_\alpha(\xi)\tilde{x}_\alpha^{(m)}(t);\;\; x^{(s)} = N_\beta(\xi)\tilde{x}_\beta^{(s)}(t);\;\; \lambda = N_c(\xi)\tilde{\lambda}_c(t), \tag{15}$$

where N_α, N_β are shape functions for system (10), N_c is shape functions for Lagrange multiplier. There are three methods for interpolation in this paper — Node-to-surface contact, Standart mortar method, Advanced mortar method.
According to Node-to-surface contact method

$$N_c = \delta(\xi - \xi_c), \tag{16}$$

where ξ_c — conquerent point on slave.
Only one point from master and one point from slave are used for each element.
According to mortar method, integral carried out by quadrature on subsegments. Master shape function $N_c = \delta_{c\beta}N_\beta(\xi)$ gives standart mortar method. Partitioning subsegments in accordance with slave finite elements gives advanced mortart method. In this case (14) can be written as

$$W_C = -\sum_m \tilde{\lambda}_c^T \left[G_{c\alpha}^m \tilde{x}_\alpha^{(m)} - G_{c\beta}^s \tilde{x}_\beta^{(s)}\right] d\Gamma, \tag{17}$$

where

$$G_{c\alpha}^m = \int\limits_{\Gamma_C} N_c(\xi)N_\alpha(\xi)d\Gamma I,$$

$$G_{c\beta}^m = \int\limits_{\Gamma_C} N_c(\xi)N_\beta(\xi_s)d\Gamma I,$$

I — identity matrix.
There is vector $\lambda = \{\lambda_n, \lambda_\tau\}$ in each node of master.

2.3 Numerical results

Two numerical examples are solved for comparison by the previously describe node-to-surface, standart mortar and advanced mortar methods.

2.3.1 Two solid bars contact

One solid bar lies on another one. The first one lies on the smooth surface. The second bar is loaded with pressure p.

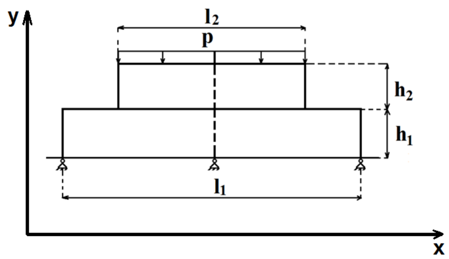
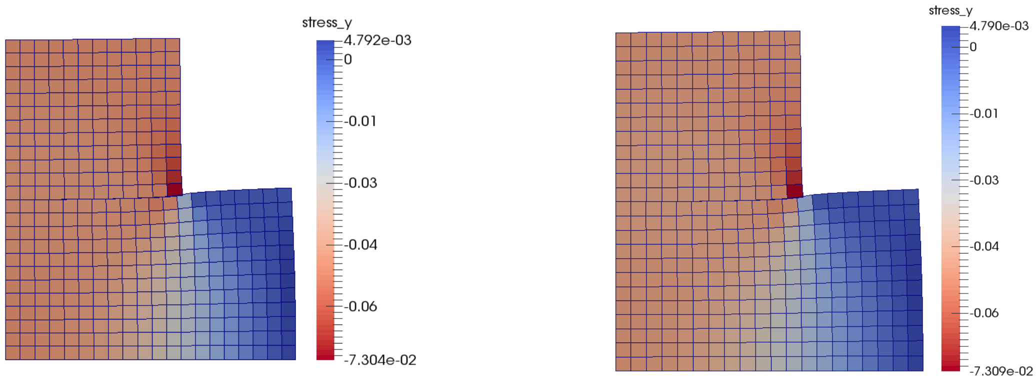


Figure 2: System of bars

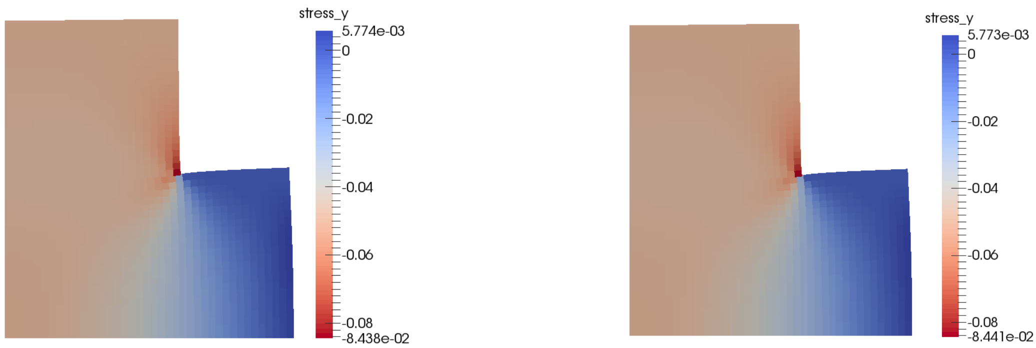
Problem is symmetrical, so we consider only a half of the area. This way, x component of displacement in the left side of both bars and y component of displacement in the bottom of first bar are equal to zero.
In Fig. 3 - 4, we present the distribution of contact stresses σ_y over the contact surface for Node-to-surface cotact and standart mortar methods. Advanced mortar method gives almost the same results as standart. Displacements scaling factor is equal to 100.



