Scattering of flexural–gravitational waves by a periodic array of obstacles in an elastic plate floating on a thin fluid layer

M.G. Zhuchkova
m.zhuchkova@list.ru

Abstract

Low-frequency time-harmonic flexural-gravitational waves of a small amplitude in a floating elastic plate are considered. Straight-line obstacles in the plate are periodically spaced in the horizontal coordinate with an equal separation. Propagation of flexural-gravitational waves through the plate is analytically studied under a thin fluid layer approximation. We are concerned with stopping and passing frequency bands, of which the boundaries are found.

1 Introduction

In [1] under a shallow-water approximation transmission of flexural-gravitational waves through a periodic floating elastic plate was investigated. The plate simulated an ice cover of ocean shelf with two kinds of straight-line obstacles: areas of broken ice and ice hummocks. The paper [2] was devoted to scattering of flexural-gravitational waves by periodically spaced arrays of straight-line narrow cracks in ice sheets modelled by a periodic elastic plate floating on water of finite depth. In [3] wave propagation in periodic plates and cylindrical shells composed by alternating segments of different elastic materials was considered.

A mathematically similar problem of transmission of waves along a periodically weighted strings and beams was studied in [4]. It was pointed out that two oncoming decaying waves can transfer energy. An elastically supported string with point-wise defects and waves in it were considered in [5]. Flexural waves in an elastic beam with periodic system of weights were explored in [6].

Note that the study and development of periodic models such as strings, rods, mass-springs, beams, plates, shells have increasing attention due to a wealth of different engineering applications, e.g., vibration protection, noise isolation, railway track dynamics simulation etc.

In our previous work [7] we investigated transmission of flexural-gravitational waves through multiple straight-line obstacles in a floating elastic plate. In the case of equidistant obstacles complete transmission of an incident wave was obtained at certain frequencies. The plot of energy transmission coefficient against the separation between the obstacles is featured by consecutive peaks reaching 1 in limited

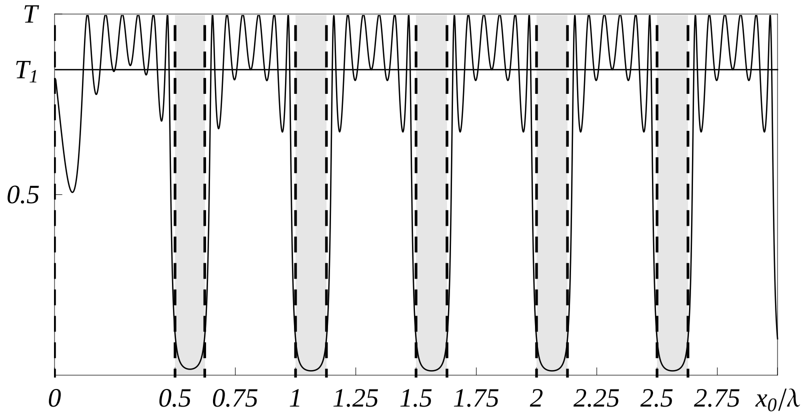

Figure 1: Energy transmission coefficient T of an incident flexural-gravitational wave against the fractional separation x_0/λ between the movable clamps. $N = 7$. A thin fluid layer approximation. The horizontal line displays energy transmission coefficient T_1 in a case of one movable clamp. Hereafter, we use the values of parameters taken from the work [8]. In particular, the plate density $\rho = 917$ kg/m^3, the Young's module $E = 4.2 \cdot 10^9$ N/m^2, the Poisson ratio $\nu = 0.3$, the thickness of the plate $h = 1.6$ m, the thickness of the fluid layer $h_1 = 50$ m, the density of the fluid $\rho_1 = 1000$ kg/m^3. Wave frequency is $\omega = 1.2\ s^{-1}$, wave length is $\lambda = 139.4$ m. Vertical dashed lines show the boundaries of the stopping bands. The stopping bands are shaded.

intervals (Fig. 1). The number of the consecutive peaks is equal to $N - 1$, where N is the number of obstacles. As the number of obstacles increases, the peaks fill in these intervals. Intervals with the peaks alternated with another intervals inside which the energy transmission coefficient is closed to zero. We suggested in [7] that in the case of a periodic array of obstacles passing and stopping bands would occur. The present paper is devoted to this case and organized as follows. We begin with the study of flexural-gravitational waves in an elastic periodically supported plate floating on a thin fluid layer. After that, we consider flexural waves in an elastic periodically supported beam and compare obtained results with each other.

