On correctness of gradient plasticity theory

Lurie S.A., Qi Chengzhi, Belov P.A.
salurie@mail.ru

Abstract

In contrast to classical elasticity in which there are no scale parameters that characterize the internal structure of the material, in nonlocal theories of elasticity such parameters appear in a natural way. Therefore, gradient theories are suitable for modeling multi-scale effects and are used to solve numerous applications for inhomogeneous structures with an extended internal surface where the degree of influence of scale effects is related to the density of interfaces. Non-local models of continuous media are especially appealing in the simulation of properties of structured materials with submicron and nanoscale internal structures, where the effective properties are largely determined by the scale effects (the cohesion and adhesion interaction effects). There is no doubt that the non-local models are also promising with the study of elastic-plastic deformations. Generalized elasticity theories even for isotropic materials include many additional physical constants, experimental determination of which is difficult or even impossible. Therefore, applied theories with a small number of additional physical parameters are of considerable interest. However, the process of reduction of nonlocal theories, which has the goal to reduce the number of additional parameters, is not quit trivial and can lead to incorrect theories. In this paper we consider non-local theories of medias with defects fields, gradient theories of elasticity and plasticity.

We give a construction of kinematic model of media with the fields of conserved defects, determine the free and constrained deformations, formulate the basic kinematic relations, and make up complete lists of arguments for the formulation of variational models of deformation of different options of media models. The variational equation for dissipative models and Lagrangian for reversible models also proposed.

It is noted that the kinematic models built on the basis of kinematics restrictions for the constrained and free distortions (consistent and inconsistent distortions/deformations), that may be considered as the criteria of the correctness of theories of defective medias and gradient plasticity theories. We give a revision of versions of theories, of media with defects fields, where the properties of integrability and generalized Cauchy relations are unjustifiably attributed to the total strains.

1 Introduction

Generalized models of continuous medias, the development of which is related to the fundamental works [1]–[3], are promising for modeling the properties of various micro/nanostructures in which the effects of both interaction of cohesion and adhesion and other manifestations of scale effects may be of crucial importance [4]–[14]. The applications of applied gradient theories are associated not only with the problems of the theory of elasticity, but also with the objectives of thermo-elastic-plasticity [15], thermal conductivity and diffusion [16], multiphase materials [17]. In formulation of models, the variational approach is the most correct and complete, the variational model allows us to formulate a concerted equations of balance (movement) of the media and the boundary problem as a whole. At the same time, kinematic model of the media and the physical model are linked and coordinated. In this paper we formulate the kinematic model, identify free and constrained deformations and draw up the lists of items for the formulation of variational models of deformation of different options of media models. It is proposed to use the kinematic models constructed on the basis of definitions of consistent and inconsistent generalized deformations as criterias for the correctness of theories of defective medias and gradient plasticity theories. We formulate the variational equation for reversible models of medias. We write the conditions allowing to formulate formally the conditions of potentiality or its absence for density of energy. It is presented the general form of variational equations for the description of deformation processes of the dissipative medias, consistent in terms of the variational approach. It is shown that the proposed approach makes it possible to formulate correct physical models of dissipative processes of deformation and write down a complete system of equations and boundary conditions determining the mathematical model.

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In formulation of models, the variational approach is the most correct and complete, the variational model allows us to formulate a concerted equations of balance (movement) of the media and the boundary problem as a whole. At the same time, kinematic model of the media and the physical model are linked and coordinated. In this paper we formulate the kinematic model, identify free and constrained deformations and draw up the lists of items for the formulation of variational models of deformation of different options of media models. It is proposed to use the kinematic models constructed on the basis of definitions of consistent and inconsistent generalized deformations as criterias for the correctness of theories of defective medias and gradient plasticity theories. We formulate the variational equation for reversible models of medias. We write the conditions allowing to formulate formally the conditions of potentiality or its absence for density of energy. It is presented the general form of variational equations for the description of deformation processes of the dissipative medias, consistent in terms of the variational approach. It is shown that the proposed approach makes it possible to formulate correct physical models of dissipative processes of deformation and write down a complete system of equations and boundary conditions determining the mathematical model.

2 Kinematics of defective media

We suppose that the defective mediums are considered. Kinematics of a defective medium is determined not only by the vector field of displacements R_i, but also by the second rank tensor field of inconsistent (free) distortions D_{ij}. The inconsistency D_{ij} is given by the equation: $\Xi_{ij} = D_{im,n}e_{mnj} \neq 0$, where Ξ_{ij} is the tensor of density of dislocations(tensor De Vita). The symmetric part of the field D^2_{ij} is a field of inconsistent (free and non-integrable) strains $\varepsilon^{\text{free}}_{ij} = (D_{ij} + D_{ji})/2$. By definition, a tensor field of free distortions D_{ij} can not be represented in the form of a gradient of some vector field. Simultaneously with the free (inconsistent) distortions there are also integrable distortions. In contrast, the field of constrained (integrable) distortions is defined as the gradient of the displacements field. Such a field of distortions will be denoted by the superscript equal to unity: $d_{ij} = R_{i,j}$. Consequently, the total field of distortions (deformations) along with an inconsistent part must also contain the consistent part:

$$D^{\text{full}}_{ij} = d_{ij} + D_{ij} = R_{i,j} + D_{ij}, \quad d_{ij} = R_{i,j}. \tag{1}$$

