Modified strain gradient theory and Timoshenko beam assumptions—A direct approach

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Abstract

Materials with intrinsic micro- or nano-structure may show size-dependent material behavior, which is reflected by a stiffer elastic response to external forces when the size of the material body is reduced. In order to account for the so-called size effect strain gradient theories are applied. In this paper a modified strain gradient theory of elasticity for isotropic materials is investigated by discussing a higher order model for static beam bending. With the analytical solution for the EULER-BERNOULLI beam model being known, we apply the TIMOSHENKO beam assumptions in the present work. In contrast to the EULER-BERNOULLI beam model the TIMOSHENKO model takes independent rotations of the cross-sections of the beam into account. A system of coupled differential equations for the functions of beam deflections and rotations is derived. TIMOSHENKO's shear coefficient and the shear modulus are involved. A non-dimensional form of the functions and coefficients is provided and first numerical results are discussed. Deflections and rotations are calculated for a straight beam with a unified load distribution. The solution includes the size-effect and the refined results of a TIMOSHENKO beam.

1 Introduction

When modeling Micro- and Nanoelectromechanical Systems (MEMS/NEMS), a quantitative understanding of a possible size effect in micro- and submicrostructures is of great importance. A size effect is reflected in a varying elastic response to external loads if the size of the material body is reduced. Experimental validation is given in, *e.g.*, [16, 4, 7, 13, 1, 9]. It was shown by LAM *et al.* (2003) [7] that in the absence of strain gradients (in uniaxial tensile tests) the elastic behavior of epoxy is independent of the thickness of the sample. This is confirmed by the results of strain gradient models. Since conventional continuum theories based on the BOLTZMANN (a.k.a. CAUCHY) continuum are not able to predict size effects, the present work deals with the *Strain Gradient theory* (SG) developed by several authors, *e.g.*, [17, 14, 6, 15]. By employing second order derivatives of the displacement vector, the SG-theories are able to account for quantities like curvature or rotation. A generalization of strain gradient continua has been promoted by ERINGEN, who proposed "nonsimple materials of gradient type [2]" in order to derive the corresponding higher-order material dependencies in a rational manner. One of the advantages of the Timoshenko beam assumptions is the inclusion of an additional rotational freedom w.r.t. the cross-sections of the beam. The solution of this model yields two functions, describing deflection and rotation of the beam. For a modified couple stress continuum, the Timoshenko beam assumptions have already been analyzed by [5]. The scope of this work is to combine these assumptions with the modified strain gradient theory.

2 Modified strain gradient theory

The present study starts with one of the three reduced forms of the strain energy densities for small deformations u^{SG}, postulated by Mindlin (1962) [14]. Einstein summation convention on repeated indices is used and spatial partial derivatives in the Cartesian coordinate system are denoted by comma-separated indices. Mindlin's second form of the strain energy density reads:

$$u^{\mathrm{SG}} = u(\varepsilon_{ij}\,, \eta_{ijk}) \ , \quad \sigma_{ij} = \frac{\partial u^{\mathrm{SG}}}{\partial \varepsilon_{ij}} \ , \quad \mu_{ijk} = \frac{\partial u^{\mathrm{SG}}}{\partial \eta_{ijk}} \ , \tag{1}$$

where σ_{ij} denotes the Cauchy stress tensor, $\varepsilon_{ij} = u_{(i,j)} = 1/2(u_{i,j} + u_{j,i})$ the small strain tensor, μ_{ijk} the higher-order stress tensor and $\eta_{ijk} = 1/2(u_{k,ij} + u_{j,ki}) = \varepsilon_{kj,i}$ the gradient of strain. As in isotropic strain gradient elasticity the decomposition of the strain gradient tensor η_{ijk} yields five irreducible parts [8]. The inclusion of the macroscopic rotation vector φ_i leads to a reduction of independent additional material parameters from five to three. Fleck & Hutchinson (1997) [3] introduced independent expressions of η_{ijk} and decomposed the second order displacement gradient into its symmetric and anti-symmetric part, $\bar{\bar{\eta}}_{ijk}$ and η^{A}_{ijk}:

$$\bar{\bar{\eta}}_{ijk} = \tfrac{1}{3}(u_{k,ij} + u_{i,jk} + u_{j,ki}) \ , \quad \eta^{\mathrm{A}}_{ijk} = \tfrac{2}{3}(\epsilon_{ikl}\,\bar{\eta}_{lj} + \epsilon_{jkl}\,\bar{\eta}_{li}) \ , \tag{2}$$