2 An elastic periodically supported floating plate

A thin elastic plate floats on the surface of an ideal incompressible fluid of the thickness h_1 and the density ρ_1. A periodic array of movable clamps at lines $x_n = x_0 n$ ($n \in \mathbb{Z}$) divides the plate into infinite number of identical strips of the width x_0. Configuration of the model is shown in Fig. 2. We deal with time-harmonic flexural-gravitational waves of a small amplitude propagating at a low frequency ω. Time-dependent factor $e^{-i\omega\tau}$ is omitted everywhere. The fluid layer is thin in comparison to all wave lengths.

The velocity potential $\Phi(x, z)$ is developed as a series in $z + h_1$

$$\Phi(x, z) = \alpha_0(x) + \alpha_1(x)(z + h_1) + \alpha_2(x)(z + h_1)^2 + \ldots, \ -h_1 < z < 0. \quad (1)$$

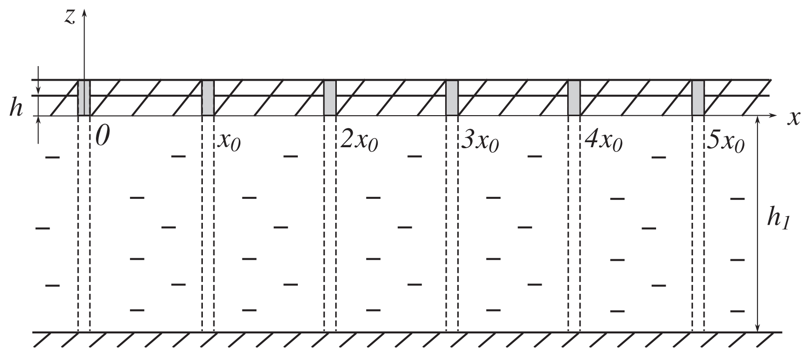

Figure 2: Cross-section of a periodic elastic plate floating on a thin fluid layer.

From the Laplace equation $\triangle\Phi(x,z)=0$, where $\triangle=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}$, we derive

$$\alpha_{2k}(x)=\frac{(-1)^k}{(2k)!}\frac{\partial^{2k}\alpha_0(x)}{\partial x^{2k}},\qquad \alpha_{2k+1}(x)=\frac{(-1)^k}{(2k+1)!}\frac{\partial^{2k}\alpha_1(x)}{\partial x^{2k}},\qquad k=1,2,\ldots \tag{2}$$

In particular, we have

$$\alpha_2(x)=-\alpha_0''(x)/2. \tag{3}$$

From the boundary condition $\frac{\partial\Phi(x,-h_1)}{\partial z}=0$ on the rigid bottom $z=-h_1$ we obtain

$$\alpha_1(x)=0. \tag{4}$$

With account of the terms of the first order of smallness inclusively, in view of (4), we have

$$\Phi(x,0)=\alpha_0(x). \tag{5}$$

From the kinematic boundary condition $\frac{\partial\Phi(x,0)}{\partial z}=-i\omega\zeta(x)$ at the interface $z=0$, in view of (3), we get

$$\zeta(x)=-\frac{i}{\omega}\alpha_0''(x)h_1, \tag{6}$$

where $\zeta(x)$ is the elevation of the plate. The dynamic boundary condition

$$D\zeta''''(x)+(\rho_1 g-\rho h\omega^2)\zeta(x)-i\omega\rho_1\Phi(x,0)=0,\ x\neq x_n$$

at the interface $z=0$ gives

$$D\frac{d^6\alpha_0(x)}{dx^6}+(\rho_1 g-\rho h\omega^2)\frac{d^2\alpha_0(x)}{dx^2}+\frac{\rho_1\omega^2}{h_1}\alpha_0(x)=0,\ x\neq x_n, \tag{7}$$

where D is the flexural rigidity, ρ is the density, h is the thickness of the plate. Thus, under a thin fluid layer approximation we will employ the equation (7) instead of the Laplace equation and the boundary conditions at $z=0$ and $z=-h_1$. The matching conditions are imposed at $x=x_0 n$ ($n\in\mathbb{Z}$).

