Assessing Deformation due to Self-Gravitation - Treacherous Pathways of Continuum Mechanics

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Abstract

In an attempt of modeling the state of deformation in self-gravitating terrestrial bodies linear Hookean elasticity formulated in terms of linear strain measures is used initially. In this case the solutions for the stresses, the strains, and the displacements are unique and can be presented in closed form. The equations will be evaluated by using data for Mercury and Earth in order to show that the displacements can be enormous. This illustrates the limit of a geometrically linear theory.

In order to improve the situation we will then present an "extended model" and study the influence of linear terms of displacement gradients in the body force density, hence, we assume the body forces to adjust with the current configuration. This non-standard approach to linear elasticity serves as a bridge to a consistently performed large-deformation-analysis while keeping the advantage of a closed-form expression.

Next we choose a geometrically nonlinear version of Hooke's law, the Saint-Venant-Kirchhoff constitutive equation written in terms of the current configuration. By doing that we should be able to handle large strains. However, even this relation creates problems, since it suffers from instability at very high compressive strains, *i.e.*, if the mass of the self-gravitating object becomes too large. There will be a limit mass beyond which stresses will go to infinity, similarly to the Chandrasekhar limit.

Finally we will turn to time-dependent modeling of deformation in terms of a deformation-wise linear viscoelastic model of the Kelvin-Voigt type. Surprisingly this model allows for a closed-form solution. As a new result it will turn out that in the early days of planet formation the so-called Love radius, which is the demarcation line between the completely compressive interior of a planet from a radial strain-wise tensile exterior, does not exist initially and requires time for its development.

1 Introduction

The purpose of this paper is quite a fundamental one: We wish to draw attention to the fact that modeling nature by using continuum theory can be based on treacherous assumptions. Whilst we may have some confidence in the applicability of the fundamental laws of classical physics, such as the conservation of mass, linear and angular momentum, or energy, the use of constitutive relations requires special care and sound scepticism, even if they follow from principles of rational thermodynamics. As a matter of fact, many engineers of daily practice are not even aware of the fundamental difference between conservation laws and “material equations” as the latter are sometimes innocently called. They believe that they are “true laws of nature.” Surely, there might be limits to their applicability, if strains become too high or temperatures are too low (say), but very often this is attributed to numerical inaccuracy rather than a principal internal deficiency. This dilemma is nicely depicted in a recent textbook [7].

In our paper we will use self-gravitation in massive celestial bodies as a concrete example. We will concentrate on “solid bodies,” whatever a solid planet may be, since we all know that it will likely show an onion-like internal structure and some of the “onion skins” will be more on the liquid than on the solid side.

We will start on seemingly firm ground, so-to-speak: Linear Hookean elasticity formulated in terms of linear strain measures is the simplest way of modeling deformation in self-gravitating terrestrial bodies. In this case the solutions for the stresses, the strains, and the displacements are unique and can be presented in closed form, as was first shown by the great (linear) elastician A.E.H. Love around the beginning of last century [1]. We will, first, present the underlying theory in modern form. Second, solutions to the resulting equations will be obtained. Third, the limits of the equations will be illustrated by using physical data Mercury and Earth. This will show that under certain circumstances the displacements may be enormous. Consequently, the limits of linear strain theory will become evident.

As a special feature we will then leave the canonical pathway of linear elasticity, where it is conventionally assumed that the body forces are applied to the undeformed configuration [2]. In contrast to conventional (engineering) literature, we will present an вЃЁextended modelвЃ№ and study the influence of linear terms of displacement gradients in the body force density. In fact, this approach may serve as a bridge between linear elasticity at small strains and elasticity at large deformations. Moreover, it has the advantage of keeping its closed-form solution character.

In an attempt to remedy the problem of large deformations once and for all we will choose a geometrically nonlinear version of HookeвЃЄs law in the current configuration. More precisely, the Cauchy stress will be related to the nonlinear deformation measure of the current configuration, the Euler-Almansi finite strain, which replaces the linear strain tensor of the ordinary Hooke’s law. This is known as the Saint-Venant-Kirchhoff constitutive equation in the literature [3], albeit stated in the current configuration. Using this type of stress-strain relation in context with self-gravitating objects was first suggested in the sixties of last century, see for example [4]. However, this approach has drawbacks: As we shall see, we will run into modeling and numerical problems again, if the mass of the self-gravitating object becomes too large. There will be a limit mass beyond which stresses will go to infinity, similarly to the case of the Chandrasekhar limit for the mass of white dwarf stars. However, this phenomenon is an artifact of the constitutive law we chose for the stress-strain relation: It can lead to a unique, two, three, or no solutions for the problem. In fact, today it is known (see [3], Section 4.3 and [5]) that this constitutive model suffers from material instability in large compressive deformation case.

