Mathematical model of static deformation of micropolar elastic circular thin bar

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Abstract

The boundary value problem of statics of plane stress state of micropolar theory of elasticity is considered in a thin circular area. Rather adequate hypotheses of general nature are formulated [1-3] and on the basis of these hypotheses applied (one-dimensional) model of micropolar elastic circular thin bar is constructed. Mechanic balance equation is obtained. It is confirmed that all energy theorems and Ritz, Bubnov-Galerkin, FEM variation methods are applicable for the constructed applied model of micropolar elastic circular thin bar and for solutions of corresponding boundary value problems of the applied model. One-dimensional variation functional is constructed and it is proved that all basic equations and natural boundary conditions of applied model of micropolar elastic circular thin bar will be obtained from the corresponding variation equation (as Euler equations).

1 Introduction

Papers [1-4] are devoted to the construction of applied theories of micropolar elastic thin plates and shells. Review of works in this direction has been implemented in the paper [5]. In the papers [6-9] general applied theory of micropolar elastic thin plates and shells is constructed on the basis of the hypotheses method, which are formulated adequately with the results of the asymptotic method of integration of three-dimensional boundary-value problem in thin domain [10]. In the paper [11] the applied theory of micropolar elastic thin straight bars is constructed in the same way.

In this paper, analogous hypotheses are formulated, on the basis of which general applied model of static deformation of micropolar elastic circular bar is constructed.

2 Problem statement.

A bar with a curved axis is considered (Fig. 1), which has a constant transverse cut with height $2h = r_2 - r_1$ and with width b so small that the problem of the bending of this bar can be viewed as plane (i.e. there is a plane stress state). The axis of

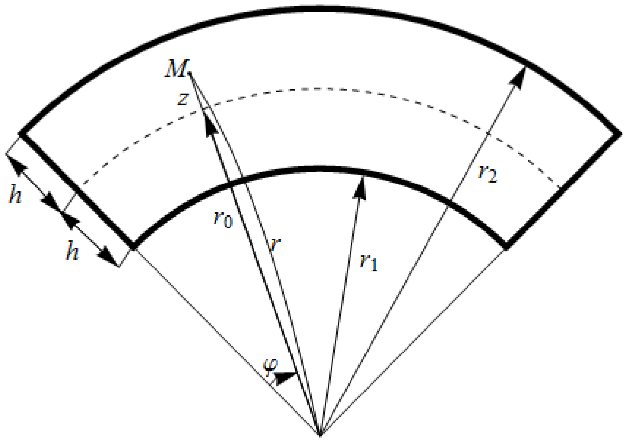

Fig. 1

the bar is the arc of a circle with radius r_0; the material of the bar is isotropic, micropolar-elastic.
Equations of the plane stress state of micropolar theory of elasticity are applicable in the middle plane of the bar, where the polar system of coordinates (r, φ) will be introduced: $r_1 \le r \le r_2,\ 0 \le \varphi \le \varphi_1$ [12,13]:

Equilibrium equations

$$\frac{1}{r}\frac{\partial \sigma_{11}}{\partial \varphi} + \frac{\partial \sigma_{21}}{\partial r} + \frac{1}{r}(\sigma_{21} + \sigma_{12}) = 0,\ \frac{\partial \sigma_{22}}{\partial r} + \frac{1}{r}(\sigma_{22} - \sigma_{11}) + \frac{1}{r}\frac{\partial \sigma_{12}}{\partial \varphi} = 0$$

$$\frac{1}{r}\frac{\partial \mu_{13}}{\partial \varphi} + \frac{\partial \mu_{23}}{\partial r} + \frac{1}{r}\mu_{23} + \sigma_{12} - \sigma_{21} = 0 \qquad (1.1)$$

Elasticity relations

$$\gamma_{11} = \frac{1}{E}[\sigma_{11} - \upsilon\sigma_{22}],\ \gamma_{22} = \frac{1}{E}[\sigma_{22} - \upsilon\sigma_{11}],\ \gamma_{12} = \frac{\mu+\alpha}{4\mu\alpha}\sigma_{12} - \frac{\mu-\alpha}{4\mu\alpha}\sigma_{21}$$

$$\gamma_{12} = \frac{\mu+\alpha}{4\mu\alpha}\sigma_{21} - \frac{\mu-\alpha}{4\mu\alpha}\sigma_{12},\ \chi_{13} = \frac{1}{B}\mu_{13},\ \chi_{23} = \frac{1}{B}\mu_{23} \qquad (1.2)$$

