# Functionally invariant solutions of the nonlinear nonautonomic Klein-Fock-Gordon equation

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**Abstract**

We develop methods of finding of exact analytical solutions of the nonlinear nonautonomic Klein-Fock-Gordon equation. These methods are based on finding of functionally invariant solutions of the partial differential equations. Solutions are received in the form of some arbitrary function $f(W)$ with $W$ depending on one $\tau(x, y, t)$ or two specially certain functions $\alpha(x, y, t)$, $\beta(x, y, t)$, which are called as ansatzes. Methods of construction of the ansatzes are developed.

## 1 Introduction

The nonlinear Klein-Fock-Gordon (NKFG) equation

$$U_{xx} + U_{yy} - \frac{U_{tt}}{v^2} = F(U) \tag{1}$$

appears in many field of modern natural sciences. Here $F(U)$ is arbitrary function, and the subscript means the derivative with respect to the corresponding variable. Eq. (1) is widely applied in fundamental and applied physics, mechanics, biology, chemistry and other field of science, when the right-hand side is a part of the exponential ($e^{nU}$) series, or hyperbolic function ($\sinh nU$, $\cosh nU$) series, or Fourier ($\sin nU$, $\cos nU$) series, or Taylor ($U^n$, $n = 1, 2, \ldots$) series. These equations describe the different physical phenomena and are modeling various technological processes. In case of nonhomogeneous external media or external fields the corresponding equation is the nonlinear nonautonomic Klein-Fock-Gordon equation

$$U_{xx} + U_{yy} - \frac{U_{tt}}{v^2} = p(x, y, t)\, F(U). \tag{2}$$

Here $p(x, y, t)$ is some function.
In literature there are practically no methods of obtaining exact analytical solutions of the nonautonomic NKFG equation. The methods for finding of the solutions based on construction of the functionally invariant solutions of the partial differential equations are offered below.

# 2 Methods of construction of exact analytical solutions of nonautonomic nonlinear Klein-Fock-Gordon equation

The solution $U = W$ of the differential equation is called functionally invariant if for arbitrary function $f$ the complex function of $U = f(W)$ is also the solution. The function $W$ is called as ansatzes. For the first time the idea about existence of the functionally invariant solutions were stated by C. Jacobi [1]. A. Forsyth has found the functionally invariant solutions of the Laplace equation, the wave equation and the Helmholtz equation [2]. Idea of C. Jacobi has fundamentally developed H. Bateman in relation to the theory of propagation of electromagnetic waves [3]. S.L. Sobolev and V.I. Smirnov have successfully applied method of construction of the functionally invariant solutions to the problems of diffraction and distribution of a sound in homogeneous and layered media [4]– [6]. N.P. Erugin has made a big contribution to development of the theory [7]. Functionally invariant solutions of the autonomic NKFG equation and sine-Gordon equation are found by authors [8]– [12]. We will seek the solution of Eq. (2) in the form of complex function $U = f(W)$. Then Eq. (2) takes the form

$$f'' \left[ W_x^2 + W_y^2 - \frac{W_t^2}{v^2} \right] + f' \left[ W_{xx} + W_{yy} - \frac{W_{tt}}{v^2} \right] = p(x,y,t)\, F[f(W)]. \tag{3}$$

Here and further the prime designates an ordinary derivative with respect to argument. From (3) it is possible to make two obvious statements.
**Proposition 1.** If function $W$ satisfies the equations

$$W_x^2 + W_y^2 - \frac{W_t^2}{v^2} = 0, \qquad W_{xx} + W_{yy} - \frac{W_{tt}}{v^2} = p(x,y,t), \tag{4}$$

then the solution of Eq. (2) is given by the inversion of integral

$$\int \frac{df}{F(f)} = W(x,y,t). \tag{5}$$

**Proposition 2.** If function $W$ satisfies the equations

$$W_x^2 + W_y^2 - \frac{W_t^2}{v^2} = p(x,y,t), \qquad W_{xx} + W_{yy} - \frac{W_{tt}}{v^2} = 0, \tag{6}$$

then the solution of Eq. (2) is given by the inversion of integral

$$\int \frac{df}{\sqrt{E + V(f)}} = \pm\sqrt{2}\, W(x,y,t). \tag{7}$$

In integral (7) $F(U) = V'(U)$, and $E$ is a constant of integration.
So, the problem of solution of the nonautonomic Eq. (2) is reduced to the finding of the function $W$, which satisfies Eqs. (4), (6). This problem can be solved by the methods of construction of functionally invariant solutions of the partial differential equations.