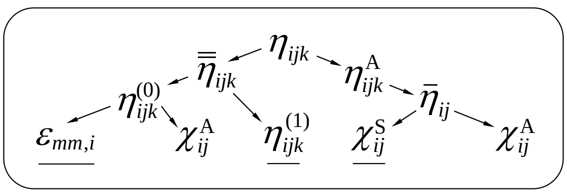

Figure 1: Scheme of decomposition of the second order gradient of displacement

where $\bar{\eta}_{ij} = \varphi_{i,j}$ is the gradient of rotation (decomposed into its symmetric and anti-symmetric part, χ^{S}_{ij} and χ^{A}_{ij} as well):

$$\varphi_i = \tfrac{1}{2}\epsilon_{ilk}\,u_{k,l}\,, \quad \bar{\eta}_{ij} = \tfrac{1}{2}\epsilon_{ilk}\,u_{k,lj}\,, \quad \chi^{\mathrm{A}}_{ij} = \tfrac{1}{2}\left(\varphi_{i,j} - \varphi_{j,i}\right), \quad \chi^{\mathrm{S}}_{ij} = \tfrac{1}{2}\left(\varphi_{i,j} + \varphi_{j,i}\right). \tag{3}$$

Here, ϵ_{ilk} is the Levi-Civita symbol. The tensor $\bar{\bar{\eta}}_{ijk}$ is further decomposed into its spherical and deviatoric part, $\eta^{(0)}_{ijk}$ and $\eta^{(1)}_{ijk}$, *cf.* Fig. 1. The quantity $\eta^{(0)}_{ijk}$ is related to χ^{A}_{ij} and the dilatation gradient $\varepsilon_{mm,i}$ in the following manner [3]:

$$\eta^{(0)}_{ijk} = \tfrac{1}{5}\left(\delta_{ij}\,\bar{\bar{\eta}}_{mmk} + \delta_{jk}\,\bar{\bar{\eta}}_{mmi} + \delta_{ki}\,\bar{\bar{\eta}}_{mmj}\right) \quad \text{and} \quad \bar{\bar{\eta}}_{mmi} = \varepsilon_{mm,i} + \tfrac{2}{3}\epsilon_{iln}\,\chi^{\mathrm{A}}_{ln} \ . \tag{4}$$

As shown in [10, 18] the anti-symmetric part of the gradient of rotation does not influence the strain energy, if symmetry of the couple stress tensor μ_{ij} is assumed. Consequently, a linear strain energy density for non-simple isotropic materials of the modified gradient type reads:

$$u^{\mathrm{MSG}} = \hat{u}(\varepsilon_{ij}, \varepsilon_{mm,i}, \eta^{(1)}_{ijk}, \chi^{\mathrm{S}}_{ij}) = \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij} + \tfrac{1}{2}p_i\varepsilon_{mm,i} + \tfrac{1}{2}\mu^{(1)}_{ijk}\eta^{(1)}_{ijk} + \tfrac{1}{2}\mu_{ij}\chi^{\mathrm{S}}_{ij} \ . \quad (5)$$

The corresponding work-conjugated stress measures are:

$$\begin{aligned} \sigma_{ij} &= \frac{\partial u^{\mathrm{MSG}}}{\partial \varepsilon_{ij}} = \lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij} \ , \qquad & p_i &= \frac{\partial u^{\mathrm{MSG}}}{\partial \varepsilon_{nn,i}} = 2G\ell_0^2\varepsilon_{mm,i} \ , \\ \mu^{(1)}_{ijk} &= \frac{\partial u^{\mathrm{MSG}}}{\partial \eta^{(1)}_{ijk}} = 2G\ell_1^2\eta^{(1)}_{ijk} \ , & \mu_{ij} &= \frac{\partial u^{\mathrm{MSG}}}{\partial \chi^{\mathrm{S}}_{ij}} = 2G\ell_2^2\chi^{\mathrm{S}}_{ij} \ . \end{aligned} \quad (6)$$

Here, λ and G are LAMÉ's constants, whereas ℓ_0, ℓ_1 and ℓ_2 denote additional material length scale parameters.