Node-to-surface contact

Standart mortar meethod

Figure 3: Step $h = 0.25$



Node-to-surface contact

Standart mortar meethod

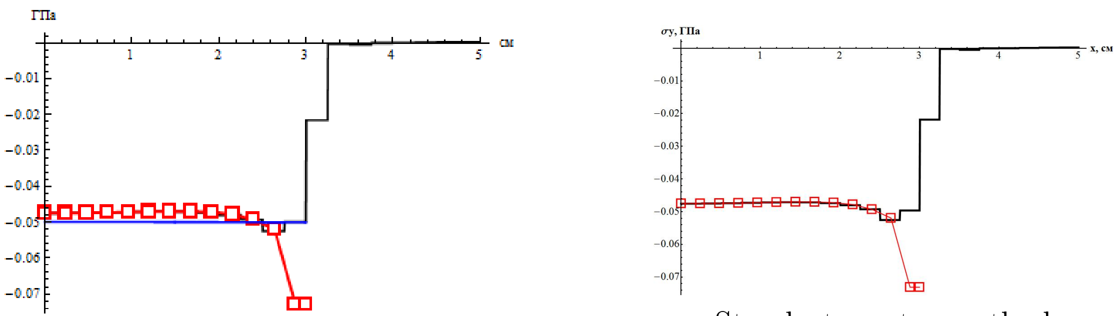
Figure 4: Step $h = 0.125$

Table 4: Input parametres

l_1	l_2	h_1	h_2	ν_1	ν_2	E_1	E_2	p
10 cm	6 cm	3 cm	3 cm	0.3	0.3	70 GPa	70 GPa	50 MPa

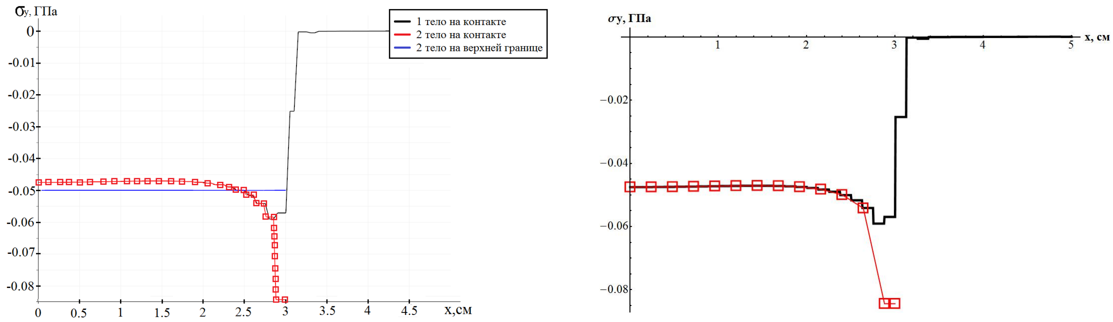
Due to geometry of the problem vertical components of displacements and stresses at the contact surface are close to the corresponding normal components. There are infinite normal stress in the corner point according to the analitical solution. Numerically obtained values of normal stresses are growing in absolute value when the step is decreasing.

Difference between results for both methods are small (0.1 percent). But the second method is more fit the physic parametres of the problem.
In Fig. 5 - 6 we present the distribution on the contact surface. Black line is used for distribution of contact stress for the first body, red line - for distribution of contact stress for the second body.



Node-to-surface contact
Standart mortar method

Figure 5: Distribution of contact stress, step $h = 0.25$



Node-to-surface contact
Standart mortar method

Figure 6: Distribution of contact stress, step $h = 0.125$

Consider the problem with mismatching meshes. Step is 0.08 cm for bottom body and 0.1 cm for top body.
In graphics 8 we present the distribution of contact stresses σ_y over the contact surface for Node-to-surface cotact and standart mortar methods. Black line is used for distribution of contact stress for the first body, red line - for distribution of contact stress for the second body.
As seen in the graphics, node-to-surface contact method gives difference of stress to the entire contact surface. Standart mortar method gives accurate contact stresses in the center of the system of bars, however,corner point, than bigger oscillations are and its amplitude is greater than in the node-to-surface contact method.