Finally we will turn to time-dependent modeling of deformation. We will use a linear viscoelastic model of the Kelvin-Voigt type [6]. Surprisingly it allows for a closed-form solution for a solid as well as for a hollow sphere. As a new result it will turn out that in the early days of planet formation the so-called Love radius, which is the demarcation line between the completely compressive interior of a planet from a radially strain-wise tensile exterior, does not exist initially and requires time for its development. Interestingly the solution for the solid sphere will not lead to zero deformation in the limit of initial time. Rather it jumps abruptly to finite values varying linearly throughout the sphere. If the same limit is considered in the solution for the hollow sphere with a very small hole at the center one can see the reason for this behavior: The transition from zero to finite deformation is extremely fast. In other words: If gravitation is "switched on," large amounts of mass will start moving and it is inapt to use the static form of the balance of momentum. Inertial forces should be taken into account. Hence, this time it is *not* a fault of the constitutive equation but an inappropriate simplification of the equations of motion, which creates a problem.

2 Linear Elasticity

We base our analysis on the static balance of momentum:

$$\nabla \cdot \boldsymbol{\sigma} = -\rho \boldsymbol{f}, \tag{1}$$

where ρ denotes the local current mass density, and $\boldsymbol{\sigma}$ the Cauchy stress tensor. The specific body force, $\boldsymbol{f}$, *i.e.*, the gravitational acceleration, is conservative and originates from self-gravity. Consequently, a gravitational potential $U^{\text{grav}}(\boldsymbol{x})$ exists, where $\boldsymbol{x}$ denotes an arbitrary (current) position within the body, and we may write:

$$\boldsymbol{f}(\boldsymbol{x}) = -\nabla U^{\text{grav}}(\boldsymbol{x}). \tag{2}$$

The gravitational potential obeys Poisson's equation:

$$\Delta U^{\text{grav}}(\boldsymbol{x}) = 4\pi G \rho(\boldsymbol{x}). \tag{3}$$

If we consider a perfectly spherical case, where we have only radial dependencies, we find from the last two relations that:

$$\rho(r)\boldsymbol{f}(r) = -G\frac{\rho(r)m(r)}{r^2}\,\boldsymbol{e}_r, \tag{4}$$

where $\boldsymbol{e}_r$ is the radial unit vector, $m(r)$ denotes the total mass within a spherical region of radial extension r:

$$m(r) = 4\pi \int\limits_{\tilde{r}=0}^{\tilde{r}=r} \rho(\tilde{r})\tilde{r}^2 \mathrm{d}\tilde{r}\,,\ 0 \leq r \leq r_\mathrm{o}, \tag{5}$$

and r_o stands for the current outer radius of the spherical body.

However, in linear elasticity it is customary to use the body force acting on the *undeformed* body. Consequently, if we pretend everything is initially homogeneous and use a constant mass density, ρ_0, we obtain:

$$\rho(r)\boldsymbol{f}(r) \approx -G\frac{\rho_0 m(r)}{r^2}\,\boldsymbol{e}_r \approx -\frac{4\pi G\rho_0^2}{3} r\,\boldsymbol{e}_r. \tag{6}$$

For the stress tensor we assume Hooke's law for an isotropic body to hold:

$$\boldsymbol{\sigma} = \lambda\ \mathrm{Tr}\boldsymbol{\epsilon}\ \mathbf{1} + 2\mu\ \boldsymbol{\epsilon}. \tag{7}$$

Moreover, in this context we use the *linear* strain tensor:

$$\boldsymbol{\epsilon} = \tfrac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^\top). \tag{8}$$

$\boldsymbol{u}$ refers to the displacement vector, *i.e.*, to $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$, $\boldsymbol{X}$ being the reference position of a material point of the sphere. λ and μ are Lamé's elastic constants.
If spherical coordinates are used and these relations are inserted into each other the following differential equation of second order for the dimensionless radial displacement, $u(x) := u_r(r)/r_\mathrm{o}$, results:

$$u'' + 2\frac{u'}{x} - 2\frac{u}{x^2} = \frac{\alpha}{2}x, \tag{9}$$

where the dash refers to differentiation w.r.t. dimensionless position, $x := r/r_\mathrm{o}$, and with a dimensionless factor, the gravity-stiffness constant:

$$\alpha = \frac{8\pi G\rho_0^2 r_\mathrm{o}^2}{3(\lambda + 2\mu)}. \tag{10}$$

If regularity at $x = 0$ and a traction-free outer surface is prescribed, its solution reads (for further details see [8], $\nu = \frac{\lambda}{2(\lambda+\mu)}$ is Poisson's ratio):