The total field of distortions D_{ij} is not consistent:

$$D^{\text{full}}_{im,n}e_{mnj} = d_{im,n}e_{mnj} + D_{im,n}e_{mnj} = R_{i,mn}e_{mnj} + D_{im,n}e_{mnj} = 0 + D_{im,n}e_{mnj} \neq 0. \tag{2}$$

Quite often the physical sense of plastic deformations is prescribed to the tensor $\varepsilon^{\text{free}}_{ij}$, and the physical sense of the elastic deformations is prescribed to the tensor $\varepsilon^{e}_{ij} = (d_{ij} + d_{ji})/2 = (R_{i,j} + R_{j,i})/2$.
We will show that both of these statements should be used with extreme caution. Below we'll give the examples of theories with reversible deformation processes, containing non-integrable (free or, as they are interpreted, plastic) deformations / distortions. At the same time, it is quite justified to interpret the free distortions as plastic, for two reasons. Firstly, they are associated with the field of defects – dislocations, and therefore, they are free or not integrable and are not related to the displacements field (although, they are continuous). Secondly, in the classical theory of plasticity, dissipative processes are usually associated exactly with plastic deformations/distortions. Here we note that the Cauchy relation is valid only for constrained distortions, and not for the total ones, as it is often treated in many studies. This follows directly from the definition of constrained and free distortions. Therefore, equations (1), (2) may be a criterion for the correctness of the kinematic theory.
Now let's go back to the interpretation of free distortions as plastic in terms of dissipative processes. Using the feature of dissipativeness for such an interpretation is not entirely correct. Indeed, in this work, we will show that there are models of dissipative processes of deformations, where only constrain deformations/distortions

are used as the kinematic variables. Thus, dissipative processes can occur in completely defect-free environments where the traditional kinematic model of elasticity determined only by constrained, integrable distortions is used. The reversibility and irreversibility of the deformation process is determined only by the physical model (not by the kinematic model) which ultimately allows you to conclude whether or not there is the potential energy of deformation.

3 Physical models of defective media

We use a variational approach which by the selected kinematic model allows to construct the variational equation recorded in variations (Sedov's equation), to obtain Green's formulas and to use them to determine force model of the theory. Regardless of the choice of dissipative or reversible model of deformation, we establish from Sedov variational equation the equilibrium equations and the whole range of boundary value problems.
We present an algorithm for constructing physical models of medias with the fields of conserved [18], [19] dislocations, ie the medias, the kinematic model of which is determined by the vector of displacements R_i, by two kinds of distortions d_{ij}, D_{ij} and two kinds of curvatures $d_{ij,k}$, $D_{ij,k}$. For such a kinematic model, the principle of possible displacements allows to present uniquely the linear variational form of possible work of external and internal forced factors:

$$\bar{\delta}L = \delta A - \iiint [\sigma^{\alpha}_{ij}\delta D^{\alpha}_{ij} + m^{\alpha}_{ijk}\delta D^{\alpha}_{ij,k}]\, dV. \tag{3}$$

Here $\bar{\delta}L$ is generally non integrable linear variation form which in the case of integrability coincides with the Lagrangian variation. $A = \iiint P^V_i R_i\, dV + \oiint P^F_i R_i\, dF$ is the work of the external volume P^V_i and external surface forces P^F_i, $D^1_{ij} \equiv d_{ij}$, $D^2_{ij} \equiv D_{ij}$, σ^{α}_{ij} are the stresses of two types, producing work respectively on constrained and free of distortions; m^{α}_{ijk} are moment stresses of two kinds; $\alpha$is the grade index that takes values 1 and 2.
The condition for the existence of bulk density of the potential energy U_V in (3) are the generalized Green's formulas:

$$\sigma^{\alpha}_{ij} = \frac{\partial U_V}{\partial D^{\alpha}_{ij}}, \quad m^{\alpha}_{ijk} = \frac{\partial U_V}{\partial D^{\alpha}_{ij,k}}. \tag{4}$$

When the conditions (4) are satisfied, linear variation form is integrable and provides the definition of the Lagrangian $L$of corresponding reversible theory:

$$\begin{aligned}\delta L &= \delta\Big(A - \iiint U_V\, dV\Big) = \delta A - \iiint [\sigma^{\alpha}_{ij}\delta D^{\alpha}_{ij} + m^{\alpha}_{ijk}\delta D^{\alpha}_{ij,k}]\, dV = \\ &= \delta A - \iiint \left[\frac{\partial U_V}{\partial D^{\alpha}_{ij}}\delta D^{\alpha}_{ij} + \frac{\partial U_V}{\partial D^{\alpha}_{ij,k}}\delta D^{\alpha}_{ij,k}\right] dV.\end{aligned} \tag{5}$$

If the linear variation form in (5) is non integrable, the model defines the theory of some dissipative processes of deformation. Then, instead of (4), there are more general relations:

$$\sigma_{ij}^{\alpha} = \frac{\partial U_V}{\partial D_{ij}^{\alpha}} + \bar{\sigma}_{ij}^{\alpha}, \quad m_{ijk}^{\alpha} = \frac{\partial U_V}{\partial D_{ij,k}^{\alpha}} + \bar{m}_{ijk}^{\alpha}. \tag{6}$$

