On homoclinics and heteroclinics of Lagrangian systems in a non-stationary force field

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Abstract

We study homoclinic and heteroclinic orbits of a natural Lagrangian systems defined on a complete Riemannian manifold being subjected to action of a non-stationary potential force field. It is assumed that Lagrangian of such a system can be written in the following form $L(q, \dot{q}, t) = K(q, \dot{q}) - U(q, t)$, where the kinetic energy K is a positive definite quadratic form in velocity $\dot{q}$ and the potential $U(q, t)$ has special representation $U(q, t) = f(t)V(q)$. We also assume that there exists $t_0 \in \mathbb{R}$ such that $f(t_0) = 0$, i.e. at the moment t_0 the system becomes free and $|f(t)|$ is monotonic on both intervals $t > t_0$ and $t < t_0$. Let X_+, X_- denote the set of isolated critical points of $V(x)$ at which $U(x, t)$ distinguishes its maximum for $t > t_0$ and $t < t_0$, respectively. Under nondegeneracy conditions on points of $X_\pm$ we prove the existence of infinitely many doubly asymptotic trajectories connecting X_- and X_+.

1 Introduction

During the last two decades many authors studied connecting (i.e. homoclinic and heteroclinic) orbits of Lagrangian systems by use of variational methods and critical points theory [1]-[4]. Being a part of intersection of invariant manifolds associated with some hyperbolic objects such trajectories usually lead to chaotic dynamics of a system. In the present work we study a natural Lagrangian system on a complete Riemannian manifold. Namely, we consider a compact Riemannian manifold $\mathcal{M}$ to be configuration space of a Lagrangian system with Lagrangian $L \in C^2(T\mathcal{M} \times \mathbb{R}, \mathbb{R})$ such that

$$L(q, \dot{q}, t) = K(q, \dot{q}) - U(q, t), \tag{1}$$

where the kinetic energy $K \in C^2(T\mathcal{M}, \mathbb{R})$ is a positive definite quadratic form in velocity $\dot{q}$, while the potential energy $U(q, t) \in C^2(\mathcal{M} \times \mathbb{R}, \mathbb{R})$ has the following special form

$$U(q, t) = f(t)V(q). \tag{2}$$

The equations of motion take the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = -f(t)\frac{\partial V}{\partial q}. \tag{3}$$

In addition we suppose that f satisfies the following assumptions:
(A_1) there exists $t_0 \in \mathbb{R}$ such that $f(t_0) = 0$,
(A_2) the function $|f|$ is strictly monotone for $t > t_0$ and $t < t_0$,
(A_3) $|f(t)| \to +\infty$ as $t \to \pm\infty$,
(A_4) there exist constants $\mu, T > 0$ such that $|f(t)| > \mu|f'(t)|$ for all $|t| > T$.
Making a time shift and changing, if necessary, the time sign we may always suppose that $t_0 = 0$ and the function f is increasing on the interval $\mathbb{R}_+ = (0, +\infty)$ and, hence, $f(+\infty) = +\infty$. We also denote $\mathbb{R}_- = (-\infty, 0)$. The simplest examples of functions f satisfying $A_1 - A_4$ may be given by t^m, $sinh(\alpha t)$ and $\log(1+\alpha t^2)$ for some $m \in \mathbb{N}$, $\alpha > 0$.
Since $\mathcal{M}$ is a compact manifold, the function V has minimum and maximum on $\mathcal{M}$. Denote by X_+, X_- the subsets of $\mathcal{M}$ on which $U(x,t)$ distinguishes global maximum for $t > 0$ and $t < 0$, respectively. We suppose that $X_\pm$ consist of isolated critical points of V and that they are nondegenerate.
We will say that a solution $q : \mathbb{R} \to \mathcal{M}$ is a *homoclinic* (*heteroclinic*) solution if there exist $z_1, z_2 \in \mathcal{M}$ (for homoclinic solution $z_2 = z_1$) such that q joins z_1 to z_2, i.e. $\lim\limits_{t\to-\infty} q(t) = z_1$, $\lim\limits_{t\to+\infty} q(t) = z_2$ and $\lim\limits_{t\to\pm\infty} \dot{q}(t) = 0$.

Theorem 1. *Under assumptions A_1 - A_4 for any $z_1 \in X_-$ and $z_2 \in X_+$ there exist infinitely many heteroclinic (homoclinic) trajectories emanating from z_1 and terminating at z_2.*

Remarks. - 1.One may note that the sets X_+, X_- do not coincide in the case $f(-\infty) = -\infty$ what lead to connection between different regions of the configuration space.
2. If we denote by Y_+, Y_- the subsets of $\mathcal{M}$ consisting of minima of $U(x,t)$ for positive and negative t, then under weak additional assumptions almost each trajectory $q(t)$ tends to a point from $Y_\pm$ as $t \to \pm\infty$ [6]. However, $\limsup_{t\to\pm\infty} |\dot{q}(t)| = \infty$ and these solutions cannot be called asymptotic.
Systems of type (3) often arise in mechanical applications. In particular, this work was inspired by [1], where the authors applied Routh's reduction to a Lagrangian system with symmetry which led to a system (3) with $f(t) = t^2$. In particular, they studied the Kirchhoff problem. We also refer to [5], [6], [9], [3] where the systems of type (3) are discussed.