3 Differential equations of the problem

In this section, the TIMOSHENKO beam theory is analyzed in context of the modified strain gradient theory. Following this, the resulting differential equations of the beam are transformed to a non-dimensional form in order to analyze the results numerically.

3.1 TIMOSHENKO beam assumptions

The TIMOSHENKO beam model—in contrast to a the EULER-BERNOULLI model—distinguishes between the current angle of the cross-section $\phi(x)$ and the derivative of the bending line $w(x)$, see [11, 5]. The displacement field u_i of a straight TIMOSHENKO beam reads:

$$u_x = -z\phi(x) \quad , \quad u_y = 0 \quad , \quad u_z = w(x) \ , \quad (7)$$

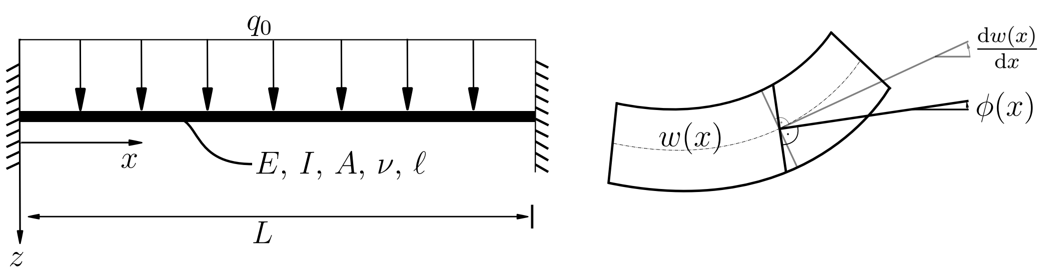

Figure 2: Loading of the TIMOSHENKO beam.

where E, I, A, L, ν and q_0 are YOUNG's modulus, second moment of inertia, area of cross-section, length of the beam, POISSON's ratio and load distribution, respectively, *cf.*, Fig. 2. The non-zero components of the small strain tensor ε_{ij} read:

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \Rightarrow \quad \varepsilon_{xx} = -z\phi' \ , \quad \varepsilon_{xz} = \tfrac{1}{2}\left(w' - \phi\right) = \varepsilon_{zx} \ , \quad (\cdot)' = \tfrac{\mathrm{d}(\cdot)}{\mathrm{d}x} \ , \tag{8}$$

where a prime denotes a derivative with respect to x. Using Hooke's law for isotropic linear-elastic materials, the non-zero components of the Cauchy stress tensor σ_{ij} read:

$$\sigma_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} \quad \Rightarrow \quad \sigma_{xx} = -Ez\phi' \quad , \quad \sigma_{xz} = G\left(w' - \phi\right) = \sigma_{zx} \ . \tag{9}$$

In what follows, the displacement field of Eq. (7) is used to derive the non-zero components of the higher-order stress and strain measures. From

$$\varepsilon_{mm,i} = \left(-z\phi'\right)_{,i} \ , \quad p_i = 2G\ell_0^2\varepsilon_{mm,i} \ , \quad \text{and} \quad \mu_{ij} = 2G\ell_2^2\chi_{ij}^{\mathrm{S}} \tag{10}$$

it follows that:

$$\begin{aligned} \varepsilon_{mm,x} &= -z\phi'' \ , & \varepsilon_{mm,z} &= -\phi' \ , \\ p_x &= -2G\ell_0^2 z\phi'' \ , & p_z &= -2G\ell_0^2\phi' \ , \\ \chi_{xy} &= -\tfrac{1}{4}(\phi' + w'') = \chi_{yx} \ , & \mu_{xy} &= -\tfrac{1}{2}G\ell_2^2(\phi' + w'') = \mu_{yx} \ . \end{aligned} \tag{11}$$