Geometrical relations

$$\gamma_{11} = \frac{1}{r}\frac{\partial V_1}{\partial \varphi} + \frac{1}{r}V_2,\ \gamma_{22} = \frac{\partial V_2}{\partial r},\ \gamma_{12} = \frac{1}{r}\frac{\partial V_2}{\partial \varphi} - \frac{1}{r}V_1 - \omega_3,\ \gamma_{21} = \frac{\partial V_1}{\partial r} + \omega_3$$

$$\chi_{13} = \frac{1}{r}\frac{\partial \omega_3}{\partial \varphi},\ \chi_{23} = \frac{\partial \omega_3}{\partial r} \qquad (1.3)$$

Here $\sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{21}$ are power (normal) stresses; μ_{13}, μ_{23} are moment stresses; $\gamma_{11}, \gamma_{22}, \gamma_{12}, \gamma_{21}$ are deformations; χ_{13}, χ_{23} are bending-torsions; V_1, V_2 are displacements; ω_3 is free rotation; $E, \nu, \mu = \frac{E}{2(1+\nu)}, \alpha, B$ are elastic constants of the micropolar material.
It is assumed that on the front lines $r = r_1,\ r = r_2$ the external forces and moments are given:

$$on\ r = r_1,\ \sigma_{21} = q_1^-;\ \sigma_{22} = q_2^-;\ \mu_{23} = m^-,$$

$$on\ r = r_2,\ \sigma_{21} = q_1^+;\ \sigma_{22} = q_2^+;\ \mu_{23} = m^+, \tag{1.4}$$

and one of the following variants of the boundary conditions can take place:

$$a)\ on\ \varphi = 0,\ \sigma_{11} = \sigma_{11}^{'},\ \sigma_{12} = \sigma_{12}^{'},\ \mu_{13} = \mu_{13}^{'};$$

$$on\ \varphi = \varphi_1,\ \sigma_{11} = \sigma_{11}^{''},\ \sigma_{12} = \sigma_{12}^{''},\ \mu_{13} = \mu_{13}^{''}; \tag{1.5}$$

$$b)\ on\ \varphi = 0,\ V_1 = V_1^{'},\ V_2 = V_2^{'},\ \omega_3 = \omega_3^{'};$$

$$on\ \varphi = \varphi_1,\ V_1 = V_1^{''},\ V_2 = V_2^{''},\ \omega_3 = \omega_3^{''}; \tag{1.6}$$

$$c)\ on\ \varphi = 0,\ \sigma_{11} = \sigma_{11}^{'},\ V_2 = V_2^{'},\ \mu_{13} = \mu_{13}^{'};$$

$$on\ \varphi = \varphi_1,\ \sigma_{11} = \sigma_{11}^{''},\ V_2 = V_2^{''},\ \mu_{13} = \mu_{13}^{''}; \tag{1.7}$$

Further it is assumed that $2h \ll r$ and $2h \ll l$, where l is the length of the bar middle line (i.e. the bar is thin). The aim of the paper is the construction of the applied (one-dimensional) model of micropolar elastic circular thin bar. For this purpose the radius vector r of an arbitrary point of the domain is represented as follows: $r = r_0 + z$, where $-h \le z \le h\,(r = r_0 - h,\ r = r_0 + h)$, and on the basis of the fact that the bar is thin-walled, it will be assumed that

$$1 + \frac{h}{r_0} \doteq 1 \tag{1.8}$$

3 Hypotheses, displacements and rotation, deformations and bending-torsions, power and moment stresses.

Following assumptions (hypotheses) are formulated for the construction of the applied model of micropolar elastic circular thin bar [6-11]:

1. Hypothesis of the straight line is accepted as an initial kinematic hypothesis for displacements, i.e. Timoshenko's hypothesis. This means that the linear element, which is initially perpendicular to the middle plane of the circular bar before the deformation, remains straight and is rotated to a certain angle after the deformation, without changing its length and without remaining perpendicular to the deformed middle line. In addition, it is accepted that the free rotation ω_2 is a constant function along the coordinate z. As a result following linear law for displacements and free rotation will be obtained along the thickness of the middle plane of the circular bar:

$$V_1 = u(\varphi) + z\psi(\varphi),\ V_2 = w(\varphi),\ \omega_3 = \Omega_3(\varphi), \tag{2.1}$$

where $u(\varphi)$ and $w(\varphi)$ are displacements of points of the middle line in directions of its tangent and normal (i.e. $w(\varphi)$ is deflection of the bar); $\psi(\varphi)$ is the angle of rotation of initially normal element; $\Omega_3(\varphi)$ is the free rotation of this element points. As in the paper [6-11], the kinematic hypothesis (2.1) in general is called Timoshenko's generalized hypothesis in the case of micropolar thin bar (in this case, for the circular thin bar).

2. The hypothesis of the circular bar thin-wallness, where the approximate equality (1.8) is accepted, and as well as,

$$\frac{1}{r}=\frac{1}{r_0+z}=\frac{1}{r_0(1+\frac{z}{r_0})}\doteq\frac{1}{r_0}. \tag{2.2}$$

3. Assumptions about smallness of the normal stress σ_{22}, compared with the normal stress σ_{11} in the first equation of Hooke's law ($(1.2)_1$).
4. During the determination of the deformations, bending-torsions, force and moment stresses, it is accepted

$$\sigma_{21}=\sigma_{21}^0(\varphi). \tag{2.3}$$

After the determination of the mentioned quantities the formula for σ_{21} will be corrected in the following way. The second equilibrium equation from (1.1) ($(1.1)_2$) is integrated by z and during the determination of the constant of integration (functions from φ), it will be required that the integral from $-h$ to h of the obtained expression is equal to zero. After this specified integration the obtained final expression will be added to the formula (2.3).
In accordance with the accepted law (2.1) of distribution of displacements and rotation, substituting them into formulas (1.3), following expressions will be obtained for deformations and bending-torsions:

$$\gamma_{11}=\left(\frac{1}{r_0}\frac{du}{d\varphi}+\frac{1}{r_0}w\right)+z\frac{1}{r_0}\frac{d\psi}{d\varphi},\ \gamma_{22}=0,\ \gamma_{12}=\frac{1}{r_0}\frac{dw}{d\varphi}-\frac{1}{r_0}u-\Omega_3,$$

$$\gamma_{21}=\psi+\Omega_3,\ \chi_{13}=\frac{1}{r_0}\frac{d\Omega_3}{d\varphi},\ \chi_{23}=0 \tag{2.4}$$

Following notations are accepted:

$$\Gamma_{11}=\frac{1}{r_0}\frac{du}{d\varphi}+\frac{1}{r_0}w,\ \Gamma_{12}=\frac{1}{r_0}\frac{dw}{d\varphi}-\frac{1}{r_0}u-\Omega_3,\ \Gamma_{21}=\psi+\Omega_3$$

$$K_{11}=\frac{1}{r_0}\frac{d\psi}{d\varphi},\ k_{13}=\frac{1}{r_0}\frac{d\Omega_3}{d\varphi}, \tag{2.5}$$

Then for deformations, bending-torsions will be obtained following formulas:

$$\gamma_{11}=\Gamma_{11}+zK_{11},\ \gamma_{22}=0,\ \gamma_{12}=\Gamma_{12},\ \gamma_{21}=\Gamma_{21},\ \chi_{13}=k_{13},\ \chi_{23}=0. \tag{2.6}$$

Here Γ_{11} is the relative longitudinal deformation of the middle line; K_{11} is the change of middle line curvature (from the force stresses); Γ_{12},Γ_{21} are shift deformations; k_{13} is the change of curvature of the middle line (from the moment stresses).
Using the hypothesis 3) and formula $(2.6)_1$, following expression will be obtained for the stress σ_{11} from the formula $(1.2)_1$:

$$\sigma_{11}=\sigma_{11}^0(\varphi)+z\sigma_{11}^1(\varphi), \tag{2.7}$$

where

$$\sigma_{11}^0(\varphi)=E\Gamma_{11},\ \sigma_{11}^1(\varphi)=EK_{11}. \tag{2.8}$$

The formulas $(1.2)_3$, $(2.4)_3$, $(2.4)_4$ will be used for determination of the power stress σ_{12}:

$$\sigma_{12} = (\mu + \alpha)\Gamma_{12} + (\mu - \alpha)\Gamma_{21}. \tag{2.9}$$