2 Variational settings

We use variational arguments to prove Theorem 1. Consider a smooth embedding of the manifold $\mathcal{M}$ into $\mathbb{R}^N$ for $N = 2n+1$ with $n = \dim \mathcal{M}$ and denote by $\langle \cdot, \cdot \rangle$ the Euclidean structure in $\mathbb{R}^N$ together with its restriction to $\mathcal{M}$. Let ∇ stands for the gradient operator with respect to the variable x. First we study the case when each of the subsets $X_\pm$ consists of one point, i.e. $X_\pm = x_\pm$.
Denote

$$\mathcal{E}_f = \left\{ v \in AC(\mathbb{R}, \mathbb{R}^N) : \|v\|_f^2 = \int\limits_{-\infty}^{+\infty} \left(|\dot{v}(t)|^2 + |f(t)| \cdot |v(t)|^2\right) \mathrm{d}t < \infty \right\}, \tag{4}$$

where $AC(\mathbb{R},\mathbb{R}^N)$ is the set of absolutely continuous curves from $\mathbb{R}$ to $\mathbb{R}^N$.
Let $\mathcal{E}_1 = W^{1,2}(\mathbb{R},\mathbb{R}^N)$ be the Sobolev space with the norm $\|\cdot\|_1$ such that $\|v\|_1^2 = \int\limits_{-\infty}^{+\infty}\big(|\dot v(t)|^2 + |v(t)|^2\big)\mathrm{d}t$.
One may prove the following lemma.

Lemma 1. *$\mathcal{E}_f \subset \mathcal{E}_1$ and $\|v\|_1 \le C_f\|v\|_f$ with $C_f^2 = \max\{(b-a)^2+1, 2\}$, where $a, b \in \mathbb{R}$ such that $a<0$, $b>0$ and $|f(a)| = |f(b)| = 1$.*

Since $\mathcal{E}_1$ is continuously embedded into $C^0(\mathbb{R},\mathbb{R}^N)$ with $\|v\|_\infty = \sup_{t\in\mathbb{R}}|v(t)| \le \|v\|_1$ we arrive at the following lemma.

Lemma 2. *$\mathcal{E}_f$ is a Hilbert space.*

PROOF: - Let $\{v_n\}, v_n \in \mathcal{E}_f$ be a Cauchy sequence. Then due to Lemma 1 $\{v_n\}$ is a Cauchy sequence in $\mathcal{E}_1$. Since $\mathcal{E}_1$ is complete there exists $v = \lim_{n\to\infty} v_n \in \mathcal{E}_1$. One may extract a subsequence $\{v_{n_k}\}$ such that $\|v_{n_k} - v\|_\infty \to 0$ and $\dot v_{n_k} \to \dot v$ almost everywhere. Fix $\varepsilon > 0$ and take $k(\varepsilon)$ such that for any $l, m > k(\varepsilon)$ $\|v_{n_l} - v_{n_m}\|_f < \varepsilon$. Then by the Fatou lemma we have

$$\|v_{n_l} - v\|_f^2 = \int\limits_{-\infty}^{\infty} \lim_{m\to\infty}\big(|\dot v_{n_l}(t) - \dot v_{n_m}(t)|^2 + |f(t)|\cdot|v_{n_l}(t) - v_{n_m}(t)|^2\big)\mathrm{d}t \le$$
$$\le \liminf_{m\to\infty}\|v_{n_l} - v_{n_m}\|_f^2 \le \varepsilon^2.$$

Due to arbitrary choice of ε one concludes that $\|v_{n_l} - v\|_f \to 0$ as $l\to\infty$. □
The next lemma gives an estimate on the absolute value of an element of the space $\mathcal{E}_f$.

Lemma 3. *If $v \in \mathcal{E}_f$ then*

$$|v(t)| \le \frac{\sqrt{2}}{|f(t)|^{1/4}}\|v\|_f.$$

PROOF: - Define $g(t) = |f(t)|^{1/2}$. For any $v \in \mathcal{E}_f$ consider

$$g(t)|v(t)|^2 = \int\limits_{-\infty}^{t} \dot g(s)|v(s)|^2\mathrm{d}s + 2\int\limits_{-\infty}^{t} g(s)\langle v(s), \dot v(s)\rangle\mathrm{d}s. \tag{5}$$

Note that (5) is valid if $-\infty$ is replaced by $+\infty$. Then substituting $t=0$ into (5) and using monotonicity of g on $\mathbb{R}_\pm$ together with the Schwartz inequality one gets

$$\Big|\int\limits_{\mathbb{R}_\pm} \dot g(s)|v(s)|^2\mathrm{d}s\Big| = 2\Big|\int\limits_{\mathbb{R}_\pm} g(s)\langle v(s), \dot v(s)\rangle\mathrm{d}s\Big| \le \int\limits_{\mathbb{R}_\pm}\big(|\dot v(s)|^2 + g^2(s)|v(s)|^2\big)\mathrm{d}s. \tag{6}$$

Finally, (5) and (6) give $g(t)|v(t)|^2 \le 2\int\limits_{-\infty}^{+\infty}\big(|\dot v(s)|^2 + g^2(t)|v(s)|^2\big)\mathrm{d}s = 2\|v\|_f^2$. □

Denote

$$\mathfrak{M} = \Big\{ q \in AC(\mathbb{R}, \mathbb{R}^N) : q(t) \in \mathcal{M} \text{ for each } t \in \mathbb{R} \text{ and } \int_{-\infty}^{+\infty} \Big(|\dot{q}(t)|^2 + |f(t)| \cdot \ |q(t) - \chi(t)|^2 \Big) \mathrm{d}t < \infty \Big\},$$

where the function $\chi(t) = x_-$ for $t < 0$ and $\chi(t) = x_+$ for $t \geq 0$.