$$\begin{gathered}\lim_{x\to x_n^+}\alpha_0(x)=\lim_{x\to x_n^-}\alpha_0(x),\ \lim_{x\to x_n^+}\alpha_0'(x)=\lim_{x\to x_n^-}\alpha_0'(x),\ \lim_{x\to x_n^+}\zeta(x)=\lim_{x\to x_n^-}\zeta(x),\\ \lim_{x\to x_n}\zeta'(x)=0,\ \lim_{x\to x_n^+}\zeta'''(x)=\lim_{x\to x_n^-}\zeta'''(x).\end{gathered} \tag{8}$$

Let us consider the equation (7) on the interval $(0, 2x_0)$. According to the theorem of Floquet [9], the general solution of the problem takes the form

$$\alpha_0(x) = \begin{cases} \sum\limits_{n=0}^{2} \left(C_n^+ e^{i\lambda_n x} + C_n^- e^{-i\lambda_n x}\right), \; x \in (0, x_0), \\ \sum\limits_{n=0}^{2} \left(C_n^+ e^{i\lambda_n (x-x_0)} + C_n^- e^{-i\lambda_n (x-x_0)}\right) e^{i\varphi}, \; x \in (x_0, 2x_0), \end{cases} \tag{9}$$

where λ_0, λ_1, λ_2 are the roots of the dispersion equation

$$D\lambda^6 + (\rho_1 g - \rho h\omega^2)\lambda^2 - \frac{\rho_1\omega^2}{h_1} = 0. \tag{10}$$

λ_0 is the positive root, λ_1 and λ_2 are two complex roots located in the upper half plane ($\lambda_2 = -\overline{\lambda_1}$). Satisfying the matching conditions at $x = x_0$ leads to an inhomogeneous linear system of equations $Pu = v$, with a matrix P equals to

$$P = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_0 & \lambda_1 & \lambda_2 \\ -\lambda_0^2 & -\lambda_1^2 & -\lambda_2^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\lambda_0^3 & -i\lambda_1^3 & -i\lambda_2^3 \\ \lambda_0^4 & \lambda_1^4 & \lambda_2^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\lambda_0^5 & i\lambda_1^5 & i\lambda_2^5 \end{pmatrix}. \tag{11}$$

Variables u are introduced by the formulae

$$u_n = C_n^+(e^{i\varphi} - e^{i\lambda_n x_0}) + C_n^-(e^{i\varphi} - e^{-i\lambda_n x_0}),$$
$$u_{n+3} = C_n^+(e^{i\varphi} - e^{i\lambda_n x_0}) - C_n^-(e^{i\varphi} - e^{-i\lambda_n x_0}), \; n = 0, 1, 2.$$

A column-vector v is determined by $v = \frac{i\omega}{H}\left(0, 0, 0, 0, \frac{B}{D}, 0\right)^t$. The first and second equations of the system describe continuity of $\alpha_0(x)$ and its first derivative $\alpha_0'(x)$ at $x = x_0$. The third, forth and sixth equations show continuity of the elevation $\zeta(x)$ and its first $\zeta'(x)$ and third $\zeta'''(x)$ derivatives at $x = x_0$, respectively. According to the firth equation, the jump of the second derivative $\zeta''(x)$ is non-zero. The quantity B represents the bending moment at $x = x_0$. Solution of the system is

$$u_0 = \frac{i\omega B}{HD(\lambda_0^2 - \lambda_1^2)(\lambda_0^2 - \lambda_2^2)}, \quad u_1 = \frac{i\omega B}{HD(\lambda_1^2 - \lambda_0^2)(\lambda_1^2 - \lambda_2^2)},$$
$$u_2 = \frac{i\omega B}{HD(\lambda_2^2 - \lambda_0^2)(\lambda_2^2 - \lambda_1^2)}, \quad u_3 = u_4 = u_5 = 0. \tag{12}$$

Coefficients $C_0^\pm$, $C_1^\pm$, $C_2^\pm$ are calculated by

$$C_0^\pm = \frac{i\omega B v_0^\pm}{2DH(\lambda_0^2 - \lambda_1^2)(\lambda_0^2 - \lambda_2^2)}, \quad C_1^\pm = \frac{i\omega B v_1^\pm}{2DH(\lambda_1^2 - \lambda_0^2)(\lambda_1^2 - \lambda_2^2)},$$
$$C_2^\pm = \frac{i\omega B v_2^\pm}{2DH(\lambda_2^2 - \lambda_0^2)(\lambda_2^2 - \lambda_1^2)}, \tag{13}$$