Where $\bar{\sigma}_{ij}^{\alpha}$ and $\bar{m}_{ijk}^{\alpha}$ are components of tensors of the second and third rank, respectively, which can not be presented as partial derivatives of some function of distortions and their gradients. In this respect, there is some analogy with (1): the first term in it has the potential, and the second one has no potential. From (4) and (6) we have:

$$\begin{aligned} \frac{\partial \sigma_{ij}^{\alpha}}{\partial D_{mn}^{\beta}} - \frac{\partial \sigma_{mn}^{\beta}}{\partial D_{ij}^{\alpha}} = 2\bar{C}_{ijmn}^{\alpha\beta} = -2\bar{C}_{mnij}^{\beta\alpha}, \\ \frac{\partial m_{ijk}^{\alpha}}{\partial D_{mn,l}^{\beta}} - \frac{\partial m_{mnl}^{\beta}}{\partial D_{ij,k}^{\alpha}} = 2\bar{C}_{ijkmnl}^{\alpha\beta} = -2\bar{C}_{mnlijk}^{\beta\alpha}. \end{aligned} \tag{7}$$

Upper Greek indices are grade indices (take the values 1 and 2).
It is easily seen that for reversible deformation processes $\bar{C}_{ijmn}^{\alpha\beta} = 0$ and $\bar{C}_{ijkmnl}^{\alpha\beta} = 0$, and for the irreversible (dissipative processes) $\bar{C}_{ijmn}^{\alpha\beta} \neq 0$ and $\bar{C}_{ijkmnl}^{\alpha\beta} \neq 0$.
We rewrite (7) for reversible processes in the following form:

$$\begin{aligned} \frac{\partial \sigma_{ij}^{\alpha}}{\partial D_{mn}^{\beta}} = \frac{\partial \sigma_{mn}^{\beta}}{\partial D_{ij}^{\alpha}} = C_{ijmn}^{\alpha\beta} = C_{mnij}^{\beta\alpha}, \\ \frac{\partial m_{ijk}^{\alpha}}{\partial D_{mn,l}^{\beta}} = \frac{\partial m_{mnl}^{\beta}}{\partial D_{ij,k}^{\alpha}} = C_{ijkmnl}^{\alpha\beta} = C_{mnlijk}^{\beta\alpha}. \end{aligned} \tag{8}$$

From equations (8) it follow that tensors of the fourth $C_{ijmn}^{\alpha\beta}$ and sixth grade $C_{ijkmnl}^{\alpha\beta}$ have the physical meaning of tensors of elastic moduli. By analogy, it is possible to treat the tensors of the fourth $\bar{C}_{ijmn}^{\alpha\beta}$ and sixth grade $\bar{C}_{ijkmnl}^{\alpha\beta}$ as tensors that have a physical meaning of tensors of dissipative modules. Taking into account (7) and (8), we can obtain:

$$\begin{aligned} \frac{\partial \sigma_{ij}^{\alpha}}{\partial D_{mn}^{\beta}} = C_{ijmn}^{\alpha\beta} + \bar{C}_{ijmn}^{\alpha\beta}, \\ \frac{\partial m_{ijk}^{\alpha}}{\partial D_{mn,l}^{\beta}} = C_{ijkmnl}^{\alpha\beta} + \bar{C}_{ijkmnl}^{\alpha\beta}. \end{aligned} \tag{9}$$

The sort of distortions does not influence at all the fact whether the physical model describes the reversible or irreversible processes. It is determined only by the symmetry of the tensors $C_{ijmn}^{\alpha\beta}$, $\bar{C}_{ijmn}^{\alpha\beta}$, and also of the tensors $C_{ijkmnl}^{\alpha\beta}$ and $\bar{C}_{ijkmnl}^{\alpha\beta}$:

$$C_{ijmn}^{\alpha\beta} = C_{mnij}^{\beta\alpha}, \qquad \bar{C}_{ijmn}^{\alpha\beta} = -\bar{C}_{mnij}^{\beta\alpha},$$

$$C_{ijkmnl}^{\alpha\beta} = C_{mnlijk}^{\beta\alpha}, \quad \bar{C}_{ijkmnl}^{\alpha\beta} = -\bar{C}_{mnlijk}^{\beta\alpha}.$$

Note that the reversible and irreversible models (9) in a particular case can be formulated in a physically linear statement. Indeed, if we hypothesize that in (9) tensors of four and six rank are independent of distortions and curvatures components, the relations (9) can be integrated and we can obtain the corresponding special case of linear equations of Hooke's law:

$$\sigma^{\alpha}_{ij} = (C^{\alpha\beta}_{ijmn} + \bar{C}^{\alpha\beta}_{ijmn})D^{\beta}_{mn}, \quad m^{\alpha}_{ijk} = (C^{\alpha\beta}_{ijkmnl} + \bar{C}^{\alpha\beta}_{ijkmnl})D^{\beta}_{mn,l}. \tag{10}$$