Proposition 1. *The set $\mathfrak{M}$ is a Hilbert manifold of class C^2 with tangent space at q given by*

$$T_q\mathfrak{M} = \Big\{ v \in \mathcal{E}_f : v(t) \in T_{q(t)}\mathcal{M} \ \ \textit{for all } t \in \mathbb{R} \Big\}.$$

PROOF: - It follows from lemmae 1 and 3 that if a sequence $\{q_k\}$ converges to q in $\mathfrak{M}$ with respect to (4) then it also converges to q uniformly on $\mathbb{R}$. Since $q(t) \to x_\pm$ as $t \to \pm\infty$ for any $q \in \mathfrak{M}$ we may use local coordinates in $\mathbb{R}^N$ and apply the classical arguments [7] as in the case of the space

$$\Omega(\mathcal{M}, x_-, x_+) = \{ q \in C^0([0,1], \mathcal{M}) : \ q(0) = x_-, \ q(1) = x_+ \}.$$

Further there will be given an alternative description of the Hilbert structure through the representation (10). □

Define $\hat{L}(q, \dot{q}, t) = L(q, \dot{q}, t) + f(t)V(\chi(t))$ and consider the Hamilton action I defined on $\mathfrak{M}$:

$$I(q) = \int_{-\infty}^{+\infty} \hat{L}(q, \dot{q}, t) \mathrm{d}t. \tag{7}$$

Since $K(q, \dot{q})$ is a homogeneous quadratic form in $\dot{q}$ and $U(q,t)$ has unique maximum at $x_\pm$ for all $t \in \mathbb{R}_\pm$ there exist constants $\alpha, \beta > 0$ such that

$$\alpha(|\dot{q}|^2 + |f(t)||q - \chi(t)|^2) \leq |L_\pm(q, \dot{q}, t)| \leq \beta(|\dot{q}|^2 + |f(t)||q - \chi(t)|^2) \tag{8}$$

for all $(q, \dot{q}) \in T\mathcal{M}$ and consequently the integral (7) converges for any $q \in \mathfrak{M}$. The following lemma plays an important role in further analysis.

Lemma 4. *For any constants $r, c > 0$ there exists $T(r,c)$ such that $q(t) \in B_r(x_\pm)$ for all $q \in \mathfrak{M}_c$ and $|t| > T(r,c)$, where $\mathfrak{M}_c = \{ q \in \mathfrak{M} : I(q) < c \}$, $B_r(x) = \{ x \in \mathbb{R}^N : |x| < r \}$.*

PROOF: - First we observe that if $I(q) < c$ then $\int_{-\infty}^{+\infty} \big(|\dot{q}(s)|^2 + |f(s)||q(s) - \chi(s)|^2 \big) \mathrm{d}s < \alpha^{-1}c$. Applying to the function $q(t) - \chi(t)$ the same arguments as in lemma 3 we get

$$|q(t) - \chi(t)|^2 < \frac{\alpha^{-1}c}{|f(t)|^{1/2}}. \tag{9}$$

Since $|f|$ is increasing and unbounded (9) proves the lemma. $\square$
Now we represent the functional I in a way suggested in [1]. Namely, one may construct an embedding of $\mathcal{M}$ into $\mathbb{R}^N$ such that small neighborhoods of $x_\pm$ lie in linear subspaces $\mathbb{R}^{n,\pm} \subset \mathbb{R}^N$. Taking $r > 0$ to be sufficiently small we may always assume that these neighborhoods coincide with the balls $B_r(x_\pm) = \{x \in \mathbb{R}^{n,\pm} : |x - x_\pm| < r\}$.
Then for any $c > 0$ set

$$\Lambda = W^{1,2}\big([-T,T],\mathcal{M}\big),$$
$$\Lambda_- = \big\{q \in AC((-\infty,-T],B_r(x_-) : \|q - x_-\|_{f,-} < \infty\big\},$$
$$\Lambda_+ = \big\{q \in AC([T,+\infty),B_r(x_+) : \|q - x_+\|_{f,+} < \infty\big\},$$

where $T = T(r,c)$ from lemma 4 and $\|v\|^2_{f,-} = \int\limits_{-\infty}^{-T} \big(|\dot v(s)|^2 + |f(s)||v(s)|^2\big)\mathrm{d}s$,

$\|v\|^2_{f,+} = \int\limits_{T}^{+\infty} \big(|\dot v(s)|^2 + |f(s)||v(s)|^2\big)\mathrm{d}s$.
Define mappings $g_\pm : \mathfrak{M} \to \Lambda_\pm$ and $g : \mathfrak{M} \to \Lambda$ as

$$g_-(q) = q|_{(-\infty,-T]}, \qquad g(q) = q|_{[-T,T]}, \qquad g_+(q) = q|_{[T,+\infty)}.$$