$$u = -\frac{\alpha}{20}\left(\frac{3-\nu}{1+\nu} - x^2\right)x. \tag{11}$$

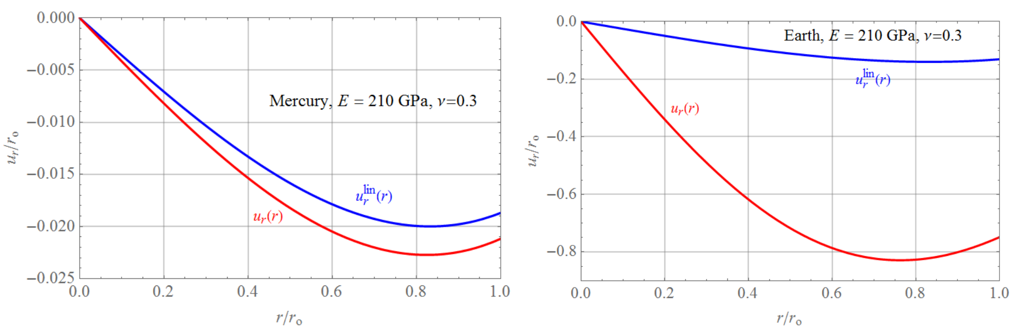

Figure 1: Radial displacement *vs.* normalized radius: Fully linear solution (blue) and "extended" solution (red); for data see text.

The blue curves in Fig. 1 show plots of Eqn. (11) when data of Mercury (index M) and Earth (index E) are used for evaluation. Specifically we have values for

the average mass densities of $\rho_0^{\mathrm{E}} = 5500\frac{\mathrm{kg}}{\mathrm{m}^3}$ and $\rho_0^{\mathrm{M}} = 5400\frac{\mathrm{kg}}{\mathrm{m}^3}$ and (average) outer radii of $r_{\mathrm{o}}^{\mathrm{E}} = 6370\,\mathrm{km}$ and $r_{\mathrm{o}}^{\mathrm{M}} = 2440\,\mathrm{km}$, respectively. For the elastic data we assume in both cases the values of iron, *i.e.*, $E \equiv \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ =210 GPa and $\nu = 0.3$. Consequently, the strains for Mercury are below 2% but the ones for Earth are huge and amount to a maximum of 14%. Hence one may question the validity of the use of linear elasticity in case of very large self-gravitating masses and turn to a non-linear formulation instead. Indeed, this has been done by the Indian school of Seth, who was one of the pioneers in large strain measures. We will explore this in the next section.
Note that, as it should be, the radial displacement is negative and that the curves show a minimum. The slope of the curves in Fig. 1 is proportional to radial strain. Hence the only positive (=tensile) radial strains can be found to the right of the minimum. The transition point between positive and negative strains, identifiable by locating the minimum of the displacement, is a.k.a. the *Love radius*:

$$r_{\mathrm{Love}} = r_0\sqrt{\frac{3-\nu}{3(1+\nu)}}. \tag{12}$$

The first edition of his monograph [1] makes it perfectly clear that Love was aiming at a failure criterion in context with this transition point, namely specifically at what is known today as *maximum principal strain theory*: Tensile strains might lead to damage. However, if one inserts real numbers into the expression for the corresponding strain (see [8], pp. 30) it turns out that the corresponding numbers are completely unrealistic. Nevertheless, so far no better model for studying damage due to severe self-gravitation has been developed. Consequently, it still has its merits, be it even just to remind us of the fact that a better model is needed.
Before turning to nonlinear theory we shall briefly depart from one of the crucial prerequisites of the linear theory of elasticity at small deformations: We shall no longer insist that the body force is applied to the undeformed configuration. In agreement with the principles of a consistently linearized theory we will now take *linear* terms of the displacement and its derivative on the right hand side of the momentum balance into account according to Eqns. $(6)_1$ and put:

$$\rho(\boldsymbol{x}) \approx \rho_0\left[1 - \mathsf{Tr}\,\boldsymbol{\epsilon}(\boldsymbol{x})\right]. \tag{13}$$

Hence it follows that:

$$\rho(r)\boldsymbol{f}(r) \approx -\frac{\lambda+2\mu}{r_{\mathrm{o}}}\frac{\alpha}{2}x\left(1 - u' - 5\frac{u}{x}\right)\boldsymbol{e}_r. \tag{14}$$

The equivalent to Eqn. (9) then reads:

$$u'' + \left(\frac{2}{x} + \frac{\alpha}{2}x\right)u' - \left(\frac{2}{x^2} - \frac{5\alpha}{2}\right)u = \frac{\alpha}{2}x. \tag{15}$$

It can be solved analytically:

$$u = \frac{1}{96\alpha}\frac{1}{x^2}\Bigg[2x\left(6 + 5\alpha x^2 + 6A\left(2 - \alpha x^2\right)\right) - \tag{16}$$

$$3(1+2A)\left(4+4\alpha x^2-\alpha^2 x^4\right)\exp\left(-\tfrac{\alpha x^2}{4}\right)\int\limits_{\tau=0}^{\tau=x}\exp\left(\tfrac{\alpha\tau^2}{4}\right)\mathrm{d}\tau\Bigg].$$

The remaining constant of integration, A, follows from the requirement of vanishing traction, *i.e.*, vanishing radial stress at the outside, r_o, of the sphere, $u'|_{x=1} + \frac{2\nu}{1-\nu}\, u|_{x=1} = 0$. The result is a very lengthy yet closed-form expression, which we decided not to present here.