Further computations yield:

$$\begin{aligned} \eta_{xxx}^{(1)} &= -\tfrac{4}{5}z\phi'' \ , & \eta_{zzz}^{(1)} &= \tfrac{1}{5}(\phi' - w'') \ , \\ \eta_{xzz}^{(1)} &= \tfrac{2}{15}z\phi'' \ , & \eta_{xxz}^{(1)} &= (\tfrac{4}{15}w'' - \tfrac{3}{5}\phi') \ , \\ \eta_{xzx}^{(1)} &= (\tfrac{4}{15}w'' - \tfrac{3}{5}\phi') \ , & \eta_{zxx}^{(1)} &= (\tfrac{4}{15}w'' - \tfrac{3}{5}\phi') \ , \\ \eta_{zzx}^{(1)} &= \tfrac{2}{15}z\phi'' \ , & \eta_{zxz}^{(1)} &= \tfrac{2}{15}z\phi'' \ , \end{aligned} \tag{12}$$

and from $\mu_{ijk}^{(1)} = 2G\ell_1^2\eta_{ijk}^{(1)}$ we have:

$$\begin{aligned} \mu_{xxx}^{(1)} &= -\tfrac{8}{5}Gz\ell_1^2\phi'' \ , & \mu_{zzz}^{(1)} &= \tfrac{2}{5}G\ell_1^2(\phi' - w'') \ , \\ \mu_{xzz}^{(1)} &= \tfrac{4}{15}G\ell_1^2 z\phi'' \ , & \mu_{xxz}^{(1)} &= G\ell_1^2(\tfrac{8}{15}w'' - \tfrac{6}{5}\phi') \ , \\ \mu_{xzx}^{(1)} &= G\ell_1^2(\tfrac{8}{15}w'' - \tfrac{6}{5}\phi') \ , & \mu_{zxx}^{(1)} &= G\ell_1^2(\tfrac{8}{15}w'' - \tfrac{6}{5}\phi') \ , \\ \mu_{zzx}^{(1)} &= \tfrac{4}{15}G\ell_1^2 z\phi'' \ , & \mu_{zxz}^{(1)} &= \tfrac{4}{15}Gz\ell_1^2\phi'' \ . \end{aligned} \tag{13}$$

The higher-order strain energy density u^{MSG} in Eq. (5) of the present beam bending problem, incorporating the terms of Eqs. (8)–(13), reads:

$$\begin{aligned} u^{\mathrm{MSG}} &= \tfrac{1}{2}\big[\sigma_{xx}\varepsilon_{xx} + \sigma_{xz}\varepsilon_{xz} + \sigma_{zx}\varepsilon_{zx} + p_x\varepsilon_{mm,x} + p_z\varepsilon_{mm,z} + \\ &\quad + \mu_{xy}\chi_{xy}^{\mathrm{S}} + \mu_{yx}\chi_{yx}^{\mathrm{S}} + \mu_{xzz}^{(1)}\eta_{xzz}^{(1)} + \mu_{xxx}^{(1)}\eta_{xxx}^{(1)} + \mu_{xzx}^{(1)}\eta_{xzx}^{(1)} + \\ &\quad + \mu_{xxz}^{(1)}\eta_{xxz}^{(1)} + \mu_{zzz}^{(1)}\eta_{zzz}^{(1)} + \mu_{zxx}^{(1)}\eta_{zxx}^{(1)} + \mu_{zzx}^{(1)}\eta_{zzx}^{(1)} + \mu_{zxz}^{(1)}\eta_{zxz}^{(1)}\big] \\ &= \tfrac{1}{2}\big[E(z\phi')^2 + G(w' - \phi)^2 + 2G\ell_0^2(z\phi'')^2 + 2G\ell_0^2\phi'^2 + \\ &\quad + \tfrac{1}{4}G\ell_2^2(\phi' + w'')^2 + \tfrac{8}{225}G\ell_1^2(z\phi'')^2 + \tfrac{32}{25}G\ell_1^2(z\phi'')^2 + \\ &\quad + 2G\ell_1^2(\tfrac{4}{15}w'' - \tfrac{3}{5}\phi')^2 + 2G\ell_1^2(\tfrac{4}{15}w'' - \tfrac{3}{5}\phi')^2 + \tfrac{2}{25}G\ell_1^2(\phi' - w'')^2 + \\ &\quad + 2G\ell_1^2(\tfrac{4}{15}w'' - \tfrac{3}{5}\phi')^2 + \tfrac{8}{225}G\ell_1^2(z\phi'')^2 + \tfrac{8}{225}G\ell_1^2(z\phi'')^2\big] \\ &= \tfrac{1}{2}\big[\phi^2 G + w'^2 G + \phi'^2(Ez^2 + 2G\ell_0^2 + \tfrac{1}{4}G\ell_2^2 + \tfrac{56}{25}G\ell_1^2) + \\ &\quad + w''^2(\tfrac{1}{4}G\ell_2^2 + \tfrac{114}{225}G\ell_1^2) + \phi''^2(2G\ell_0^2 z^2 + \tfrac{312}{225}G\ell_1^2 z^2) - \\ &\quad - \phi w' 2G + \phi' w''(\tfrac{1}{2}G\ell_2^2 - \tfrac{156}{75}G\ell_1^2)\big] \ . \end{aligned} \tag{14}$$