**1-st way.** The solution of Eqs. (4) we seek in the form

$$W = \varphi(\tau). \tag{8}$$

Here $\varphi(\tau)$ is arbitrary function, $\tau$ is a root of the algebraic equation

$$x\,\xi(\tau) + y\,\eta(\tau) - v^2 t\tau = \frac{s^2+q^2}{2}, \tag{9}$$
$$s^2 = x^2 + y^2 - v^2t^2, \qquad q^2 = \xi^2(\tau) + \eta^2(\tau) - v^2\tau^2,$$

$\xi(\tau)$, $\eta(\tau)$ are arbitrary functions of $\tau$.
Eq. (9) defines implicit dependence $\tau$ on time and space coordinates. From (9), we find

$$\tau_x^2 + \tau_y^2 - \frac{\tau_t^2}{v^2} = 0, \qquad \tau_{xx} + \tau_{yy} - \frac{\tau_{tt}}{v^2} = \frac{2}{\nu}, \tag{10}$$

$$\nu = \xi_\tau(x-\xi) + \eta_\tau(y-\eta) - v^2(t-\tau). \tag{11}$$

In Eq. (10) it was taking into account that

$$\xi_\tau\tau_x + \eta_\tau\tau_y + \tau_t = 1, \qquad \nu_x\tau_x + \nu_y\tau_y - \frac{\nu_t\tau_t}{v^2} = 1. \tag{12}$$

On the basis of Proposition 1 and Eqs. (10) we come to a conclusion that (8) will be the solution of Eq. (2), if $f(W)$ is an inversion of the integral (5) and

$$p(x,y,t) = \frac{\varphi_\tau}{\nu}. \tag{13}$$

For the purpose of obtaining new solutions of Eq. (2), we will introduce the function

$$\lambda = l(\tau)(x-\xi) + m(\tau)(y-\eta) - v^2w(\tau)(t-\tau). \tag{14}$$

here $l(\tau)$, $m(\tau)$, $w(\tau)$ are arbitrary functions. We will impose on them the following restrictions

$$l\xi_\tau + m\eta_\tau - v^2w = 0, \qquad l^2 + m^2 = v^2w^2. \tag{15}$$

The function $\lambda$ satisfies the equations

$$\lambda_x^2 + \lambda_y^2 - \frac{\lambda_t^2}{v^2} = 2\omega\frac{\lambda}{\nu}, \tag{16}$$

$$\lambda_{xx} + \lambda_{yy} - \frac{\lambda_{tt}}{v^2} = 3\frac{\omega}{\nu}, \tag{17}$$

$$\omega = l_\tau(x-\xi) + m_\tau(y-\eta) - v^2w_\tau(t-\tau). \tag{18}$$

Besides, it is possible to prove that function $\tau, l, m, w, \nu, \lambda$ are connected by the following relations

$$l\tau_x + m\tau_y + \omega\tau_t = \frac{\lambda}{\nu}, \tag{19}$$

$$l_\tau \tau_x + m\tau_y + \omega_\tau \tau_t = \frac{\omega}{\nu}, \tag{20}$$

$$\lambda_x \tau_x + \lambda_y \tau_y - \frac{\lambda_t \tau_t}{v^2} = \frac{\lambda}{\nu}, \tag{21}$$

$$\lambda_x \nu_x + \lambda_y \nu_y - \frac{\lambda_t \nu_t}{v^2} = \omega + (\sigma_0 - q_1^2)\frac{\lambda}{\nu}, \tag{22}$$

$$\sigma_0 = \xi_{\tau\tau}(x-\xi) + \eta_{\tau\tau}(y-\eta) - q_1^2, \qquad q_1^2 = \xi_\tau^2 + \eta_\tau^2 - v^2. \tag{23}$$