Fig. 1 gives us a foretaste of what to expect if the gravitating mass becomes really large. Two curves are presented, the fully linear solution according to Eqn. (11) in blue and the "extended" solution based on Eqn. (16) in red. The equations were evaluated for Mercury and for Earth according to the data presented before. Although α is already equal to 0.347 in the case of Mercury (*i.e.*, it is *not* small but in the range of more than 30 percent) the difference between the two predictions for the normalized displacement are close together, and in the range of 2%. This is very different for Earth: Both predictions are far apart and we should expect normalized displacements up to 90%. We will get back to this issue in the next section.

3 Geometrically Nonlinear Elasticity

We start our discourse on nonlinear effects by mentioning that the original left hand side of the differential equation (15) consisted of three small quantities of first order, namely of u and its spatial derivatives. Consequently, its right hand side should also be small, *i.e.*, consist of first order terms comparable in magnitude to $u(x)$, $u'(x)$, and $u''(x)$. Hence, we must ask the question as to whether the factor α is "small" or not. Unfortunately the size of α is dictated by physics and not by mathematics. It cannot really be chosen freely and it is as large as required by the celestial object we wish to model. At the end of the last section we have seen that even for small gravitating objects like Mercury it is of the order of thirty percent, which cannot really be considered as small. We wish to improve upon the situation and use the most simple counterpart to Eqn. (7) by replacing $\boldsymbol{\epsilon}$ with the Euler-Almansi finite strain tensor, e:

$$2\boldsymbol{e} := 1 - \boldsymbol{F}^{-1^\top}\cdot\boldsymbol{F}^{-1}\ ,\quad \boldsymbol{F} := \nabla_{\boldsymbol{X}}\boldsymbol{x}(\boldsymbol{X}), \tag{17}$$

$\boldsymbol{F}$ being the deformation gradient joining current and reference positions, $\boldsymbol{x}$ and $\boldsymbol{X}$, respectively. The corresponding stress-strain relation could be called Saint-Venant-Kirchhoff law in the current configuration. It is physically linear but geometrically nonlinear and was originally suggested by Seth [9]:

$$\boldsymbol{\sigma} = \lambda\ \mathsf{Tr}\boldsymbol{e}\ \mathbf{1} + 2\mu\ \boldsymbol{e} \tag{18}$$

A group of Indian scholars (see for example [4]) used it to study selfgravitation during the sixties. The corresponding differential equation for the normalized radial displacement reads (see [8], Section 3.1.3 for a proof):

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[u'\left(1-\tfrac{1}{2}u'\right)+\frac{2\nu}{1-\nu}\frac{u}{x}\left(1-\tfrac{1}{2}\frac{u}{x}\right)\right]+ \tag{19}$$

$$\frac{2(1-2\nu)}{1-\nu}\frac{1}{x}\left[u'\left(1-\tfrac{1}{2}u'\right)-\frac{u}{x}\left(1-\tfrac{1}{2}\frac{u}{x}\right)\right]=\frac{\alpha}{2}x(1-u')\left(1-\frac{u}{x}\right)^5.$$

It must be solved numerically by taking into account vanishing displacement (or better "regularity") at the center and vanishing radial stress at the outer surface. Various techniques are possible ranging from Runge-Kutta shooting methods, series expansions, up to finite differences and finite elements. Two results are shown in Fig. 2. For these plots the geometry and mass data for Mercury and Earth were chosen. There is a slight variation in the choice of Poisson's ratio in the case of Earth compared to our study in Fig. 1.: If the original elastic data were used for Earth α increases up to 1.96. For this value a solution could no longer be obtained (see the discussion below). Hence, Poisson's ratio was raised to 0.38, which is equivalent to $\alpha \approx 1.75$. The reason for the lack of solution becomes immediately apparent by looking at the plots: In the case of Mercury, there is already a slight discrepancy to the analytical solution, $u_r^{\text{anal}}(r)$, shown in Eqn. (7). The analytical solution (slightly) *underestimates* the displacement. This is not surprising since the radial strain, which is roughly given by u_r/r_o, is more than two percent, hence, probes the limits of a geometrically linear theory.

In the case of Earth the situation is much more dramatic. First, the difference between the analytical and the numerical solution is huge and, second, even the analytical solution already predicts strains of almost 10%, whereas the numerical solution amounts to 30% and more. Note that the curvature of $u_r(r)$ "on the left" becomes more and more pronounced when the α-values increase. For larger values of α, *i.e.*, for large values of reference density and small values of Young's modulus and/or Poisson's ratio, the $u_r(r)$-curve will first decline very steeply and then show an essentially linear behavior with a moderate slope.