Lemma 5. *The mappings $g, g_\pm$ are of class C^∞.*

PROOF: - Note that the map $g_- \times g \times g_+ : \mathfrak{M} \to \Lambda_- \times \Lambda \times \Lambda_+$ identifies $\mathfrak{M}_c$ with an open set in the Hilbert submanifold $\{(z_-, z_0, z_+) \in \Lambda_- \times \Lambda \times \Lambda_+ : z_\pm(\pm T) = z_0(\pm T)\}$ of codimentsion $2n$ in $\Lambda_- \times \Lambda \times \Lambda_+$. Hence the lemma immediately follows from the equality $\|q\|^2_f = \|g_-(q)\|^2_{f,-} + \|g(q)\|^2_{f,0} + \|g_+(q)\|^2_{f,+}$ for all $q \in \mathfrak{M}_c$, where $\|v\|^2_{f,0} = \int\limits_{-T}^{-T} \big(|\dot v(s)|^2 + |f(s)||v(s)|^2\big)\mathrm{d}s$. $\square$
Now we may represent the restriction $I|_{\mathfrak{M}_c}$ in the following way [1]:

$$I|_{\mathfrak{M}^c} = J_- \circ g_- + J \circ g + J_+ \circ g_+, \tag{10}$$

where the functionals $J : \Lambda \to \mathbb{R}$, $J_\pm : \Lambda_\pm \to \mathbb{R}$ are

$$J_-(q) = \int\limits_{-\infty}^{-T} \hat L(q,\dot q,t)\mathrm{d}t, \quad J(q) = \int\limits_{-T}^{T} \hat L(q,\dot q,t)\mathrm{d}t, \quad J_+(q) = \int\limits_{T}^{\infty} \hat L(q,\dot q,t)\mathrm{d}t.$$

Proposition 2. *The functional $I \in C^1(\mathfrak{M})$ with locally Lipschitz derivative.*

PROOF: - The proof of this proposition is rather straightforward. First we observe that the functional J is of class C^2 on $\mathfrak{M}$ (see e.g. [2]). Hence it remains to consider the funcionals $J_\pm$. They can be studied in analogous ways, so we consider only J_+. To prove smoothness of J_+, take sufficiently small r, $q \in \Lambda_+(r,c)$ and $v \in T_x\Lambda_+$ such that $\|v\|_f < r/C_f$. Then due to lemma 3 we see that $\|v\|_\infty < r$ and $q(t) + v(t) \in B_{2r}(x_+)$ for all $t > T(r,c)$. Due to smallness of r we may assume to be on $\mathbb{R}^n$ and use Taylor's formula:

$$L(q(t)+v(t),\dot q(t)+\dot v(t),t) - L(q(t),\dot q(t),t) =$$
$$= \langle K_q(q(t),\dot q(t)),v(t)\rangle + \langle K_{\dot q}(q(t),\dot q(t)),\dot v(t)\rangle - \langle U_q(q(t),t),v(t)\rangle +$$
$$+\frac{1}{2}\langle K_{qq}v(t),v(t)\rangle + \langle K_{q\dot q}\dot v(t),v(t)\rangle + \frac{1}{2}\langle K_{\dot q\dot q}\dot v(t),\dot v(t)\rangle - \frac{1}{2}\langle U_{qq}v(t),v(t)\rangle,$$

where the second derivatives of K and U are evaluated at a point $(q(t) + \theta(t)v(t), \dot{q}(t) + \theta(t)\dot{v}(t), t)$ with measurable function $\theta(t) \in [0,1]$. By assumptions on K and U, for any $(q, \dot{q}, t) \in B_{2r}(x_+) \times \mathbb{R}^n \times \mathbb{R}$ one has

$$\|K_{qq}\| \leq c_1|\dot{q}|^2,\ \|K_{q\dot{q}}\| \leq c_2|\dot{q}|,\ \|K_{\dot{q}\dot{q}}\| \leq c_3,\ \|U_{qq}\| \leq c_4|f(t)| \cdot |q|^2,$$

for some positive constants c_i and $\|\cdot\|$ standing for the operator norm. Using these estimates, lemma 3 and the Schwartz inequality we get

$$\begin{aligned}
\left|\int_T^{+\infty} \langle K_{qq}v(t), v(t)\rangle \mathrm{d}t\right| &\leq c_1 \int_T^{+\infty} |\dot{q}(t) + \theta(t)\dot{v}(t)|^2 |v(t)|^2 \mathrm{d}t \leq \\
&\leq c_1\|v\|_\infty^2 (\|q - x_+\|_{f,+}^2 + \|v\|_{f,+}^2) \leq c_5\|v\|_{f,+}^2 \\
\left|\int_T^{+\infty} \langle K_{q\dot{q}}v(t), v(t)\rangle \mathrm{d}t\right| &\leq c_2 \int_T^{+\infty} |\dot{q}(t) + \theta(t)\dot{v}(t)||\dot{v}(t)||v(t)| \mathrm{d}t \leq \\
&\leq c_2\|v\|_\infty (\|q - x_+\|_{f,+}\|v\|_{f,+} + \|v\|_{f,+}^2) \leq c_6\|v\|_{f,+}^2 \qquad (11) \\
\left|\int_T^{+\infty} \langle K_{\dot{q}\dot{q}}v(t), v(t)\rangle \mathrm{d}t\right| &\leq c_3 \int_T^{+\infty} |\dot{v}(t)|^2 \mathrm{d}t \leq c_7\|v\|_{f,+}^2 \\
\left|\int_T^{+\infty} \langle U_{q\dot{q}}v(t), v(t)\rangle \mathrm{d}t\right| &\leq c_4 \int_T^{+\infty} |f(t)||q(t) + \theta(t)v(t)|^2 |v(t)|^2 \mathrm{d}t \leq \\
&\leq c_4\|q - x_+\|_{f,+}^2 \|v\|_{f,+}^2 \leq c_8\|v\|_{f,+}^2
\end{aligned}$$