We want to make use of the principle of virtual work and find the strain energy of the body and the work of the external loads that minimize $\delta U - \delta W \to 0$. The variation of the strain energy $\delta U = \int_V \delta u^{\mathrm{MSG}} \,\mathrm{d}V$ of the problem reads:

$$\begin{aligned}\delta U = \int\limits_x \int\limits_y \int\limits_z \big[&\phi\delta\phi G + w'\delta w' G + \phi'\delta\phi'(Ez^2 + 2G\ell_0^2 + \tfrac{1}{4}G\ell_2^2 + \tfrac{56}{25}G\ell_1^2) + \\ &+ w''\delta w''(\tfrac{1}{4}G\ell_2^2 + \tfrac{114}{225}G\ell_1^2) + \phi''\delta\phi''(2G\ell_0^2 z^2 + \tfrac{312}{225}G\ell_1^2 z^2) - \\ &- (\delta\phi w' + \phi\delta w')G + (\delta\phi' w'' + \phi'\delta w'')(\tfrac{1}{4}G\ell_2^2 - \tfrac{156}{150}G\ell_1^2)\big]\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\ .\end{aligned} \tag{15}$$

We take TIMOSHENKO's shear factor κ into account, where $\int_{A^\star} \mathrm{d}y\,\mathrm{d}z = A\kappa$ and $A^\star$ denotes the current cross-section. Integration over the variables y and z leads to:

$$\begin{aligned}\delta U = \int\limits_0^L \big[&\phi\delta\phi GA\kappa + w'\delta w' GA\kappa + \phi'\delta\phi'(EI + 2GA\kappa\ell_0^2 + \tfrac{1}{4}GA\kappa\ell_2^2 + \tfrac{56}{25}GA\kappa\ell_1^2) + \\ &+ w''\delta w''(\tfrac{1}{4}GA\kappa\ell_2^2 + \tfrac{114}{225}GA\kappa\ell_1^2) + \phi''\delta\phi''(2GI\ell_0^2 + \tfrac{312}{225}GI\ell_1^2) - \\ &+ (\delta\phi w' + \phi\delta w')GA\kappa + (\delta\phi' w'' + \phi'\delta w'')(\tfrac{1}{4}GA\kappa\ell_2^2 - \tfrac{156}{150}GA\kappa\ell_1^2)\big]\,\mathrm{d}x \\ = \int\limits_0^L \big[&\delta\phi(T\phi - Tw') + \delta w'(Tw' - T\phi) + \delta\phi'(K\phi' + Nw'') + \\ &+ \delta w''(Pw'' + N\phi') + \delta\phi''\phi'' S\big]\,\mathrm{d}x\ ,\end{aligned} \tag{16}$$

where the following substitutions are used:

$$\begin{aligned}T &= GA\kappa\ , \\ K &= EI + 2GA\kappa\ell_0^2 + \tfrac{1}{4}GA\kappa\ell_2^2 + \tfrac{56}{25}GA\kappa\ell_1^2\ , \\ P &= \tfrac{1}{4}GA\kappa\ell_2^2 + \tfrac{114}{225}GA\kappa\ell_1^2\ , \\ S &= 2GI\ell_0^2 + \tfrac{312}{225}GI\ell_1^2\ , \\ N &= \tfrac{1}{4}GA\kappa\ell_2^2 - \tfrac{156}{150}GA\kappa\ell_1^2\ .\end{aligned} \tag{17}$$