For the physically linear medias (10), the expression of possible work (3) takes the form:

$$\begin{aligned}
\bar{\delta} L &= \delta A - \iiint [\sigma^{\alpha}_{ij}\delta D^{\alpha}_{ij} + m^{\alpha}_{ijk}\delta D^{\alpha}_{ij,k}]\, dV = \\
&= \delta A - \iiint [(C^{\alpha\beta}_{ijmn} + \bar{C}^{\alpha\beta}_{ijmn})D^{\beta}_{mn}\delta D^{\alpha}_{ij} + \\
&\quad + (C^{\alpha\beta}_{ijkmnl} + \bar{C}^{\alpha\beta}_{ijkmnl})D^{\beta}_{mn,l}\delta D^{\alpha}_{ij,k}]\, dV = \\
&= \delta\left\{ A - \frac{1}{2}\iiint [C^{\alpha\beta}_{ijmn} D^{\alpha}_{ij} D^{\beta}_{mn} + C^{\alpha\beta}_{ijkmnl} D^{\alpha}_{ij,k} D^{\beta}_{mn,l}]\, dV \right\} - \\
&\quad - \iiint [\bar{C}^{\alpha\beta}_{ijmn} D^{\beta}_{mn}\delta D^{\alpha}_{ij} + \bar{C}^{\alpha\beta}_{ijkmnl} D^{\beta}_{mn,l}\delta D^{\alpha}_{ij,k}]\, dV = \\
&= \delta L - \frac{1}{2}\iiint [\bar{C}^{\alpha\beta}_{ijmn}(D^{\beta}_{mn}\delta D^{\alpha}_{ij} - D^{\alpha}_{ij}\delta D^{\beta}_{mn}) + \\
&\quad + \bar{C}^{\alpha\beta}_{ijkmnl}(D^{\beta}_{mn,l}\delta D^{\alpha}_{ij,k} - D^{\alpha}_{ij,k}\delta D^{\beta}_{mn,l})]\, dV.
\end{aligned} \tag{11}$$

Relation (11) can be formulated as a **theorem 1:**
"Any physical linear reversible model can be associated with dissipative model by the addition of possible work of all dissipation channels that are permissible within the selected kinematic model to the Lagrangian variation of the reversible model". Here, by the channel of dissipation we understand non integrable linear variational form (see also [20]):

$$\begin{aligned}
&\frac{1}{2}\iiint e^{A}_{ijmn}(D^{\beta}_{mn}\delta D^{\alpha}_{ij} - D^{\alpha}_{ij}\delta D^{\beta}_{mn})\, dV, \\
&\frac{1}{2}\iiint e^{B}_{ijkmnl}(D^{\beta}_{mn,l}\delta D^{\alpha}_{ij,k} - D^{\alpha}_{ij,k}\delta D^{\beta}_{mn,l})\, dV.
\end{aligned} \tag{12}$$

Here e^{A}_{ijmn} and e^{B}_{ijkmnl} are the components of the orthonormal bases in the space of tensors of fourth and sixth order. Each of $3(A+B)$ channels of dissipation is associated with its own dissipative module. Indeed, let's present the tensors of dissipative modules in the form of expansions on the basis tensors:

$$\bar{C}^{\alpha\beta}_{ijmn} = {}^{-\alpha\beta}_{A} e^{A}_{ijmn}, \quad \bar{C}^{\alpha\beta}_{ijkmnl} = \bar{G}^{\alpha\beta}_{B} e^{B}_{ijkmnl}. \tag{13}$$

Substituting (13) into (11), (12) we obtain:

$$\begin{aligned}
\bar{\delta} L &= \delta L - \\
&\quad - \bar{C}^{\alpha\beta}_{A}\frac{1}{2}\iiint [e^{A}_{ijmn}(D^{\beta}_{mn}\delta D^{\alpha}_{ij} - D^{\alpha}_{ij}\delta D^{\beta}_{mn})]\, dV - \\
&\quad - \bar{G}^{\alpha\beta}_{B}\frac{1}{2}\iiint [e^{B}_{ijkmnl}(D^{\beta}_{mn,l}\delta D^{\alpha}_{ij,k} - D^{\alpha}_{ij,k}\delta D^{\beta}_{mn,l})]\, dV
\end{aligned} \tag{14}$$

From equation (14) it follows that the number of channels of dissipation is equal to the number of dissipative modules $\bar{C}^{\alpha\beta}_{A}$ and $\bar{G}^{\alpha\beta}_{B}$.