Estimates (11) prove differentiability of J_+ on $\mathfrak{M}$ and provide an expression for J'_+:

$$J'_+(q)v = \int_T^{+\infty} \Big(\langle K_q(q(t), \dot{q}(t)), v(t)\rangle + \langle K_{\dot{q}}(q(t), \dot{q}(t)), \dot{v}(t)\rangle - |f(t)|\langle V_q(q(t)), v(t)\rangle \Big) \mathrm{d}t.$$

Now take $q_1, q_2 \in \Lambda_+(2r)$, $v \in T\Lambda_+$ and consider the difference

$$J'_+(q_2)v - J'_+(q_1)v = \int_T^{+\infty} \Big(\big\langle K_{qq}\Delta q(t), v(t)\big\rangle + \big\langle K_{q\dot{q}}v(t), \Delta\dot{q}(t)\big\rangle +$$

$$\big\langle K_{q\dot{q}}\Delta q(t), \dot{v}(t)\big\rangle + \big\langle K_{\dot{q}\dot{q}}\Delta\dot{q}(t), \dot{v}(t)\big\rangle - |f(t)|\big\langle V_{qq}\Delta q(t), v(t)\big\rangle \Big) \mathrm{d}t,$$

where $\Delta q(t) = q_2(t) - q_1(t)$ and the second derivatives of K and V are evaluated at a point $(q(t) + \theta(t)v(t), \dot{q}(t) + \theta(t)\dot{v}(t), t)$. Using (11) one may get

$$\left| J'_+(q_2)v - J'_+(q_1)v \right| \leq c_9\|q_2 - q_1\|_{f,+}\|v\|_{f,+}$$

with some positive constant c_9, what proves that J'_+ is locally Lipschitz. $\square$

Proposition 3. *Critical points of I correspond to doubly asymptotic trajectories of (3) such that $q(t) \to x_\pm$ and $\dot{q}(t) \to 0$ as $t \to \pm\infty$.*

PROOF:- Assume $q \in \mathfrak{M}$ is a critical point of I. Then $I'(q)v = 0$ for any $v \in T_q\mathfrak{M}$. Using classical arguments (see e.g. [7]) one may show that q is of class $C^2(\mathbb{R})$ and satisfies the Lagrange equations (3). By lemma 3 $q(t) \to x_\pm$ as $t \to \pm\infty$. Hence it remains to prove that $\dot{q}(t) \to 0$ as $t \to \pm\infty$. The following lemma admits this statement:

Lemma 6. *If $q(t)$ satisfies the Lagrange equations (3) and $\lim\limits_{t\to\pm\infty} |q(t) - x_\pm| = 0$ then $\lim\limits_{t\to\pm\infty} |\dot{q}(t)| = 0$.*

PROOF: - Introduce a symmetric positive definite operator $A(x) : T\mathcal{M} \to T^*\mathcal{M}$ by the formula $K(x, \dot{x}) = \frac{1}{2}\langle A(x)\dot{x}, \dot{x}\rangle$ and rewrite the Lagrange equations as

$$A(q)\ddot{q} + \frac{1}{2}\langle B(q)\dot{q}, \dot{q}\rangle + f(t)\nabla V(q) = 0, \tag{12}$$

where $B(x) = A'(x)$.
We consider the case $t \to +\infty$ (the case $t \to -\infty$ can be treated similarly). Let $\mathcal{A} = A(x_+)$ and $h(t) = \langle \mathcal{A}(q(t) - x_+), q(t) - x_+\rangle$. Then $\dot{h}(t) = \langle \mathcal{A}\dot{q}(t), q(t) - x_+\rangle$ and

$$\ddot{h}(t) = \langle \mathcal{A}\ddot{q}(t), q(t) - x_+\rangle + \langle \mathcal{A}\dot{q}(t), \dot{q}(t)\rangle. \tag{13}$$

Taking sufficiently small $r > 0$ we define T such that $q(t) \in B_r(x_+)$ for all $t > T$. We note that there exist positive constants $C_A^{(1)}, C_A^{(2)}, C_B$ such that for all $x \in B_r(x_+)$

$$C_A^{(1)} \le \|A(x)\| \le C_A^{(2)}, \;\; \|B(x)\| \le C_B. \tag{14}$$

. Using (12), (13) we conclude

$$\ddot{h}(t) \ge \delta\big(|\dot{q}(t)|^2 + f(t)|q(t) - x_+|^2\big) \tag{15}$$

with some positive constant δ. Integrating (15) we get for any $t_2 > t_1 > T$

$$\left|\dot{h}(t_2) - \dot{h}(t_1)\right| \ge \delta \int\limits_{t_1}^{t_2} \big(|\dot{q}(s)|^2 + f(s)|q(s) - x_+|^2\big)\mathrm{d}s. \tag{16}$$