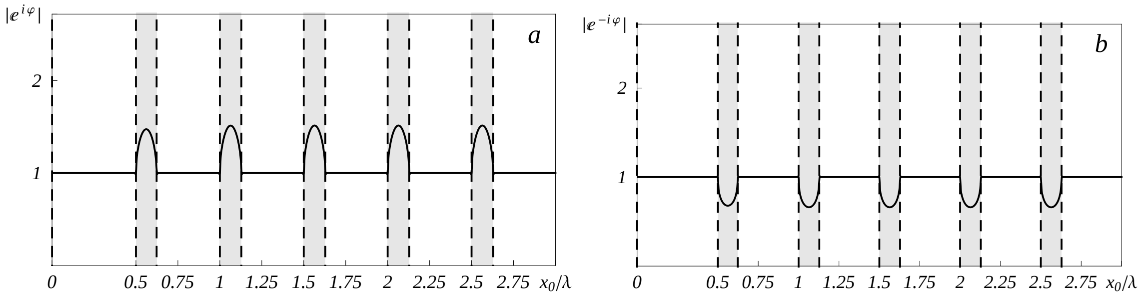

Figure 3: Moduli $|e^{i\varphi}|$ and $|e^{-i\varphi}|$ against the fractional separation x_0/λ between the movable clamps. Wave frequency is $\omega = 1.2\ s^{-1}$, wave length is $\lambda = 139.4$ m. Vertical dashed lines show the boundaries of the stopping bands. The stopping bands are shaded

with $\upsilon_n^{\pm} = 1/(e^{i\varphi} - e^{\pm i\lambda_n x_0})$, $n = 0, 1, 2$. We set $\zeta'(x)$ equal to zero at $x = x_0$ and derive a simple algebraic equation

$$\frac{\upsilon_0^+ - \upsilon_0^-}{(\lambda_0^2 - \lambda_1^2)(\lambda_0^2 - \lambda_2^2)}\,\lambda_0^3 + \frac{\upsilon_1^+ - \upsilon_1^-}{(\lambda_1^2 - \lambda_0^2)(\lambda_1^2 - \lambda_2^2)}\,\lambda_1^3 + \frac{\upsilon_2^+ - \upsilon_2^-}{(\lambda_2^2 - \lambda_0^2)(\lambda_2^2 - \lambda_1^2)}\,\lambda_2^3 = 0. \quad (14)$$

Let us consider the cases a) $|e^{i\varphi}| = 1$ and b) $|e^{i\varphi}| \neq 1$ separately. In the case a) the solution $\alpha_0(x)$ undergoes a change in phase φ across two neighboring cells of the width x_0 and represents a wave propagating without attenuation throughout the periodic array of the movable clamps. The phase velocity of the wave is directed to the side of increasing or decreasing of coordinate x. The frequency is said to lie in a passing band. Under condition b) relation $\alpha_0(x + x_0) = e^{i\varphi}\alpha_0(x)$ indicates that amplitude of the wave process exponentially grows or decays when passing from one cell to another. The frequency is said to lie in a stopping band.
It can be shown straightforwardly that the boundaries of passing and stopping bands satisfy the equations

$$\left[\begin{array}{l} \lambda_0^3(\lambda_1^2 - \lambda_2^2)\operatorname{ctg}\dfrac{\lambda_0 x_0}{2} + \lambda_1^3(\lambda_2^2 - \lambda_0^2)\operatorname{ctg}\dfrac{\lambda_1 x_0}{2} + \lambda_2^3(\lambda_0^2 - \lambda_1^2)\operatorname{ctg}\dfrac{\lambda_2 x_0}{2} = 0, \\ \lambda_0^3(\lambda_1^2 - \lambda_2^2)\operatorname{tg}\dfrac{\lambda_0 x_0}{2} + \lambda_1^3(\lambda_2^2 - \lambda_0^2)\operatorname{tg}\dfrac{\lambda_1 x_0}{2} + \lambda_2^3(\lambda_0^2 - \lambda_1^2)\operatorname{tg}\dfrac{\lambda_2 x_0}{2} = 0, \\ \sin(\lambda_0 x_0) = 0. \end{array}\right. \quad (15)$$

If $e^{i\varphi}$ is a solution of the equation (14) then so is $e^{-i\varphi}$. It means that waves have no preference in the direction in which they propagate. Dependencies of $|e^{\pm i\varphi}|$ from the fractional separation x_0/λ between the movable clamps are shown in Fig. 3.