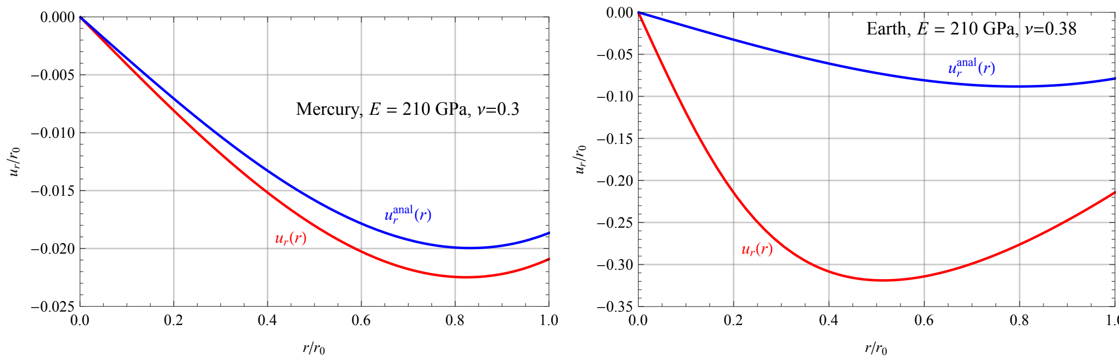

Figure 2: Radial displacement as a function of radial position.

We proceed to investigate this in more detail. The plots shown in Fig. 3 present a study of solutions to the nonlinear differential equation (19) in the following manner: For a given choice of Poisson's ratio, ν, and gravity-stiffness-parameter, α, a numerical solution was found by using a shooting technique. To this end use was made of the boundary condition $u(0) = 0$ with $u(x) = x[1-\beta(x)]$, where $\beta'(x=0) = 0$ and $\beta(0)$ was varied within a reasonable range. This allowed to calculate numerical values for the radial stresses σ_{rr} on the interval $x \in \{0,1\}$ based on Eqn. (18) and, in particular, at the position $x = 1$, *i.e.*, at the right boundary, where it must vanish. Hence we have to insist on the condition that $\sigma_{rr}(x=1) \overset{!}{=} 0$ among all the values

found. During this procedure Poisson's ratio was kept constant, $\nu = 0.3$, while the value of α increased steadily.
The first inset where $\alpha = 0.35$ depicts the situation for Mercury. As we can see the nonlinear differential equation has only one well-defined solution, $\beta(x=0) \approx 1.041$, and this would essentially lead to the displacement distribution shown in Fig. 2_1. If we increase α to 1 and to 1.3, the situation stays the same: There is only one solution. However, the curves become more and more non-monotonic. Finally at $\alpha \approx 1.3887$ a second solution appears and if we increase the parameter even a bit more, $\alpha = 1.39$, we even end up with three solutions. These disappear and only two solutions result again if $\alpha \approx 1.39618$. If we keep increasing the parameter, *i.e.*, we choose $\alpha \approx 1.3962$ and $\alpha \approx 1.4$, respectively, we find first only one and finally no solution at all. This is essentially what we observe in Fig. 2: There is an end to calculating a displacement distribution for large values of α. Does this mean that Earth might not show strains larger than 30 percent due to its huge self-gravity? Of course not! All of this is an artifact of the constitutive equation used to connect stresses and strains, even if nonlinear strain measures are used. It should be mentioned that the malfunction of the Saint-Venant-Kirchhoff model is well known and was put into context with Ball's notion of missing polyconvexity of the strain energy density function, [3], Section 4.3. As an example one may want to study the one-dimensional analogue to our selfgravitating sphere, the compression of a one-dimensional Saint-Venant-Kirchhoff beam [10].
This shows very clearly that the cure to the problem does not come by turning to non-linear deformation measures alone, as Love and the group of Seth may have thought. It is much more complex: We need to find a physically sound connection between stresses and strains *and* nonlinear strain measures.
Fig. 4_1 presents a study of the movement of various material points from the perspective of the reference configuration. Imagine a sphere of homogeneous mass density, ρ_0, and outer radius, R_o, which is initially completely stress-free. In order to study its deformation after gravity has beens "switched on," we first normalize distances and displacements in contrast to our previous line of reasoning by that outer radius, $\bar{x} = \frac{r}{R_o}$, $\bar{u} = \frac{u_r}{R_o}$ and then write in view of Eqn. (5) for the current mass of a radially-symmetric planet up to a position x_0:

$$m(\bar{x}) = m_0 \bar{x}^3 \left(1 - \frac{\bar{u}}{\bar{x}}\right)^3 \;, \;\; m_0 = \tfrac{4\pi}{3} \rho_0 R_o^3. \tag{20}$$

Note that $\bar{x}$ will always be smaller than one, which is in contrast to its counterpart, x. Eqn. (19) will now take the following form:

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}\bar{x}} \left[\bar{u}' \left(1 - \tfrac{1}{2}\bar{u}'\right) + \tfrac{2\nu}{1-\nu} \tfrac{\bar{u}}{\bar{x}} \left(1 - \tfrac{1}{2}\tfrac{\bar{u}}{\bar{x}}\right)\right] + \\ &\tfrac{2(1-2\nu)}{1-\nu} \tfrac{1}{\bar{x}} \left[\bar{u}' \left(1 - \tfrac{1}{2}\bar{u}'\right) - \tfrac{\bar{u}}{\bar{x}} \left(1 - \tfrac{1}{2}\tfrac{\bar{u}}{\bar{x}}\right)\right] = \tfrac{\alpha_0}{2} \bar{x} (1 - \bar{u}') \left(1 - \tfrac{\bar{u}}{\bar{x}}\right)^5 , \end{aligned} \tag{21}$$

where a counterpart to the original parameter α has been defined as follows:

$$\alpha_0 = \tfrac{3G}{2\pi} \frac{m_0^2}{R_o^4 (\lambda + 2\mu)}. \tag{22}$$

Moreover, we have to observe the following boundary conditions:

$$\bar{u}\big|_{\bar{x}=0} = 0 \;, \tag{23}$$

$$\bar{u}'\big|_{\bar{x}=\bar{x}_o}\left(1-\tfrac{1}{2}\bar{u}'\big|_{\bar{x}=\bar{x}_o}\right)+\tfrac{2\nu}{1-\nu}\bar{u}\big|_{\bar{x}=\bar{x}_o}\left(1-\tfrac{1}{2}\bar{u}\big|_{\bar{x}=\bar{x}_o}\right)=0.$$

Summarizing we may say that the determination of the (current) outer radius, $\bar{x}_o := \frac{r_o}{R_o}$, is now part of the problem. Due to the new normalization for the radius the outer surface of the sphere is no longer at $\bar{x}=1$: The more massive the gravitating body, the smaller $\bar{x}_o$ will become. Moreover, α_0 will be particularly large in case of very massive bodies of great resilience.

This is investigated in Fig. 4_1, where the "movement" of points originally at $0 \leq\ R/R_o \leq\ 1$, within a selfgravitating body, is depicted in form of blue trajectories in increments of 0.05, as a function of the mass-stiffness parameter α_0 for $\nu = 0.36$ up to the maximum value of $\alpha \approx 1.319$ (black line): The more gravitational mass, the higher the displacement. The red curve shows the displacement of the Love radius. Note that if nonlinearity is taken into account the location of the Love radius may change considerably when compared to the value from analytical solution of Eqn. (12): Fig. 4_2.

4 Viscoelasticity

We consider a linear viscoelastic material model at small strains of the Kelvin-Voigt type with no bulk viscosity, *i.e.*, stresses $\boldsymbol{\sigma}$, linear strains $\boldsymbol{\epsilon}$, and strain rates $\dot{\boldsymbol{\epsilon}}$ are connected by (η: shear viscosity):

$$\boldsymbol{\sigma} = \lambda\,\mathsf{Tr}\boldsymbol{\epsilon}\,\mathbf{1} + 2\mu\,\boldsymbol{\epsilon} + 2\eta\left(\dot{\boldsymbol{\epsilon}} - \frac{1}{3}\mathsf{Tr}\dot{\boldsymbol{\epsilon}}\,\mathbf{1}\right). \tag{24}$$

We assume perfectly spherical conditions. Then the ordinary differential equation (9) must be replaced by the following partial differential equation:

$$u'' + 2\frac{u'}{x} - 2\frac{u}{x^2} + \tfrac{4}{3}\left(\dot{u}'' + 2\frac{\dot{u}'}{x} - 2\frac{\dot{u}}{x^2}\right) = \frac{\alpha}{2}x, \tag{25}$$

where the dot now refers to differentiation w.r.t. dimensionless time, $\tau := \frac{\lambda+2\mu}{\eta}t$, and the dash means differentiation w.r.t. dimensionless position, x.

This equation can be solved in closed form, *e.g.*, by mapping it into Laplace space and finding the back transform. Details of the involved procedures are outlined in [8], Chapter 5. For a solid sphere one finds:

$$u(x,\tau=0) = 0\ , \tag{26}$$

$$u(x,\tau>0) = -\tfrac{\alpha}{20}x\left[\tfrac{3-\nu}{1+\nu} - x^2\right]\left[1-\exp\left(-\tfrac{3}{4}\tau\right)\right] - \tfrac{\alpha}{10}\tfrac{1-\nu}{1+\nu}x\exp\left(-\tfrac{3}{4}\tau\right),$$

and for a hollow sphere (r_i inner radius):

$$u(x,\tau) = \frac{\alpha}{20}\left[-x\left(3-\nu+2(1-2\nu)\xi^5\right) + \tfrac{[3-\nu-(1+\nu)\xi^2]\xi^3}{x^2}\right]\times \tag{27}$$