Since $h(t) \to 0$ as $t \to +\infty$ we have $\liminf\limits_{t\to\infty} |\dot{h}(t)| = 0$ and by (16) $\|q - x_+\|_{f,+} < \infty$. We now prove that $f(t)|q(t) - x_+|^2 \to 0$ as $t \to +\infty$. Note that (15) also implies

$$\ddot{h}(t) \ge \frac{\delta f(t)}{C_A^{(2)}} h(t).$$

Then fix $\varepsilon > 0$ and assume that there exists a sequence $t_k \to +\infty$ such that $f(t_k)h(t_k) > \varepsilon$. Without loss of generality we may suppose that $\dot{h}(t_k) < 0$ (or $\dot{h}(t_k) > 0$) for all k. Since $\ddot{h}(t) > 0$ and

$$\dot{h}(s) = \dot{h}(t_k) - \int\limits_{s}^{t_k} \ddot{h}(p)\mathrm{d}p$$

we obtain that $\dot{h}(t) < 0$ for all $t \in [T, t_k]$. Hence $h(t)$ is monotonically decreasing for $t > T$ and $h(s) > h(t_k) > \varepsilon/f(t_k)$. By (A_4) one may also assume that

$$\int_{t_{k-1}}^{t_k} f(s)\mathrm{d}s > \mu f(t_k).$$

Then

$$\int_{t_{k-1}}^{t_k} f(s)|q(s) - x_+|^2\mathrm{d}s \geq \frac{2}{C_A^{(2)}} \int_{t_{k-1}}^{t_k} f(s)h(s)\mathrm{d}s \geq \frac{2\varepsilon}{f(t_k)} \int_{t_{k-1}}^{t_k} f(s) \geq \frac{2\varepsilon\mu}{C_A^{(2)}}.$$

Summarizing over k we get a contradiction with $\|q - x_+\|_{f,+} < \infty$.
To prove that $|\dot{q}(t)| \to 0$ as $t \to +\infty$ introduce the Hamiltonian $H(q, \dot{q}, t) = K(q, \dot{q}) + f(t)(V(q) - V(x_+))$. If q satisfies the Lagrange equations then $\dot{H} = -f'(t)(V(q) - V(x_+))$ and by (A_4) we have for any $t_2 > t_1 > T$ that

$$\Big|H(q(t_2), \dot{q}(t_2), t_2) - H(q(t_1), \dot{q}(t_1), t_1)\Big| \leq \beta\mu \int_{t_1}^{t_2} f(s)|q(s) - x_+|^2\mathrm{d}s$$

or equivalently

$$\Big|\langle A(q(t_2))\dot{q}(t_2), \dot{q}(t_2)\rangle - \langle A(q(t_1))\dot{q}(t_1), \dot{q}(t_1)\rangle\Big| \leq \tag{17}$$
$$\leq \beta \int_{t_1}^{t_2} f(s)|q(s) - x_+|^2\mathrm{d}s + \beta\Big(f(t_2)|q(t_2) - x_+|^2 - f(t_1)|q(t_1) - x_+|^2\Big).$$

Since $q \in \Lambda_+$ then $\liminf_{t\to\pm\infty} |\dot{q}(t)|^2 = 0$, $\int_T^\infty |f(s)||q(s) - x_+|^2\mathrm{d}s < +\infty$ and $\lim_{t\to+\infty} f(t)|q(t) - x_+| = 0$. Then by (17) for any $\varepsilon > 0$ there exists $\tau(\varepsilon)$ such that for any $t_2 > t_1 > \tau(\varepsilon)$

$$\Big|\langle A(q(t_2)\dot{q}(t_2), \dot{q}(t_2)\rangle - \langle A(q(t_1)\dot{q}(t_1), \dot{q}(t_1)\rangle\Big| \leq \varepsilon.$$

$A(x)$ is bounded from below and we conclude that $\dot{q}(t) \to 0$ as $t \to \pm\infty$. □

3 The Palais-Smale conditions

Proposition 4. *The functional I satisfies the Palais-Smale conditions.*

PROOF: - Let $q_n \in \mathfrak{M}$ be a PS sequence, i.e. a sequence for which $I(q_n)$ is bounded and $I'(q_n) \to 0$ as $n \to \infty$. Then there exists $c > 0$ such that $q_n \in \mathfrak{M}_c$ and due to lemma 4 $\{q_n\}$ is uniformly bounded. Hence, up to a subsequence, q_n converges uniformly to a function $q_\infty \in L_\infty(\mathbb{R}, \mathcal{M})$. For the constant c and some small $\varepsilon > 0$

we define $T = T(\varepsilon, c)$ as in lemma 4 and introduce the sets $\Lambda, \Lambda_\pm$ to represent the functional I in the form (10).
Denote

$$z_n = g(q_n),\ z_n^\pm = g^\pm(q_n).$$

Since $I'(q_n) \to 0$ as $n \to \infty$ we have $\big|I'(q_n)v\big| \le \varepsilon_n \|v\|_f$ for all $v \in T_{x_n}\mathfrak{M}$, where $\varepsilon_n \to 0$ as $n \to \infty$. If we consider $v \in T_{q_n}\mathfrak{M}$ such that $v|_{\mathbb{R}\setminus[-T,T]} \equiv 0$ then

$$I'(q_n)v = J'(z_n)v|_{[-T,T]}.$$

and we conclude that

$$\big|J'(z_n)v\big| \le \varepsilon_n \|v\|_{f,T} \ \text{ for all } v \in T_{z_n}\Lambda \text{ such that } v(\pm T) = 0, \tag{18}$$

where $\|v\|_{f,T}$ stands for the norm in $T_{z_n}\Lambda$. Taking $v \in T_{q_n}\mathfrak{M}$ such that $v(t) \equiv 0$ for all $t \le T$ (respectively $t \ge -T$) we get

$$I'(q_n)v = J'_+(z_n^+)v|_{[T,\infty)},\ \ I'(q_n)v = J'_-(z_n^-)v|_{(-\infty,T]}$$

and

$$\big|J'_\pm(z_n^\pm)v\big| \le \varepsilon_n \|v\|_{f,\pm} \ \text{ for all } v \in T_{z_n^\pm}\Lambda_\pm \text{ such that } v(\pm T) = 0. \tag{19}$$

Denote by z_∞, z_∞^- and z_∞^+ the restriction of x_∞ to $[-T, T], (-\infty, -T]$ and $[T, \infty)$, respectively. Since $q_n \to x_\infty$ in L_∞, $z_n \to z_\infty$ and $z_n^\pm \to z_\infty^\pm$ uniformly.