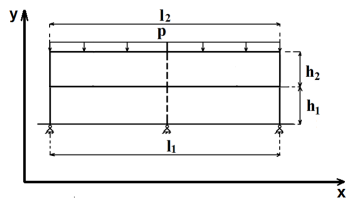
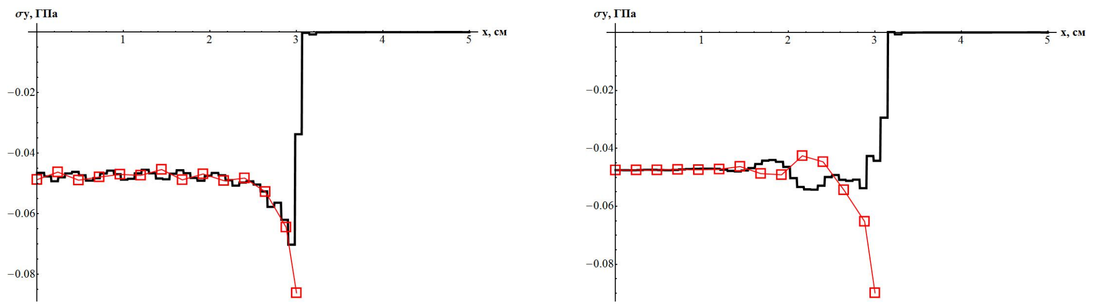


Figure 7: System of thwo bars



Node-to-surface contact method Standart mortar method

Figure 8: Distribution of contact stress, step $h = 0.08$ and $h = 0.1$

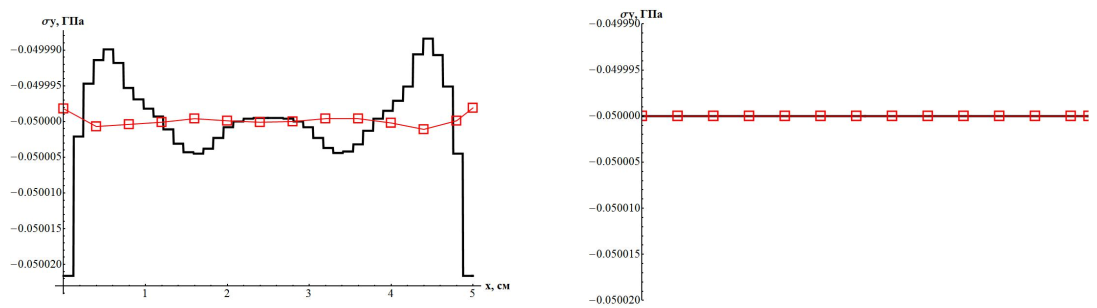
2.3.2 Two same-sized solid bars contact with mismatching meshes

Consider a contact problem with the same-sized bars to show the difference between standart mortar and advanced mortar methods (pic.7).
One solid bar lies on another one. The first one lies on the smooth surface. The second bar is loaded with a force p.
Problem is symmetrical, so we consider only a half of the area. This way, component x of displacement in the left side of both bars and component y of displacement int the bottom of first bar equals to zero.

Table 5: Input parameters

l_1	l_2	h_1	h_2	ν_1	ν_2	E_1	E_2	p
10 cm	10 cm	3 cm	3 cm	0.3	0.3	700 GPa	70 GPa	50 MPa

Consider the problem with mismatching meshes. Step is 0.12 sm for bottom body and 0.15 sm for top body.
In graphics 9 we present the distribution of contact stresses σ_y over the contact surface for Node-to-surface contact and standart mortar methods. Black line is used for distribution of contact stress for the first body, red line - for distribution of contact stress for the second body.



Standart mortar method Advanced mortar method

Figure 9: Distribution of contact stress, step $h = 0.12$ and $h = 0.15$

There is an analitic solution for this problem: $\sigma_y = p$. Standart mortar method gives oscillations in the contact surface. Advanced mortar method gives practically exact solution.

3 Conclusion

Methods for solving elastic frictionless contact problems are considered. Finite element method for numerical modeling and Lagrange multiplier method with different variants of presenting is used. Node-to-surface contact method, standart and advanced mortar method are described.
Mortar methods are more fit the physic parametres of the problem than node-to-surface contact method. But there are oscillations of stress during the contact surface. Node-to-surface contact method has difference between numerical and theoretical stress to the entire contact surface. Standart mortar method gives accurate contact stresses in the center of the system of bars, however, oscillations is appear to closer to the corner point and its amplitude is greater than in the node-to-surface contact method. Advanced mortar method gives smoother results, but in some class if problems.

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