4 Variational formulation of physically linear reversible theories

For the formulation of physically linear reversible models of media with field of defects, we will Lagrange principle. From the requirement of the stationarity of Lagrangian (5) follow the Euler equations and the entire spectrum of the boundary value problems:

$$\begin{aligned}
\delta L = \delta A - \iiint [\sigma^{\alpha}_{ij}\delta D^{\alpha}_{ij} + m^{\alpha}_{ijk}\delta D^{\alpha}_{ij,k}]\, dV = \\
= \delta A - \iiint [\sigma^{\delta}_{ij} d_{ij} + s_{ij}\delta D_{ij} + m^{\delta}_{ijk} d_{ij,k} + \beta^{\delta}_{ijk} D_{ij,k}]\, dV = \\
= \delta A - \iiint [(\sigma_{ij} - m_{ijk,k})\delta R_{i,j} + (s_{ij} - \beta_{ijk,k})\delta D_{ij}]\, dV + \\
+ \oiint \{-m_{ijk} n_k \delta R_{i,j} - \beta_{ijk} n_k \delta D_{ij}\}\, dF = \\
= \iiint [(\sigma_{ij,j} - m_{ijk,kj} + P^V_i)\delta R_i - (s_{ij} - \beta_{ijk,k})\delta D_{ij}]\, dV + \\
+ \oiint \{[P^F_i - (\sigma_{ij} - m_{ijk,k})n_j + (m_{ijk}n_k)_{,p}\delta^*_{pj}]\delta R_i - \\
- m_{ijk} n_j n_k \delta(R_{i,p} n_p) - \beta_{ijk} n_k \delta D_{ij}\}\, dF - \\
- \sum \oint m_{ijk} v_j n_k \delta R_i ds = 0.
\end{aligned} \tag{15}$$

Here $D^1_{ij} \equiv d_{ij}$, $D^2_{ij} \equiv D_{ij}$, $\sigma^1_{ij} \equiv \sigma_{ij}$, $\sigma^2_{ij} \equiv s_{ij}$, $m^1_{ijk} \equiv m_{ijk}$, $m^2_{ijk} \equiv \beta_{ijk}$.
The static model of physically linear reversible theories is determined by the particular case of the system (10):

$$\begin{cases} \sigma_{ij} = C^{11}_{ijmn} R_{m,n} + C^{12}_{ijmn} D_{mn}, \\ s_{ij} = C^{21}_{ijmn} R_{m,n} + C^{22}_{ijmn} D_{mn}, \end{cases} \quad \begin{cases} m_{ijk} = C^{11}_{ijkmnl} R_{m,nl} + C^{12}_{ijkmnl} D_{mn,l}, \\ \beta_{ijk} = C^{21}_{ijkmnl} R_{m,nl} + C^{22}_{ijkmnl} D_{mn,l}. \end{cases} \tag{16}$$

In this way, physically linear reversible theory of defective medias with the fields of conserved dislocations [18]–[20] is generally defined by twelve equilibrium equations:

$$\sigma_{ij,j} - m_{ijk,kj} + P^V_i = 0, \quad \sigma_{ij} - \beta_{ijk,k} = 0. \tag{17}$$

The spectrum of the boundary value problems is defined by fifteen pares of alternate the system of the boundary conditions in each non-singular point of the body surface:

$$\begin{cases} [P^F_i - (\sigma_{ij} - m_{ijk,k})n_j + (m_{ijk}n_k)_{,p}\delta^*_{pj}]\delta R_i = 0, \\ m_{ijk} n_j n_k \delta(R_{i,p} n_p) = 0, \\ \beta_{ijk} n_k \delta D_{ij} = 0. \end{cases} \tag{18}$$

In the special points of the body surface (on surface edges) there are additional conditions for continuity of the vector of displacements R_i and the vector of meniscus forces $m^1_{ijk} v_j n_k$at the transition of the body surface through the rib.
In the result variational equation (15) define the mathematical statement of the common enough model of media with defects for the reversible processes including constitutive equations (16), equilibrium equations (17) and the boundary conditions (18).

4.1 Particular theories

Let's consider as the particular cases of the theory (15) the gradient theory of Mindlin –Toupin [1], [3]. Assume that the kinematic model of the media does not allow the existence of concerved dislocations or fields of free distortion. Then the Lagrangian (15) coincides with the Lagrangian of the theory of Mindlin-Toupin:

$$L = A - \frac{1}{2}\iiint (C_{ijmn}R_{i,j}R_{m,n} + C_{ijkmnl}R_{i,jk}R_{m,nl})\, dV.$$

Static model (constitutive equations) of such gradient theory is determined by the particular case of the system (10), (16):

$$\sigma_{ij} = C_{ijmn}D_{mn}, \quad m_{ijk} = C_{ijkmnl}D_{mn,l}.$$

Accordingly, the Euler equations are defined by three equations of equilibrium of high order relatively to classical equations of the fourth order (in displacements):

$$\sigma_{ij,j}m_{ijk,kj} + P_i^V = 0.$$

The spectrum of the boundary value problems is defined by six pares of alternate boundary conditions in each non-singular point of the surface of the body:

$$\begin{cases} [P_i^F - (\sigma_{ij} - m_{ijk,k})n_j + (m_{ijk}n_k)_{,p}\delta^*_{pj}]\delta R_i = 0, \\ m^1_{ijk}n_j n_k \delta(R_{i,p}n_p) = 0. \end{cases}$$

As in general theory, in the singular points of the body surface (on surface edges) there are additional conditions for continuity of the vector of displacements R_i and the vector of meniscus forces $m_{ijk}v_j n_k$ at the transition of the body surface through the rib.