Lemma 7. *If $J(z_n)$ is uniformly bounded, $z_n \to z_\infty$ uniformly and satisfies (18) then $\{z_n\}$ has a subsequence converging in Λ to z_∞.*

PROOF: - The proof repeates the proof of Lemma 5.1 from [1].

Lemma 8. *If $J_\pm(z_n^\pm)$ is uniformly bounded, $z_n^\pm \to z_\infty^\pm$ uniformly and satisfies (19) then $\{z_n^\pm\}$ has a subsequence converging in $\Lambda_\pm$ to $z_\infty^\pm$.*

PROOF: - We outline the proof for the sequence $\{z_n^+\}$ whereas the case $\{z_n^-\}$ can be studied in a similar way. Using notations from [1] we set

$$\xi_{nm}(t) = z_m^+(t) - z_n^+(t),\ \ \zeta_{nm}(t) = \frac{Tf^{1/2}(T)}{tf^{1/2}(t)}\xi_{nm}(T). \tag{20}$$

Since $\|\zeta_{nm}\|_{f,+} \to 0$ as $n, m \to \infty$ and $\zeta_{nm}(T) = \xi_{nm}(T)$ one may apply (19) and get $J'_+(z_n^+)(\xi_{nm} - \zeta_{nm}) \to 0$ as $n, m \to \infty$. Now using boundness of J'_+ we see that $J'_+(z_n^+)\xi_{nm} \to 0$ as $n, m \to \infty$. Extract a subsequence, denoting also by z_n^+, such that the sequence $J_+(z_n^+)$ is converging. Then we obtain

$$\Psi_{nm} = J_+(z_m^+) - J_+(z_n^+) + J'_+(z_n^+)\xi_{nm} \to 0 \text{ as } n, m \to \infty. \tag{21}$$

We estimate separately the terms in the integeral Ψ_{nm} involving the kinetic and potential energy. Assume the radius r is so small that the estimates (14) are fulfilled. One may observe that

$$\begin{aligned} K(z_m^+, \dot z_m^+) - K(z_n^+, \dot z_n^+) \ &- \ \langle K_x(z_n^+, \dot z_n^+), \xi_{nm}\rangle - \langle K_{\dot x}(z_n^+, \dot z_n^+), \dot\xi_{nm}\rangle = \\ = K(z_n^+, \dot\xi_{nm}) \ &+ \ \langle K_x(\tau_{nm}, \dot z_m^+), \xi_{nm}\rangle - \langle K_x(z_n^+, \dot z_n^+), \xi_{nm}\rangle \ge \\ &\ge \ C_A^{(1)}|\xi_{nm}|^2 - C_B(|\dot z_m^+|^2 + |\dot z_n^+|^2)|\xi_{nm}|, \end{aligned} \tag{22}$$

where τ_{nm} is some intermediate point between z_n^+ and z_m^+. For the potential energy terms we get

$$V(z_m^+) - V(z_n^+) - \langle \nabla V(z_n^+), \xi_{nm} \rangle = \frac{1}{2} \langle \nabla^2 V(\tau_{nm}) \xi_{nm}, \xi_{nm} \rangle \geq \alpha |\xi_{nm}|^2. \tag{23}$$

Substituting (22) and (23) into (21) one obtains

$$\Psi_{nm} \geq \int_T^\infty \left(C_A^{(1)} |\xi_{nm}(s)|^2 + \alpha |\xi_{nm}(s)|^2 - C_B (|\dot{z}_m^+(s)|^2 + |\dot{z}_n^+(s)|^2) |\xi_{nm}(s)| \right) \mathrm{d}s \geq$$
$$\geq c_{10} \|\xi_{nm}\|^2 - \Theta_{nm},$$

where $c_{10} = \min\{C_A^{(1)}, \alpha\}$ and

$$\Theta_{nm} = C_B \left(\|z_m\|_{f,+}^2 + \|z_n\|_{f,+}^2 \right) \|\xi_{nm}\|_\infty.$$

Since $\{z_n^+\}$ is bounded and $\|\xi_{nm}\|_\infty \to 0$ as $n, m \to \infty$ we have $\Theta_{nm} \to 0$. Hence

$$\|\xi_{nm}\|_{f,+}^2 \leq c_{10}^{-1} \left(\Psi_{nm} - \Theta_{nm} \right) \to 0, \text{ as } n, m \to \infty.$$

Thus we showed that $\{z_n^+\}$ is a Cauchy sequence what finishes the proof. □
Combining lemmae 6 and 7 we prove proposition 4. □
To prove Theorem 1 we notice that $\mathfrak{M}$ is homotopically equivalent to the space of paths

$$\Omega(\mathcal{M}, x_-, x_+) = \{ q \in C^0([0,1]) : \; q(0) = x_-, \; q(1) = x_+ \}.$$