The virtual work of external loads and higher-order loads is assumed to be:

$$\delta W = \int\limits_0^L \big(\delta\phi m(x) + \delta w q(x)\big)\mathrm{d}x + V\delta w\Big|_0^L + M\delta\phi\Big|_0^L + M^{\mathrm{i}}\delta w'\Big|_0^L + M^{\mathrm{h}}\delta\phi'\Big|_0^L\ , \tag{18}$$

where V is the single force acting at a boundary, M the single moment acting at a boundary, M^{i} an internal moment acting at a boundary which influences $w'(\tilde{x})$ and M^{h} denotes a higher-order moment acting at a boundary that influences the gradient of the angle of the cross-sections, *cf.* Eq. (19). Double partial integration of Eq. $(16)_2$ results in:

$$\delta U = \int_0^L \Big(\delta\phi \overbrace{\left[T(\phi - w') - K\phi'' - Nw'' + S\phi^{\mathrm{IV}}\right]}^{m(x)} + \delta w \overbrace{\left[T(w'' - \phi') + Pw^{\mathrm{IV}} + N\phi'''\right]}^{q(x)}\Big)\mathrm{d}x + \underbrace{\big(T(w' - \phi) - Pw'' - N\phi''\big)}_{V}\,\delta w\Big|_0^L + \underbrace{\big(K\phi' + Nw'' - S\phi'''\big)}_{M}\,\delta\phi\Big|_0^L + \underbrace{\big(Pw'' + N\phi'\big)}_{M^{\mathrm{i}}}\,\delta w'\Big|_0^L + \underbrace{S\phi''}_{M^{\mathrm{h}}}\,\delta\phi'\Big|_0^L . \quad (19)$$

By comparison, the governing differential equations are $\forall x \in (0, L)$:

$$\begin{aligned} \left[T(\phi - w') - K\phi'' - Nw'' + S\phi^{\mathrm{IV}}\right] &= m(x) , \\ \left[T(w'' - \phi') + Pw^{\mathrm{IV}} + N\phi'''\right] &= q(x) . \end{aligned} \quad (20)$$

The boundary conditions at $x = 0$ and $x = L$ become:

$$\begin{aligned} \big(T(w' - \phi) - Pw''' - N\phi''\big) &= V(x) , \\ \big(K\phi' + Nw'' - S\phi'''\big) &= M(x) , \\ (Pw'' + N\phi') &= M^{\mathrm{i}}(x) , \\ S\phi'' &= M^{\mathrm{h}}(x) , \end{aligned} \quad (21)$$

where $m(x)$ denotes a distribution of moments and $q(x)$ the force distribution.

3.2 Non-dimensional forms

For the purpose of improving the condition of the numerical problem, a non-dimensional form of the deflection w and the beam coordinate x is chosen as follows:

$$w(x) = \frac{FL^3}{EI}\tilde{w}(\tilde{x}) , \quad x = L\tilde{x} \quad \Rightarrow \quad \frac{\partial^n(\cdot)}{\partial x^n} = \frac{1}{L^n}\frac{\partial^n(\cdot)}{\partial \tilde{x}^n} , \; n = 1, 2, 3 \ldots , \quad (22)$$

where the reference force is chosen w.r.t. the analyzed loading as $F = q_0 L$. Normalized with respect to the highest order of the derivatives Eqs. (20) read:

$$\begin{aligned} \underbrace{\frac{L^4T}{S}}_{c_1}\phi - \underbrace{\frac{L^6TF}{EIS}}_{c_2}\tilde{w}' - \underbrace{\frac{L^2K}{S}}_{c_3}\phi'' - \underbrace{\frac{L^5NF}{EIS}}_{c_4}\tilde{w}'' + \phi^{\mathrm{IV}} &= \underbrace{\frac{L^4}{S}m(\tilde{x})}_{\tilde{m}} , \\ \underbrace{\frac{TL^2}{P}}_{c_5}\tilde{w}'' - \underbrace{\frac{TEI}{PF}}_{c_6}\phi' + \tilde{w}^{\mathrm{IV}} + \underbrace{\frac{NEI}{L^2PF}}_{c_7}\phi''' &= \underbrace{q(\tilde{x})\frac{LEI}{PF}}_{\tilde{q}} , \end{aligned} \quad (23)$$