We will seek the solution of Eqs. (6) in the form of arbitrary function depending from $\tau$ and $\lambda$

$$W = f(\tau, \lambda). \tag{24}$$

Then Eqs. (6) will take a form

$$W_x^2 + W_y^2 - \frac{W_t^2}{v^2} = 2\frac{\lambda}{\nu}\left(\omega + \frac{f_\tau}{f_\lambda}\right) f_\lambda^2, \tag{25}$$

$$W_{xx} + W_{yy} - \frac{W_{tt}}{v^2} = \frac{1}{\nu}\left[2\lambda f_\lambda + f\right]_\tau + \frac{\omega}{\nu}\left[2\lambda f_\lambda + f\right]_\lambda. \tag{26}$$

From (25), (26) it follows that $W$ will satisfy Eqs. (6) if

$$2\lambda f_\lambda + f = 0, \qquad f = \frac{\varphi(\tau)}{\sqrt{\lambda}}, \tag{27}$$

$\varphi(\tau)$ are an arbitrary function.

On the basis of Proposition 2 we find that solution of Eq. (2) is an inversion of the integral (7) for

$$W = \frac{\varphi(\tau)}{\sqrt{\lambda}}, \qquad p(x,y,t) = \frac{\varphi^2}{2\nu\lambda^2}\left[\omega - 2\lambda\frac{\varphi_\tau}{\varphi}\right]. \tag{28}$$

**2-nd way.** Ansatz $\tau$, which is found from Eq. (9) depends on time and space coordinates. Set of analytical expressions for ansatz $\tau$ can be expanded assuming that $\tau$, in addition to space coordinates and time, depends on some parameters $(\alpha, \beta)$. Certainly the new equations are necessary for finding of parameters $(\alpha, \beta)$. Let's find ansatz $\tau = \tau(x, y, t, \alpha, \beta)$ from the equations

$$x\xi(\alpha,\beta,\tau) + y\eta(\alpha,\beta,\tau) - v^2 t\theta(\alpha,\beta,\tau) = \frac{s^2 + q_2^2}{2}, \tag{29}$$

$$x\xi_\alpha + y\eta_\alpha - v^2 t\theta_\alpha = \frac{1}{2}\left(q_2^2\right)_\alpha, \tag{30}$$

$$x\xi_\beta + y\eta_\beta - v^2 t\theta_\beta = \frac{1}{2}\left(q_2^2\right)_\beta, \tag{31}$$

$$q_2^2 = \xi^2 + \eta^2 - v^2\theta^2. \tag{32}$$

Eq. (29) is algebraic. This equation is linear to time and space coordinates like Eq. (9) but its coefficients are the functions of three arguments $(\tau, \alpha, \beta)$. Eqs. (30), (31) are partial differential equations of the first order. From Eqs. (29)–(31) one can calculate partial derivatives $\tau$ of the first and second orders with respect to the time and space coordinates and to receive the following relations

$$\tau_x^2 + \tau_y^2 - \frac{\tau_t^2}{v^2} = 0, \tag{33}$$

$$\tau_{xx} + \tau_{yy} - \frac{\tau_{tt}}{v^2} = \frac{1}{\nu}\left[3 - (\xi_x + \eta_y + \theta_t)\right], \tag{34}$$

On the basis of Proposition 1 and Eqs. (33), (34) we find that solution of Eq. (2) is an inversion of the integral (5), $W$ is arbitrary function of $\tau$, $W = f(\tau)$, $\tau$ is defined by Eqs. (29)–(31), and the function $p(x, y, z, t)$ is defined by the equation

$$p(x, y, t) = \frac{1}{\nu}\left[3 - (\xi_x + \eta_y + \theta_t)\right] f_\tau. \tag{35}$$

**3-d way.** Let ansatz $\alpha(x, y, t)$ is a solution of the algebraic equation

$$xl(\alpha) + ym(\alpha) - v^2 tw(\alpha) + q(\alpha) = 0. \tag{36}$$

Here $l(\alpha)$, $m(\alpha)$, $w(\alpha)$, $q(\alpha)$ are arbitrary functions coupled by the condition

$$l^2 + m^2 = v^2 w^2. \tag{37}$$

Let's introduce the function $\beta$ in according to the definition

$$\beta = xl_\alpha + ym_\alpha - v^2 tw_\alpha + q_\alpha. \tag{38}$$

From Eqs. (36), (38) one can calculate the partial derivatives of the functions $\alpha, \beta$ and verify that they satisfy the equations