3 An elastic periodically supported beam

We also concern with an infinitely long elastic beam with a periodic array of movable clamps at the points $x_n = x_0 n$, $n \in \mathbb{Z}$. The wave motion of the beam is governed by the equation

$$D\zeta''''(x) - \rho h \omega^2 \zeta(x) = 0, x \neq x_n, \quad (16)$$

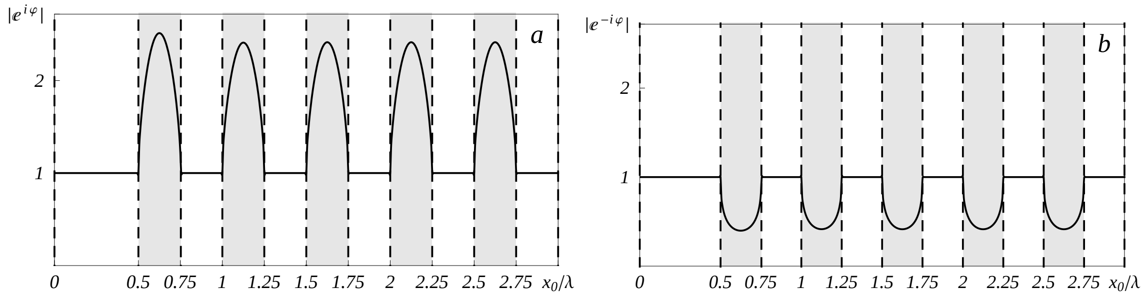

Figure 4: Moduli $|e^{i\varphi}|$ and $|e^{-i\varphi}|$ against the fractional separation x_0/λ between the movable clamps. Wave frequency is $\omega = 1.2\ s^{-1}$, wave length is $\lambda = 184.6$ m. Vertical dashed lines show the boundaries of the stopping bands. The stopping bands are shaded.

where $\zeta(x)$ is the elevation of the beam, D is the flexural rigidity, ρ is the density, h is the thickness, ω is a wave angular frequency. The elevation $\zeta(x)$ must satisfy the matching conditions at the points $x = x_n$

$$\lim_{x\to x_n^+} \zeta(x) = \lim_{x\to x_n^-} \zeta(x), \ \lim_{x\to x_n^+} \zeta'''(x) = \lim_{x\to x_n^-} \zeta'''(x), \ \lim_{x\to x_n} \zeta'(x) = 0. \tag{17}$$

The periodicity of the configuration allows us to consider the equation (16) on the interval $(0, 2x_0)$ and in accordance with the Floquet theorem write the general solution in the form

$$\zeta(x) = \begin{cases} C_1 e^{ikx} + C_2 e^{-ikx} + \alpha e^{-kx} + \beta e^{kx}, \ x \in (0, x_0), \\ (C_1 e^{ik(x-x_0)} + C_2 e^{-ik(x-x_0)} + \alpha e^{-k(x-x_0)} + \beta e^{k(x-x_0)}) e^{i\varphi}, \ x \in (x_0, 2x_0), \end{cases} \tag{18}$$

where k is the arithmetical root of the equation $Dk^4 - \rho h \omega^2 = 0$. Unknown coefficients C_1, C_2, α and β must satisfy the inhomogeneous linear system of equations

$$\begin{cases} C_1(e^{i\varphi} - e^{ikx_0}) + C_2(e^{i\varphi} - e^{-ikx_0}) + \alpha(e^{i\varphi} - e^{-kx_0}) + \beta(e^{i\varphi} - e^{kx_0}) = 0, \\ C_1(e^{i\varphi} - e^{ikx_0}) - C_2(e^{i\varphi} - e^{-ikx_0}) + i\alpha(e^{i\varphi} - e^{-kx_0}) - i\beta(e^{i\varphi} - e^{kx_0}) = 0, \\ C_1(e^{i\varphi} - e^{ikx_0}) + C_2(e^{i\varphi} - e^{-ikx_0}) - \alpha(e^{i\varphi} - e^{-kx_0}) - \beta(e^{i\varphi} - e^{kx_0}) = \dfrac{B}{D}, \\ C_1(e^{i\varphi} - e^{ikx_0}) - C_2(e^{i\varphi} - e^{-ikx_0}) - i\alpha(e^{i\varphi} - e^{-kx_0}) + i\beta(e^{i\varphi} - e^{kx_0}) = 0 \end{cases} \tag{19}$$

to which we arrive from the matching conditions (17). The solution of the system is

$$\begin{aligned} C_1 &= \frac{B}{4D(e^{i\varphi} - e^{ikx_0})}, \qquad C_2 = \frac{B}{4D(e^{i\varphi} - e^{-ikx_0})}, \\ \alpha &= -\frac{B}{4D(e^{i\varphi} - e^{-kx_0})}, \qquad \beta = -\frac{B}{4D(e^{i\varphi} - e^{kx_0})}. \end{aligned} \tag{20}$$