As the second particular case let us consider the theory of defective medium. Assume that the tensors of modules in (16) are such that $\overset{11}{}_{ijkmnl} = \overset{12}{}_{ijkmnl} = \overset{21}{}_{ijkmnl} = 0$. Then the Lagrangian (15) coincides with the Lagrangian of Mindlin theory cite2:

$$L = A - \frac{1}{2}\iiint (C^{11}_{ijmn}R_{i,j}R_{m,n} + 2C^{12}_{ijmn}R_{i,j}D_{mn} + C^{22}_{ijmn}D_{ij}D_{mn} + C_{ijkmnl}D_{ij,k}D_{mn,l})\, dV.$$

The static model of Mindlin theory is determined by the particular case of the system (10), (16):

$$\begin{cases} \sigma_{ij} = C_{ijmn}R_{m,n} + C^{12}_{ijmn}D_{mn}, \\ s_{ij} = C^{21}_{ijmn}R_{m,n} + C^{22}_{ijmn}D_{mn}, \end{cases} \qquad \begin{cases} m_{ijk} = 0 \\ \beta_{ijk} = C_{ijkmnl}D_{mn,l}, \end{cases}$$

Accordingly, the Euler equations (see (17)) are determined by twelve equations:

$$\sigma_{ij,j} + P_i^V = 0, \quad s_{ij} - \beta_{ijk,k} = 0.$$

The spectrum of the boundary value problems (see ((18)) is defined by twelve pares of alternate boundary conditions in each non-singular point of the surface body:

$$(P_i^F - \sigma_{ij}n_j)\delta R_i = 0, \quad \beta_{ijk}n_k\delta D_{ij} = 0.$$

In the Mindlin's type theory (22) there are no additional conditions on the surface edges.
Let's pay attention to the fact that, the formulated theory (15) has place for the reversible processes. However, there are free distortions and their gradients. This confirms the some incorrectness of the interpretation of the tensor of inconsistent deformities $\varepsilon_{ij}^{\text{free}} = (D_{ij} + D_{ji})/2$ as the tensor of plastic deformations, because plasticity is a dissipative property.

5 The variational formulation of physically linear irreversible theories

Using the definition of possible work of all static factors and equating it to zero, we can get instead (15) the following Sedov's variational equation for physically linear irreversible theories:

$$\begin{aligned}\bar{\delta}L = \delta\{A - \frac{1}{2}\iiint[C^{\alpha\beta}_{ijmn}D^{\alpha}_{ij}D^{\beta}_{mn} + C^{\alpha\beta}_{ijkmnl}D^{\alpha}_{ij,k}D^{\beta}_{mn,l}]\,dV\}-\\ -\frac{1}{2}\iiint[\bar{C}^{\alpha\beta}_{ijmn}(D^{\beta}_{mn}\delta D^{\alpha}_{ij} - D^{\alpha}_{ij}\delta D^{\beta}_{mn}) +\\ +\bar{C}^{\alpha\beta}_{ijkmnl}(D^{\beta}_{mn,l}\delta D^{\alpha}_{ij,k} - D^{\alpha}_{ij,k}\delta D^{\beta}_{mn,l})]\,dV = 0,\end{aligned} \tag{19}$$

here $C^{\alpha\beta}_{ijmn}(D^{\beta}_{mn}\delta D^{\alpha}_{ij} + D^{\alpha}_{ij}\delta D^{\beta}_{mn})$ is the variation of the bilinear part of the potential energy, $\bar{C}^{\alpha\beta}_{ijmn}(D^{\beta}_{mn}\delta D^{\alpha}_{ij} - D^{\alpha}_{ij}\delta D^{\beta}_{mn})$ is the non integrable variational form, which define the channels of the dissipation
The static model for physically linear irreversible theories is determined from (19) and can be written as (see (10)):

$$\begin{aligned}&\begin{cases}\sigma_{ij} = (C_{ijmn} + \bar{C}_{ijmn})R_{m,n} + (C^{12}_{ijmn} + \bar{C}^{12}_{ijmn})D_{mn},\\ \sigma^{2}_{ij} = (C^{21}_{ijmn} + \bar{C}^{21}_{ijmn})R_{m,n} + (C^{22}_{ijmn} + \bar{C}^{22}_{ijmn})D_{mn},\end{cases}\\ &\begin{cases}m_{ijk} = (C^{11}_{ijkmnl} + \bar{C}^{11}_{ijkmnl})R_{m,nl} + (C^{12}_{ijkmnl} + \bar{C}^{12}_{ijkmnl})D_{mn,l},\\ \beta_{ijk} = (C^{21}_{ijkmnl} + \bar{C}^{21}_{ijkmnl})R_{m,nl} + (C^{22}_{ijkmnl} + \bar{C}^{22}_{ijkmnl})D_{mn,l},\end{cases}\end{aligned} \tag{20}$$

By example we consider the stresses of first grade. They can be represented as a sum of reversible and irreversible parts (20):

$$\begin{aligned}\sigma_{ij} &= (C_{ijmn} + \bar{C}_{ijmn})R_{m,n} + (C^{12}_{ijmn} + \bar{C}^{12}_{ijmn})D_{mn} =\\ &= (C_{ijmn}R_{m,n} + C^{12}_{ijmn}D_{mn}) + (\bar{C}_{ijmn}R_{m,n} + \bar{C}^{12}_{ijmn}D_{mn}).\end{aligned}$$

The reversible part of the stresses depends on both constrain and free distortions with the tensors of reversible modules C^{11}_{ijmn}, C^{12}_{ijmn}. The irreversible part of the stresses also depends on both sorts of distortions, but with another set of tensors of modules – with the dashed modules $\bar{C}^{11}_{ijmn}$, $\bar{C}^{12}_{ijmn}$.