Taking into consideration formulas for σ_{11} ((2.7)), σ_{12} ((2.9)), the second equilibrium equation ($(1.1)_2$) is considered, which is integrated over r. Based on the thin-wallness of the domain and the boundary conditions from (1.4) for σ_{22} , we will finally obtain:

$$\sigma_{22} = \frac{1}{2}(q_2^+ + q_2^-) - \frac{h^2}{2}\frac{1}{r_0}\sigma_{11}^1 + z\Big(\frac{1}{r_0}\sigma_{11}^0 - \frac{1}{r_0}\frac{d\sigma_{12}^0}{d\varphi}\Big) + \frac{1}{r_0}\sigma_{11}^1\frac{z^2}{2}. \tag{2.10}$$

On the basis of formulas $(1.2)_5$ and taking into account formulas from (2.6) for χ_{13} , following formula will be obtained for the moment stress μ_{13} :

$$\mu_{13} = Bk_{13}. \tag{2.11}$$

The expression for the moment stress μ_{23} will be obtained from the third equilibrium equation ($(1.1)_3$) by integrating it by r, taking into account formulas (2.11), (2.9) and (2.3):

$$\mu_{23} = \frac{1}{2}(m^+ + m^-) - z\Big(\frac{1}{r_0}\frac{d\mu_{13}^0}{d\varphi} + \sigma_{12}^0 - \sigma_{21}^0\Big) \tag{2.12}$$

For determination of the force stress σ_{21} ,the hypothesis 4) will be taken into account, then by using the first equilibrium equation ($(1.1)_1$), as well as the formula (2.3), it will be finally obtained:

$$\sigma_{21} = \sigma_{21}^0(\varphi) + \frac{1}{r_0}\frac{h^2}{6}\frac{d\sigma_{11}^1}{d\varphi} - z\Big(\frac{1}{r_0}\frac{d\sigma_{11}^0}{d\varphi} + \frac{1}{r_0}\sigma_{12}^0\Big) - \frac{1}{r_0}\frac{z^2}{2}\frac{d\sigma_{11}^1}{d\varphi} \tag{2.13}$$

4 Forces and moments. Obtainance of the basic system of equations of micropolar elastic circular bar.

In order to bring two-dimensional problem of micropolar theory of elasticity to one-dimensional one, which has already been done for displacements and rotation, deformations and bending-torsions, force and moment stresses, following integral characteristics are introduced in the applied theory of micropolar elastic circular thin bar, which are statistically equivalent to the components of the force and moment stresses: efforts N, Q_1, Q_2 and moments M_{11}, L_{13}, which are expressed by the following formulas:

$$N = \int_{-h}^{h}\sigma_{11}dz,\ Q_1 = \int_{-h}^{h}\sigma_{12}dz,\ Q_2 = \int_{-h}^{h}\sigma_{21}dz,$$

$$M_{11} = \int_{-h}^{h}\sigma_{11}z dz,\ L_{13} = \int_{-h}^{h}\mu_{13}dz \tag{3.1}$$

Now formulas for σ_{21} ((2.13)), σ_{22} ((2.10)) and μ_{23} ((2.12)) will be accepted as basic ones. Satisfying the boundary conditions (1.4) and considering the formula (3.1), following system of equilibrium equations of the applied model of micropolar elastic circular bar will be obtained:

$$\frac{1}{r_0}N-\frac{1}{r_0}\frac{dQ_1}{d\varphi}=q_2^+-q_2^-,\ \frac{1}{r_0}Q_1+\frac{1}{r_0}\frac{dN}{d\varphi}=-(q_1^+-q_1^-)$$

$$Q_2-\frac{1}{r_0}\frac{dM_{11}}{d\varphi}=h(q_1^++q_1^-),\ Q_2-Q_1-\frac{1}{r_0}\frac{dL_{13}}{d\varphi}=m^+-m^- \qquad (3.2)$$