$$\alpha_x^2 + \alpha_y^2 - \frac{\alpha_t^2}{v^2} = 0, \qquad \alpha_{xx} + \alpha_{yy} - \frac{\alpha_{tt}}{v^2} = 0, \tag{39}$$

$$\beta_x^2 + \beta_y^2 - \frac{\beta_t^2}{v^2} = l_\alpha^2 + m_\alpha^2 - v^2 w_\alpha^2, \quad \beta_{xx} + \beta_{yy} - \frac{\beta_{tt}}{v^2} = \frac{2}{\beta}(l_\alpha^2 + m_\alpha^2 - v^2 w_\alpha^2). \tag{40}$$

$$\alpha_x \beta_x + \alpha_y \beta_y - \frac{\alpha_t \beta_t}{v^2} = 0. \tag{41}$$

In Eqs. (40) the relation

$$l\, l_{\alpha,\alpha} + m m_{\alpha,\alpha} - v^2 w w_{\alpha,\alpha} = -(l_\alpha^2 + m_\alpha^2 - v^2 w_\alpha^2) \tag{42}$$

has been taken into account.
Let's that $W$ is an arbitrary functions of the two variables $(\alpha, \beta)$

$$W = f(\alpha, \beta) \tag{43}$$

Taking into account (39)–(41), we find

$$W_x^2 + W_y^2 - \frac{W_t^2}{v^2} = f_\beta^2\left(l_\alpha^2 + m_\alpha^2 - v^2 w_\alpha^2\right), \tag{44}$$

$$W_{xx} + W_{yy} - \frac{W_{tt}}{v^2} = \left( f_{\beta\beta} + \frac{2}{\beta} f_\beta \right) \left( l_\alpha^2 + m_\alpha^2 - v^2 w_\alpha^2 \right). \tag{45}$$

Let's

$$f_{\beta\beta} + \frac{2}{\beta} f_\beta = 0, \qquad f = A(\alpha) + \frac{B(\alpha)}{\beta}, \tag{46}$$

$A(\alpha)$, $B(\alpha)$ are arbitrary functions. Therefore, the function $W$, which is given by (43), (46), satisfies Eqs. (6) with

$$p(x, y, t) = \left( l_\alpha^2 + m_\alpha^2 - v^2 w_\alpha^2 \right) \frac{B^2(\alpha)}{\beta^4}. \tag{47}$$

The obtained results allows to construct the solution of Eq. (2) on the basis of the Proposition 2.
Ansatz $\alpha$ is possible to use for construction of the function

$$W = xL + yM - v^2 tV + Q. \tag{48}$$

Here

$$L_\alpha = l, \quad M_\alpha = m, \quad V_\alpha = w, \quad Q_\alpha = q. \tag{49}$$

The function $W$, defining by (48), (49), satisfies the equations

$$W_x^2 + W_y^2 - \frac{W_t^2}{v^2} = L^2 + M^2 - v^2 V^2, \tag{50}$$

$$W_{xx} + W_{yy} - \frac{W_{tt}}{v^2} = -\frac{1}{\beta} \left( l^2 + m^2 - v^2 w^2 \right). \tag{51}$$

From (50), (51) it follows that $W$ will be the solution of Eqs. (4) if

$$L^2 + M^2 - v^2 V^2 = 0, \quad p = -\frac{1}{\beta} \left( l^2 + m^2 - v^2 w^2 \right), \tag{52}$$

or the solution of (6), if

$$l^2 + m^2 - v^2 w^2 = 0, \quad p = L^2 + M^2 - v^2 V^2. \tag{53}$$

Thus, if the functions $l, m, w, q$ satisfying to the conditions (52) or (53) are obtained, then one can find the function $W$ and to construct the solution of Eq. (2).
So, if

$$L = \frac{1}{\cosh \alpha}, \quad M = \frac{\sinh \alpha}{\cosh \alpha}, \quad vV = 1, \quad Q = 0, \tag{54}$$

then

$$W = vt - \sqrt{x^2 + y^2}, \quad p = \frac{1}{\sqrt{x^2 + y^2}}. \tag{55}$$

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