In expanded form, Sedov's equation becomes:

$$\begin{aligned}&\iiint[(\sigma_{ij,j}-m_{ijk,kj}+P_i^V)\delta R_i-(s_{ij}-\beta_{ijk,k})\delta D_{ij}]\,dV+\\&\quad+\oiint\{[P_i^F-(\sigma_{ij}-m_{ijk,k})n_j+(m_{ijk}n_k)_{,p}\delta^*_{pj}]\delta R_i-\\&\quad-m_{ijk}n_jn_k\delta(R_{i,p}n_p)-\beta_{ijk}n_k\delta D_{ij}\}\,dF-\\&\quad-\sum\oint m_{ijk}v_jn_k\delta R_i ds=0.\end{aligned}\tag{21}$$

It is easy to make sure that the dissipative (21) and reversible (15) models have the same form, being recorded in the static factors.
At the same time, the formulation of physically linear irreversible theories in kinematic variables will be different due to other equations of generalized law (20).

5.1 Model of Mindlin–Toupin gradient theory with dissipation

Let us consider Mindlin–Toupin gradient theory for the reversible processes. Then we must assume that $D_{ij}=0$. Then (19) takes the form:

$$\begin{aligned}\bar\delta L=\delta&\left\{A-\frac{1}{2}\iiint C^{11}_{ijmn}R_{i,j}R_{m,n}+C_{ijkmnl}R_{i,jk}R_{m,nl}\,dV\right\}-\\&-\frac{1}{2}\iiint[\bar C_{ijmn}(R_{m,n}\delta R_{i,j}-R_{i,j}\delta R_{m,n})+\\&+\bar C_{ijkmnl}(R_{m,nl}\delta R_{i,jk}-R_{i,jk}\delta R_{m,nl})]\,dV=0.\end{aligned}\tag{22}$$

It is easy to see that there are non-zero dissipation channels in the variational equation (22).
Let's note that the formulated theory (22) is a dissipative gradient theory. However this theory is written using only constrain distortions and has not free distortions and their gradients.
This confirms the incorrectness of the interpretation of the tensor of constrain deformations $\varepsilon_{ij}=(d_{ij}+d_{ji})/2=(R_{i,j}+R_{j,i})/2$ as the tensor of elastic deformations, since the joint deformations allows plasticity, a dissipative property, too.

5.2 Dissipative Landau theory

Taking into account theorem 1 (see (11) in section 3) we can introduce in the list of dissipation channels also the bilinear non-integrable variational forms, containing gradients of velocity fields and write Sedov's equation in the form:

$$\bar\delta L=\delta L-\iiint\int\bar E^{(}_{ijmn}\dot R_{m,n}\delta R_{i,j}-R_{i,j}\delta\dot R_{m,n})\,dV\,dt=0.\tag{23}$$

The qualitative feature of the Landau model (23) is that there are new kinematic variables in it – the distortions of rates. Accordingly, we define the new static factors – tensor of momentums p_{mn}:

$$p_{mn}=\frac{\partial L_V}{\partial\dot R_{m,n}}+\bar E_{mnij}R_{i,j}.\tag{24}$$

Along with this, the classical stresses get additional terms associated with dissipation:

$$\sigma_{ij} = \frac{\partial L_V}{\partial R_{i,j}} + \bar{E}_{ijmn}\dot{R}_{m,n}. \tag{25}$$

Sedov's equation in this case (24), (25) gives to the following variation equation:

$$\begin{aligned}
\bar{\delta} L = \delta A - \iiint \int \Bigg\{ \left(\frac{\partial L_V}{\partial R_{i,j}} + \bar{E}_{ijmn}\dot{R}_{m,n} \right) \delta R_{i,j} + \frac{\partial L_V}{\partial R_{i,jk}} \delta R_{i,jk} - \\
- \rho \dot{R}_m \delta \dot{R}_m + \left(\frac{\partial L_V}{\partial \dot{R}_{m,n}} - \bar{E}_{ijmn} R_{i,j} \right) \delta \dot{R}_{m,n} \Bigg\} dV\, dt = \\
= \iiint \int \Bigg\{ \Bigg[\frac{\partial}{\partial x_j} \frac{\partial L_V}{\partial R_{i,j}} - \frac{\partial}{\partial x_j \partial x_k} \frac{\partial L_V}{\partial R_{i,jk}} + \bar{E}^{11}_{ijmn} \dot{R}_{m,nj} - \rho \ddot{R}_i + P^V_i + \\
+ \frac{\partial}{\partial t \partial x_j} \left(\frac{\partial L_V}{\partial \dot{R}_{i,j}} - \bar{E}^{11}_{pqij} R_{p,q} \right) \Bigg] \delta R_i \Bigg\} dV\, dt - \\
+ \oiint \int \Bigg\{ \Bigg[P^F_i - \left(\frac{\partial L_V}{\partial R_{i,j}} - \frac{\partial}{\partial x_k} \frac{\partial L_V}{\partial R_{i,jk}} - \frac{\partial}{\partial t} \frac{\partial L_V}{\partial \dot{R}_{i,j}} \right) n_j + \left(\frac{\partial L_V}{\partial R_{i,jk}} n_k \right)_{,p} \delta^*_{pj} \Bigg] \delta R_i + \\
+ \left(-\frac{\partial L_V}{\partial R_{i,jk}} n_j n_k \right) \delta(R_{i,p} n_p) \Bigg\} dF dt - \sum \oint \int \left(\frac{\partial L_V}{\partial R_{i,jk}} v_j n_k \right) \delta R_i \, ds\, dt - \\
- \Bigg\{ \iiint \Bigg[\rho \dot{R}_m - \frac{\partial}{\partial x_n} \left(\frac{\partial L_V}{\partial \dot{R}_{m,n}} - \bar{E}^{11}_{ijmn} R_{i,j} \right) \Bigg] \delta R_m \, dV + \\
+ \oiint \left(\frac{\partial L_V}{\partial \dot{R}_{m,n}} - \bar{E}^{11}_{ijmn} R_{i,j} \right) n_n \delta R_m \, dF \Bigg\} \Bigg|_{t=t_0}^{t=t_k} = 0.
\end{aligned} \tag{26}$$

As a result using (26) we can write the equations of motion of dissipative medias in the form:

$$\frac{\partial}{\partial x_j} \frac{\partial L_V}{\partial R_{i,j}} - \frac{\partial}{\partial x_j \partial x_k} \frac{\partial L_V}{\partial R_{i,jk}} - \rho \ddot{R}_i + P^V_i = -\frac{\partial}{\partial t \partial x_j} \frac{\partial L_V}{\partial \dot{R}_{i,j}} - 2\bar{E}^{11}_{ijmn} \dot{R}_{m,nj} \tag{27}$$

The left side of equations (27) is the operator of the reversible theory, and the right one describes the corrections defined by the dissipation.

Boundary problem in (26) still defines six pairs of alternative boundary conditions in each non-singular point of a surface:

$$\begin{cases} \left[P^F_i - \left(\frac{\partial L_V}{\partial R_{i,j}} - \frac{\partial}{\partial x_k} \frac{\partial L_V}{\partial R_{i,jk}} - \frac{\partial}{\partial t} \frac{\partial L_V}{\partial \dot{R}_{i,j}} \right) n_j + \left(\frac{\partial L_V}{\partial R_{i,jk}} n_k \right)_{,p} \delta^*_{pj} \right] \delta R_i = 0, \\ \left(-\frac{\partial L_V}{\partial R_{i,jk}} n_j n_k \right) \delta(R_{i,p} n_p) = 0. \end{cases} \tag{28}$$

As follows from conditions (28) Sedov's equation requires continuity of the vector of displacements R_i in the specific points on the surface (on the surface edges, if any one exist) and the vector of meniscus forces $\left(\frac{\partial L_V}{\partial R_{i,jk}} v_j n_k\right)$ at the transition of the body surface through the rib:

$$\sum \oint \int \left(\frac{\partial L_V}{\partial R_{i,jk}} v_j n_k \right) \delta R_i \, ds\, dt = 0.$$

It is important that the Landau theory leads to a boundary value problem on time instead of the Cauchy problem (initial value problem).

$$\left\{\iiint\left[\rho\dot{R}_m-\frac{\partial}{\partial x_n}\left(\frac{\partial L_V}{\partial\dot{R}_{m,n}}-\bar{E}^{11}_{ijmn}R_{i,j}\right)\right]\delta R_m\,dV+\right.$$
$$\left.+\oiint\left(\frac{\partial L_V}{\partial\dot{R}_{m,n}}-\bar{E}^{11}_{ijmn}R_{i,j}\right)n_n\delta R_m\,dF\right\}\Bigg|_{t=t_0}^{t=t_k}=0.$$

6 Conclusion

We formulated the Lagrangian in the case of static problems and we proposed the applied version of the theory, which provides the reduction of the physical constants up to the maximum reduction when the physical properties are determined by the classical elastic moduli. We show that the kinematic model built on the basis of strict concepts for integrable and non-integrable generalized strains can be a criterion of correctness of theories of defective media and gradient theories of plasticity. It is proposed the revision of the variants of theories with defects fields where the integrable properties and generalized Cauchy relations are unreasonably attributed to the total strains. We consider the models of the dynamic dissipative media in which dissipative properties are associated with the strain rates. It is given the formulation of the variation of the dissipative part of energy which provides the dissipativeness properties (properties of the lack of potential). We establish the conditions under which the introduction of energy of dissipation, can be justified formally by comparing the motion equations and boundary conditions with equations obtained correctly using the variation of energy of dissipation.

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S. Lurie, Institute of Applied Mechanics of RAS, Moscow, Institute for Problems in Mechanics of RAS

Qi Chengzhi, Beijing University of Civil Engineering and Architecture

P. Belov, Scientific-innovation Center "Institute of Research, Development and Technology Transfer"