Further, with the help of formulas for σ_{11} ((2.7)), σ_{12} ((2.9)), σ_{21} ((2.13)), μ_{13} ((2.11))elasticity relations will be obtained for this model:

$$N=2Eh\Gamma_{11},\ Q_1=2h(\mu+\alpha)\Gamma_{12}+2h(\mu-\alpha)\Gamma_{21},\ Q_2=2h(\mu+\alpha)\Gamma_{21}+2h(\mu-\alpha)\Gamma_{12}.$$

$$M_{11}=\frac{2Eh^3}{3}K_{11},\ L_{13}=2Bhk_{13}. \qquad (3.3)$$

Geometric equation (2.5) will be added to the equilibrium equations (3.2) and the elasticity relations (3.3):

$$\Gamma_{11}=\frac{1}{r_0}\frac{du}{d\varphi}+\frac{1}{r_0}w,\ \Gamma_{12}=\frac{1}{r_0}\frac{dw}{d\varphi}-\frac{1}{r_0}u-\Omega_3,\ \Gamma_{21}=\psi+\Omega_3$$

$$K_{11}=\frac{1}{r_0}\frac{d\psi}{d\varphi},\ k_{13}=\frac{1}{r_0}\frac{d\Omega_3}{d\varphi}, \qquad (3.4)$$

The equilibrium equations (3.2), elasticity relations (3.3) and geometric relations (3.4) represent the system of basic equations of the applied model of micropolar elastic circular bar. The boundary conditions should be added to this system of equations:

I. Conditions of the power and moment load:

$$N|_{\varphi=0}=N^{'}=\int_{-h}^{h}\sigma_{11}^{'}dz,\ M_{11}|_{\varphi=0}=M^{'}=\int_{-h}^{h}\sigma_{11}^{'}zdz;$$

$$Q_1|_{\varphi=0}=Q_1^{'}=\int_{-h}^{h}\sigma_{12}^{'}dz,\ L_{13}|_{\varphi=0}=L_{13}^{'}=\int_{-h}^{h}\mu_{13}^{'}dz;$$

$$N|_{\varphi=\varphi_1}=N^{''}=\int_{-h}^{h}\sigma_{11}^{''}dz,\ M_{11}|_{\varphi=\varphi_1}=M^{''}=\int_{-h}^{h}\sigma_{11}^{''}zdz;$$

$$Q_1|_{\varphi=\varphi_1}=Q_1^{''}=\int_{-h}^{h}\sigma_{12}^{''}dz,\ L_{13}|_{\varphi=\varphi_1}=L_{13}^{''}=\int_{-h}^{h}\mu_{13}^{''}dz. \qquad (3.5)$$

In particular, from these conditions we will obtain conditions of free edges.

II. Conditions, when the displacements and rotation are set on the edge:

$$u|_{\varphi=0}=u^{'}=\frac{1}{2h}\int_{-h}^{h}V_1^{'}dz,\ \psi|_{\varphi=0}=\psi^{'}=\frac{3}{2h^3}\int_{-h}^{h}V_1^{'}zdz;$$

$$w|_{\varphi=0}=w^{'}=\frac{1}{2h}\int_{-h}^{h}V_2^{'}dz,\ \Omega_3|_{\varphi=0}=\Omega_3^{'}=\frac{1}{2h}\int_{-h}^{h}\omega_3^{'}dz;$$

$$u|_{\varphi=\varphi_1} = u^{''} = \frac{1}{2h}\int_{-h}^{h} V_1^{''} dz,\ \psi|_{\varphi=\varphi_1} = \psi^{''} = \frac{3}{2h^3}\int_{-h}^{h} V_1^{''} z dz;$$

$$w|_{\varphi=\varphi_1} = w^{''} = \frac{1}{2h}\int_{-h}^{h} V_2^{''} dz,\ \Omega_3|_{\varphi=\varphi_1} = \Omega_3^{''} = \frac{1}{2h}\int_{-h}^{h} \omega_3^{''} dz. \qquad (3.6)$$

Particularly, conditions of full sealing edges will be obtained from the above mentioned conditions.
III. Hinged supported conditions:

$$N|_{\varphi=0} = N^{'} = \int_{-h}^{h} \sigma_{11}^{'} dz,\ M_{11}|_{\varphi=0} = M^{'} = \int_{-h}^{h} \sigma_{11}^{'} z dz;$$

$$w|_{\varphi=0} = w^{'} = \frac{1}{2h}\int_{-h}^{h} V_2^{'} dz,\ L_{13}|_{\varphi=0} = L_{13}^{'} = \int_{-h}^{h} \mu_{13}^{'} dz;$$

$$N|_{\varphi=\varphi_1} = N^{''} = \int_{-h}^{h} \sigma_{11}^{''} dz,\ M_{11}|_{\varphi=\varphi_1} = M^{''} = \int_{-h}^{h} \sigma_{11}^{''} z dz;$$

$$w|_{\varphi=\varphi_1} = w^{''} = \frac{1}{2h}\int_{-h}^{h} V_2^{''} dz,\ L_{13}|_{\varphi=\varphi_1} = L_{13}^{''} = \int_{-h}^{h} \mu_{13}^{''} dz. \qquad (3.7)$$