We re-substitute T, K, P, S and N from Eq. (17) and choose $\ell_0 = \ell_1 = \ell_2 = \ell$, $\kappa = \frac{5}{6}$, and $G = \frac{E}{2}$ (which is equivalent to $\nu = 0$). Furthermore, we only consider beams of rectangular cross-sections. The constants $c_1, c_2, \ldots, c_7$ now read:

$$\begin{aligned} & c_1(\ell) = \frac{375}{127}\frac{L^4}{H^2\ell^2}\ , \quad c_2(\ell) = \frac{4500}{127}\frac{L^7 q_0}{EH^4A\ell^2}\ , \quad c_3(\ell) = \frac{75}{127}\left(\frac{L^2}{\ell^2} + 22.45\frac{L^2}{H^2}\right)\ , \\ & c_4 = -\frac{180}{127}\frac{L^6 q_0}{EAH^4}\ , \quad c_5(\ell) = \frac{300}{227}\frac{L^2}{\ell^2}\ , \\ & c_6(\ell) = \frac{300}{2724}\frac{EAH^2}{q_0 L \ell^2}\ , \quad c_7 = -\frac{12}{2724}\frac{EAH^2}{q_0 L^3}\ , \end{aligned} \tag{24}$$

where H is the thickness of the beam. The system of differential equations in the present nondimensionalized form reads:

$$\boxed{\begin{aligned} c_1\phi - c_2\tilde{w}' - c_3\phi'' - c_4\tilde{w}'' + \phi^{\mathrm{IV}} &= \tilde{m} \\ c_5\tilde{w}'' - c_6\phi' + \tilde{w}^{\mathrm{IV}} + c_7\phi''' &= \tilde{q}\ . \end{aligned}} \tag{25}$$

These equations are implemented using Mathematica [12] for further numerical investigation.

4 Numerical solution

According to Fig. 2 the physical boundary conditions of the problem are identified as:

$$\tilde{m} = 0\ , \quad V = 0\ , \quad \tilde{q} = \frac{EI}{P}\ , \tag{26}$$

whereas the geometrical boundary conditions are:

$$\begin{aligned} & \tilde{w}(\tilde{x}=0) = 0\ , \quad \tilde{w}(\tilde{x}=1) = 0\ , \quad \tilde{w}'(\tilde{x}=0) = 0\ , \quad \tilde{w}'(\tilde{x}=1) = 0\ , \\ & \phi(\tilde{x}=0) = 0\ , \quad \phi(\tilde{x}=1) = 0\ , \quad \phi'(\tilde{x}=0) = 0\ , \quad \phi'(\tilde{x}=1) = 0\ . \end{aligned} \tag{27}$$

Table 13: Dimensions, loads and material parameters.

L	W	H	E	G	ℓ	q_0
$80\,\mu$m	$20\,\mu$m	$20\,\mu$m	3.8 GPa	1.9 GPa	$16\,\mu$m	$10\,\frac{\mathrm{kN}}{\mathrm{m}}$

For a beam with the dimensions, loads and material parameters shown in Table 13, Mathematica numerically computes a solution to the ordinary differential equations (25) after incorporating the prescribed boundary conditions. It should be noted that the ratio of length to thickness is 4 : 1, which is no longer in the range of the classical Euler-Bernoulli beam model. For a verification of the numerical method, the material length scale parameter ℓ is set equal to zero. The resulting differential equations are derived by modifying Eqs. (20)–(21). The analytical solution w_{T} for the classical Timoshenko model,

$$w_{\mathrm{T}}(x = \tfrac{L}{2}) = \frac{q_0 L^2}{48EI}\left(\frac{L^2}{4} + \frac{H^2}{\kappa}\right)\ , \tag{28}$$

is then approximated by the numerical solution with a difference of about 10 μ% for a wide range of dimensions. In Figure 3 the deflection line $w(\tilde{x})$ and the rotation angle $\phi(\tilde{x})$ are shown, both taking into account higher-order terms.