Since $\mathcal{M}$ is compact, the Lusternik-Schnirelmann category $cat(\mathcal{M}) = \infty$ [8]. This implies the existence of infinitely many critical points for the functional I and due to propositions 4,5 the existence of infinitely many doubly asymptotic trajectories of (3) such that $q(t) \to x_\pm$, $\dot{q}(t) \to 0$ as $t \to \pm\infty$. □
To prove Theorem 1 for the case when subsets $X_\pm$ consist of more than one point we note that one may proceed in the same way except the estimate (8), which is not valid. Following [4], [3] define for $\varepsilon > 0$ a subset $\Gamma_\varepsilon(x_-, x_+) \subset \mathfrak{M}$:

$$\Gamma_\varepsilon(x_-, x_+) = \left\{ q \in \mathfrak{M} : \text{ for all } t \in \mathbb{R}_\pm \;\; q(t) \notin B_\varepsilon\big(X_\pm \setminus \{x_\pm\}\big) \right\}.$$

We also consider the restriction of the action functional $I_\varepsilon = I|_{\Gamma_\varepsilon(x_-, x_+)}$. For $\Gamma_\varepsilon(x_-, x_+)$ the estimate (8) is valid and using the same arguments one may prove that for any $\varepsilon > 0$ the functional I_ε achieves its minimum on some function $q_\varepsilon \in \Gamma_\varepsilon(x_-, x_+)$. Moreover, $q_\varepsilon(t)$ is a classical solution of (3) whenever $q_\varepsilon(t) \notin \partial B_\varepsilon\big(X_\pm \setminus \{x_\pm\}\big)$, where $\partial B_\varepsilon(A)$ stands for the ε-neighbourhood of a set A. Let us take a sequence $\varepsilon_j \to 0$ as $j \to \infty$.

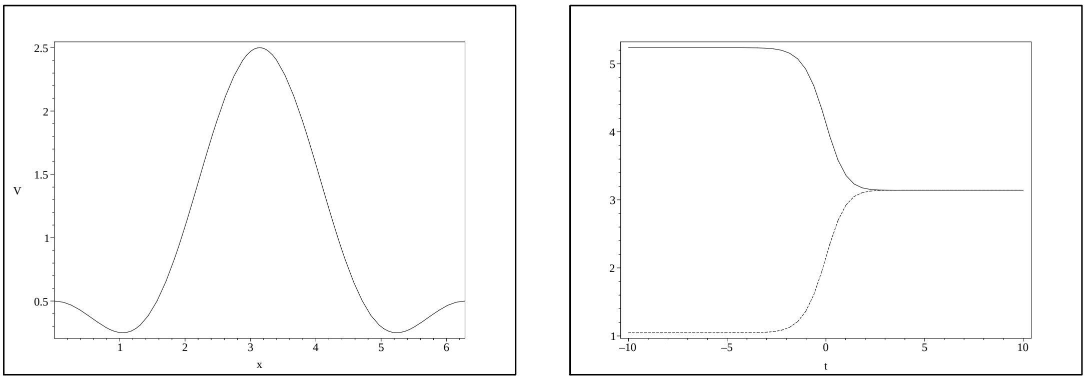
Lemma 9. *For sufficiently large $j \in \mathbb{N}$ $q_{\varepsilon_j}(t) \notin B_{\varepsilon_j}\big(X_\pm \setminus \{x_\pm\}\big)$ for all $t \in \mathbb{R}$.*

PROOF: - The proof of this lemma is similar to lemma 2.10 from [3].
This lemma guarantees that q_{ε_j} is a heteroclinic solution of (3) joining x_- to x_+ and satisfying $\dot{q}_{\varepsilon_j}(t) \to 0$ as $t \to \pm\infty$.
To illustrate the result consider a pendulum-like system with Lagrangian

$$L(\varphi, \dot{\varphi}, t) = \frac{1}{2}|\dot{\varphi}|^2 - t\left(1 - \cos(\varphi) - \frac{1}{2}\cos(2\varphi)\right). \tag{24}$$

The configuration space of this example is a circle and the system possesses four equilibria $\varphi = 0, \pi, \pm\pi/3$. Since the factor $f(t) = t$, the subset X_+ consists of one point $X_+ = \{\pi\}$ whereas the subset X_- consists of two points $X_- = \{-\pi/3, \pi/3\}$. Application of Theorem 1 to this system yields the following

Proposition 5. *For any $m \in \mathbb{Z}$ there exists a doubly asymptotic trajectory $\varphi_m^{\pm}(t)$ of the system (24) such that $\lim_{t\to-\infty} \varphi_m^{\pm}(t) = \pm\pi/3$ and $\lim_{t\to+\infty} \varphi_m^{\pm}(t) = (2m+1)\pi$, i.e. the trajectories $\varphi_m^{\pm}(t)$ join the points $\pm\pi/3$ with π via m full rotations.*



a) b)

Figure 1: a. The profile of potential $V(x) = 1 - \cos(\varphi) - \frac{1}{2}\cos(2\varphi)$; b. heteroclinic trajectories $\varphi_0^{\pm}(t)$, connecting $\pm\pi/3$ with π without rotations (the solid curve corresponds to $\varphi_0^{-}(t)$ and the dashed curve - to $\varphi_0^{+}(t)$)

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