Mathematical model of micropolar elastic circular thin bar is expressed by the system of equations (3.2)-(3.4) and boundary conditions (3.5) (or (3.6), or (3.7)).
The equation, expressing the law of conservation of mechanical energy in the case of deformation of micropolar circular bar, and corresponding general variation functional are also obtained. On the basis of the latter, finite element method will be developed in future for solving concrete boundary value problems of models (3.2)-(3.7).
It should be noted that if in the model (3.2) - (3.7) of micropolar elastic circular thin bar physical constant $\alpha(\alpha = 0)$ is equated to zero, then we will come to the model [14] of elastic circular thin bar in the classical formulation with consideration of transverse share deformations.
On the basis of the constructed applied model of micropolar elastic circular bar (3.2)-(3.7) some concrete problems of the applied character are considered. With the help of numerical analysis efficiency of micropolar material is established in comparison with classical in terms of stiffness and strength of the bar.

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References

[1] Jilin, P. A., 1982. Basic equations of non-classical theories of elastic shells // Dynamics and Rigidity of Machines. No 386. pp. 29-42. (in Russian)

[2] Palmov, V. A., 1986. The simplest consistent system of equations of theory of thin elastic shells of asymmetric elasticity // Journal of Mechanics of Deformable Bodies. Moscow: Publishing House "Science". pp. 106-112. (in Russian)

[3] Altenbach, H., Eremeyev, V., 2009. On the linear theory of micropolar plates// Z. Angew. Math. Mech (ZAMM). V. 89. No 4. pp. 242-256.

[4] Eremeyev, V. A., Zubov, L. M., 2008. Mechanics of elastic shells. M.: Sience. p. 280. (in Russian)

[5] Altenbach, J., Altenbach, H. and Eremeyev, V.A., 2010. On generalized Cosserat-tape theories of plates and shells: a short review and bibliography. Archive of Applied Mechanics, 80(1), pp.73-92.

[6] Sargsyan, S. H., 2012. Mathematical model of micropolar elastic thin plates and their strength and stiffness characteristics// Journal of Applied Mechanics and Technical Physics. Vol. 53. No 2. pp. 275-282.

[7] Sargsyan, S. H., 2011. Mathematical models of micropolar elastic thin shells// Advanced Structured Materials. Shell-like Structures. Non-classical Theories and Applications. Springer. Vol. 15. pp. 91-100.

[8] Sargsyan, S. H., 2011. The general dynamic theory of micropolar elastic thin shells// Doklady Physics, Vol. 56. No 1. pp. 39-42.

[9] Sargsyan, S. H., 2014. Energy balance equation, energetic theorems and variation equation for the general theory of micropolar elastic isotropic thin shells// International Journal of Mechanics. Vol. 8. pp. 93-100.

[10] Sargsyan, S. H., 2015. Asymptotically confirmed hypotheses method for the costruction of micropolar and classical theories of elastic thin shells // Advances in Pure Mathematics. No 5. pp. 629-642.

[11] Sargsyan, S. H., 2012. Effective manifestations of characteristics of strength and rigidity of micropolar elastic thin bars// Journal of Materials Science and Engineering. Vol.2. No 1. pp. 98-108.

[12] Morozov, N. F., 1984. Mathematical issues in theory of cracks. M.: Science. p. 256. (in Russian)

[13] Nowacki, W., 1986. Theory of asymmetric elasticity. Pergamon Press. Oxford. New York. Toronto. Sydney. Paris. Frankfurt. p. 383.

[14] Sargsyan, S.H., Khachatryan, M.V.,2016. Mathematical model of elastic plane curve beam (circular) with consideration of shear deformations on the basis of the classical theory of elasticity//Doklady NAS Armenia. V.116. No 1. pp. 34-42 (in Russian)

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