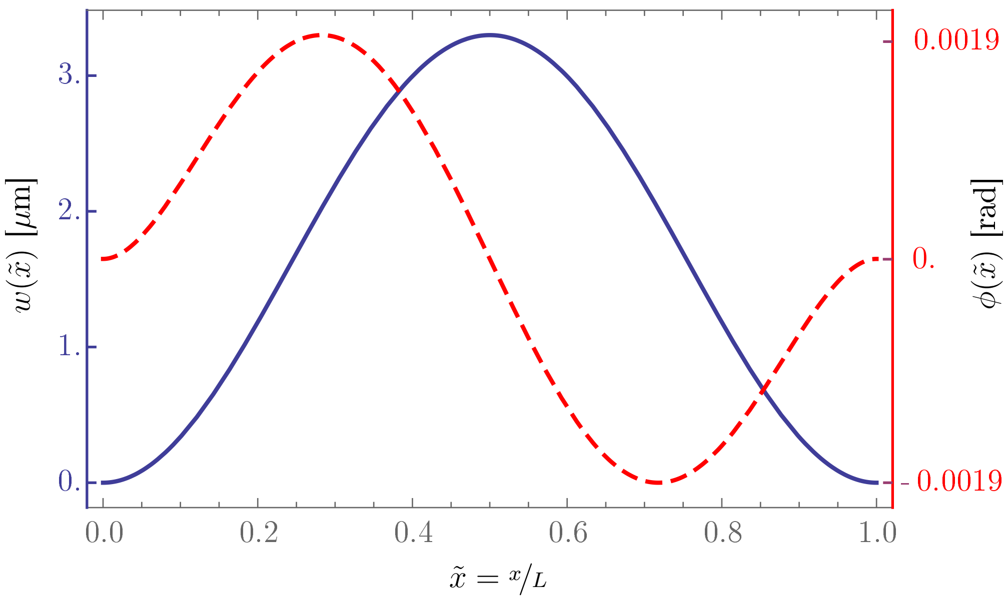

Figure 3: Plot of the deflection line (blue) and the rotation angles (dashed, red).

The solution is in good agreement to the boundary conditions stated in Eqs. (27). A size effect becomes evident when evaluating different beam thicknesses. Figure 4 shows the ratio of maximum deflection of (a) the classical analytical solution of Timoshenko's beam equation w_{T} and (b) the numerically computed maximum deflection $w_{\mathrm{num.}}^{\mathrm{MSG}}$ for the presented modification employing the strain gradient. We considered a short rectangular beam with a range of thicknesses between 10–60 μm.

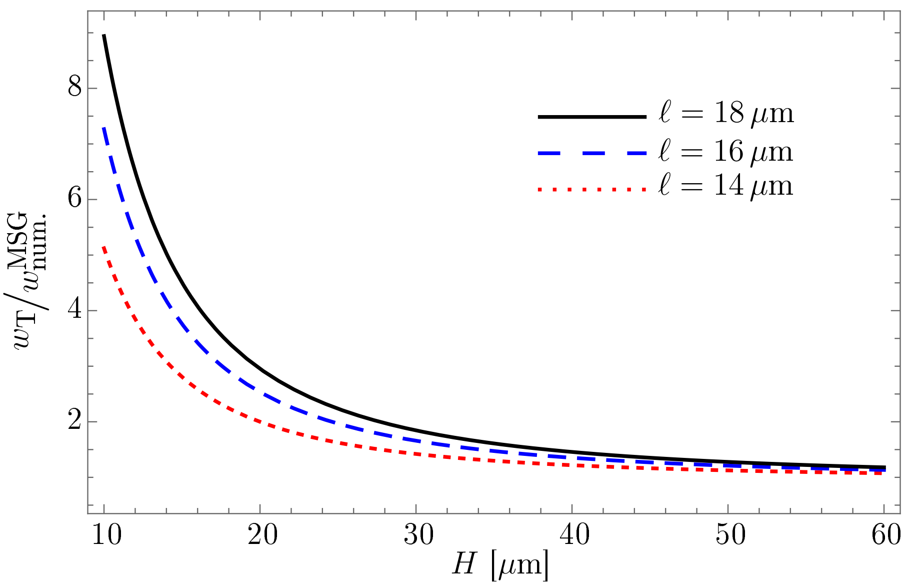

Figure 4: The ratio of maximum deflections from the classical to the higher-order model, plotted for different material length scale parameters.

The figure shows that the ratio $w_{\mathrm{T}}/w_{\mathrm{num.}}^{\mathrm{MSG}}$ increases for decreasing thicknesses. Here, the modified model yields up to eight times higher bending rigidities in comparison to the classical analytical solution. For high thickness values, both models converge to the same values. This so-called size effect is increased for larger material length scale parameters, as demonstrated with the values: $\ell = 14$, 16 and 18 μm.

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