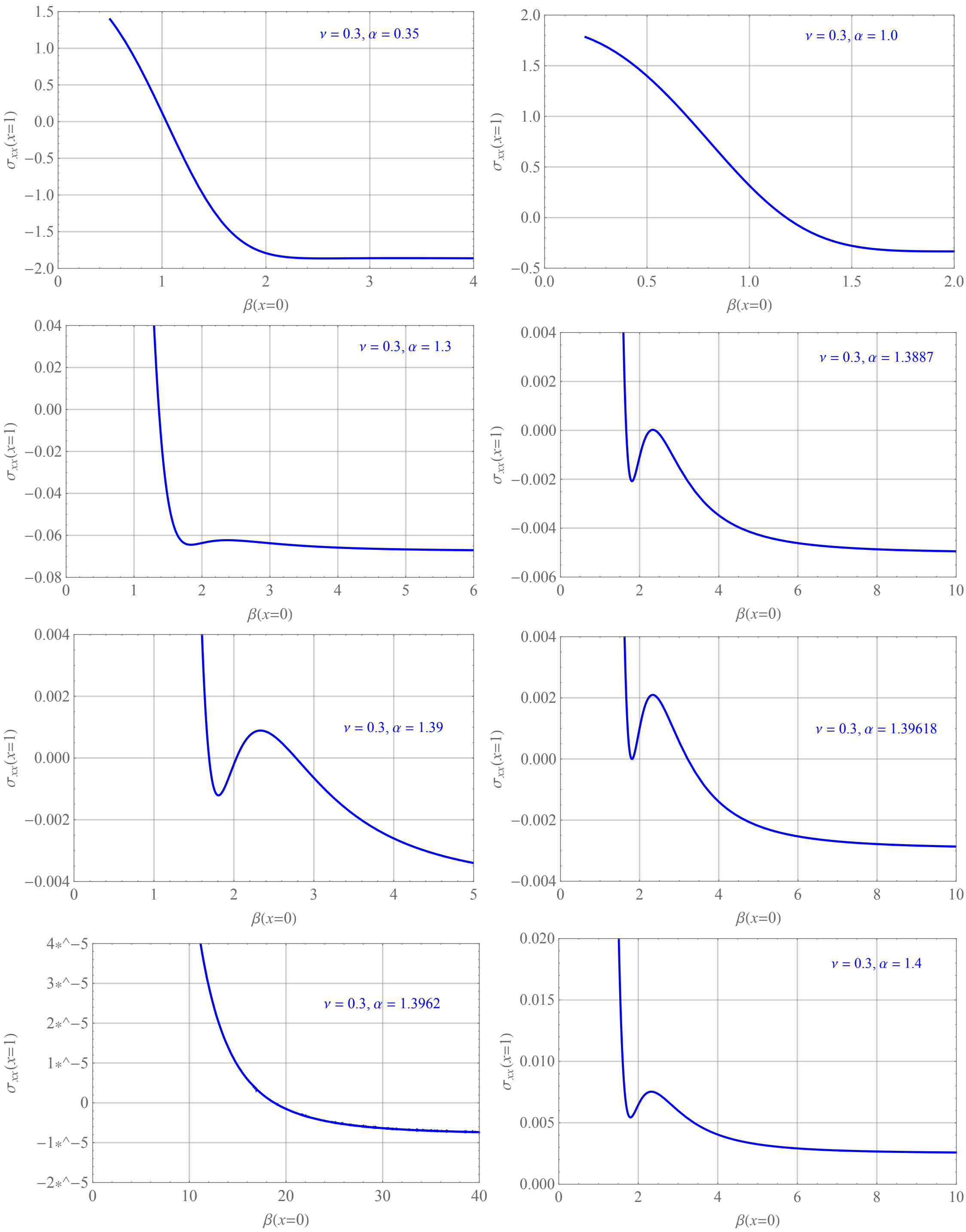

Figure 3: Behavior of the solution for increasing α-value (see text).

$$\left[\frac{1}{1+\nu+2(1-2\nu)\xi^3} - \frac{\exp\left(-\frac{3}{4}\tau\right)}{(1+\nu)(1-\xi^3)} + \frac{1-\nu}{1+\nu}\frac{3\xi^3\exp\left[-\left(\frac{1-2\nu}{1-\nu}+\frac{1}{2\xi^3}\frac{1+\nu}{1-\nu}\right)\frac{\tau}{2}\right]}{(1-\xi^3)\left[1+\nu+2(1-2\nu)\xi^3\right]}\right] +$$

$$\frac{\alpha}{20}\frac{1-\nu}{1+\nu}\frac{1}{1-\xi^2}\left[-\left(2+3\xi^5\right)x + \frac{2\xi^3}{x^2}\right]\exp\left(-\tfrac{3}{4}\tau\right) \times$$

$$\left[1-\exp\left(-\frac{1+\nu}{1-\nu}\frac{\tau}{4}\frac{1-\xi^3}{\xi^3}\right)\right] + \frac{\alpha}{20}x^3\left[1-\exp\left(-\tfrac{3}{4}\tau\right)\right],\ \xi := \frac{r_i}{r_o} \leq x \leq 1.$$

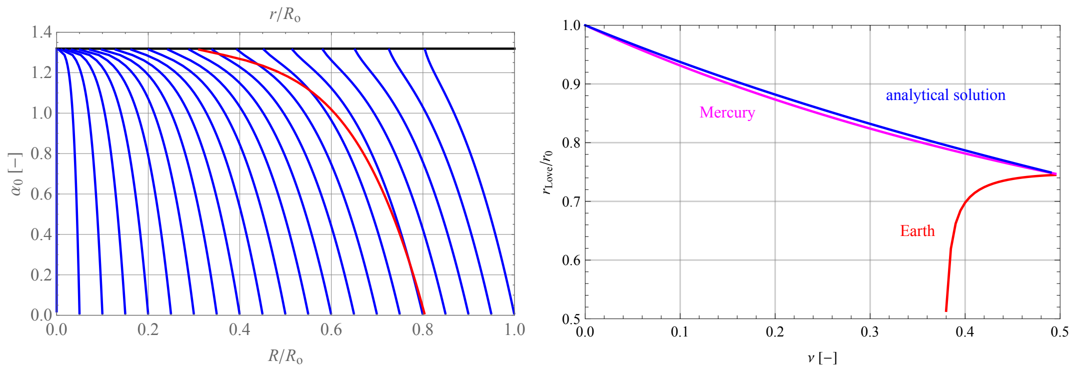

Figure 4: Studies of the nonlinear deformation due to self-gravitation (see text).

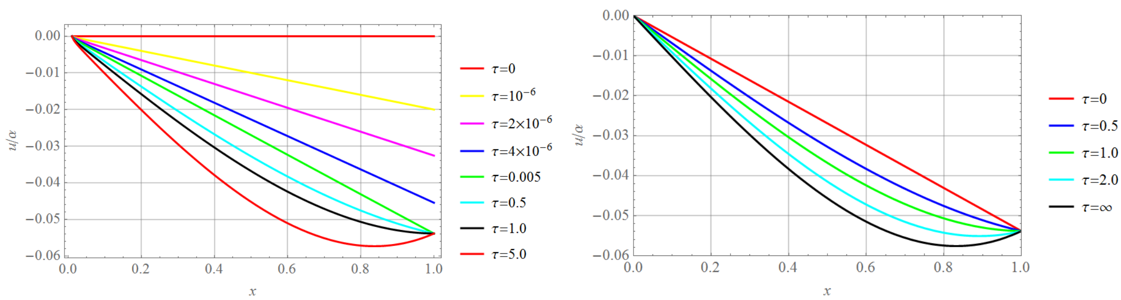

Figure 5: Temporal development of the displacement as a function of radial position for a hollow and a solid sphere (see text).

Note that special care has to be given to the case $\tau = 0$ in Eqn. $(26)_2$: If we consider the limit $\tau \to 0$ we find a non-vanishing initial displacement in contradiction to the requirement. This is not so for the more cumbersome looking Eqn. (27), where the limit leads to $u(x, \tau \to 0) = 0$. Fig. 5_1 depicts the temporal evolution of the displacement as a function of radial distance in dimensionless form as predicted by Eqn. (22) for the choice $\nu = 0.3$ for a hollow sphere with a very small hole, $\xi = 0.01$. In other words: This figure essentially represents the behavior of the solid sphere. Moreover, Fig. 5_2 depicts the temporal evolution of the displacement as predicted by Eqn. (21) for the solid sphere. Note that immediately after "gravity has been switched on" the dependence is nearly linear, in other words we observe a sudden jump, whereas the solution shown in Fig. 5_1 evolves continuously, but *fast* (observe the values for the dimensionless times).

However, this high speed gives us a clue of what is happening: Our quasistatic approximation from Eqn. (1) reaches its limits. There will be fast movements of mass if we "switch on gravity" and this would require us to solve the full balance of momentum with the inertia terms.
As far as the Love radius is concerned we may use Eqn. (26) to find the following analytical relation:

$$x_{\text{Love}} = \frac{1}{\sqrt{3}}\sqrt{\frac{3-\nu}{1+\nu} + \frac{1-\nu}{1+\nu}\frac{1}{\exp\left(\frac{3}{4}\tau\right)-1}}. \tag{28}$$

For physical reasons the (normalized) Love radius must be smaller than one. Obviously it takes some time before this is the case. Before that time has passed the Love radius and the outer radius coincide. There is no tensile strain until then, *i.e.*, there is no damage possible.

5 Conclusions

We end this essay by a quote by Albert Einstein, which summarizes the predicament of modeling in physics quite concisely: "As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality."

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