The derivative $\zeta'(x)$ must be equated to zero at $x = x_0$ to find the characteristic equation with respect to the unknown factor $e^{i\varphi}$

$$\frac{e^{ikx_0}}{e^{i\varphi} - e^{ikx_0}} - \frac{e^{-ikx_0}}{e^{i\varphi} - e^{-ikx_0}} - i\frac{e^{-kx_0}}{e^{i\varphi} - e^{-kx_0}} + i\frac{e^{kx_0}}{e^{i\varphi} - e^{kx_0}} = 0. \tag{21}$$

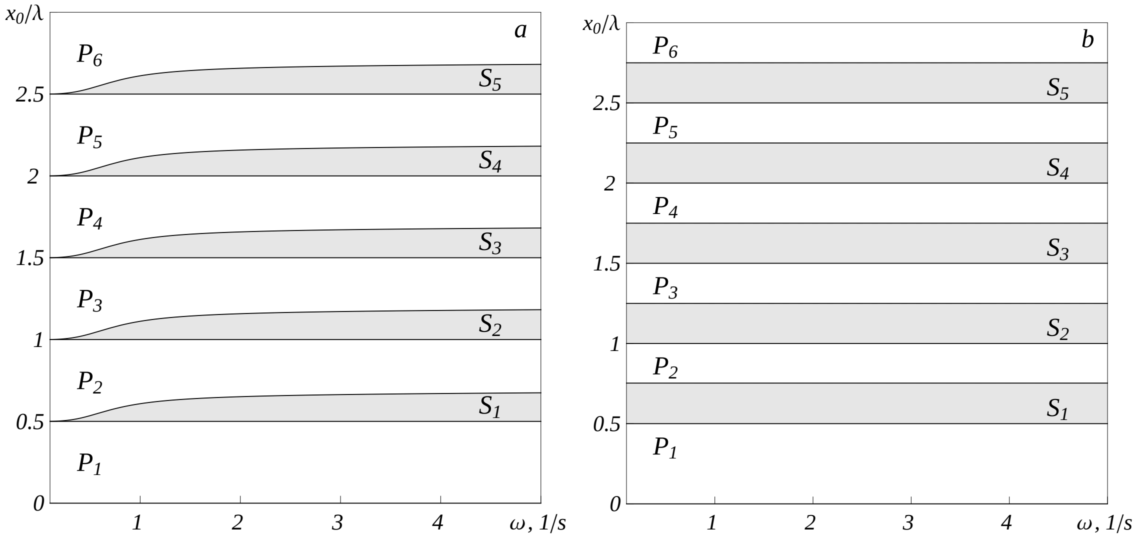

Figure 5: Passing and stopping bands in relation to the angular frequency ω and the fractional separation x_0/λ between the movable clamps. Stopping bands are shaded. λ is a wave length. a) An elastic periodic floating plate. b) An elastic periodic beam.

The boundaries of passing and stopping bands are described by equations

$$\operatorname{tg}\frac{kx_0}{2} \pm \operatorname{th}\frac{kx_0}{2} = 0, \quad \sin(kx_0) = 0. \tag{22}$$

Fig. 4 shows $|e^{\pm i\varphi}|$ versus the fractional separation x_0/λ between the movable clamps. Fig. 5 demonstrates the passing and stopping bands in relation to the frequency ω and the fractional separation x_0/λ between the movable clamps. The left subplot shows the curves obtained in the case of a floating elastic plate, the right subplot – in the case of an elastic beam. It is seen that the presence of a fluid layer markedly affects on the propagation of flexural waves. Stopping bands are narrowed. At very low frequencies they vanish. On the other hand, passing bands are broaden. Apparently this result has the following physical explanation. A fluid layer represents an additional channel of energy transfer, which facilitates the transmission of waves. Another feature is that the width of the first passing band is larger then others and is equal to a half-wave length.

4 Conclusion

Calculations performed by formulae (15) show that the maxima of energy transmission coefficient in a case of a finite number of movable clamps are reached within passing bands, and the minima – within stopping bands (Fig. 1). Since the maxima of transmission of an incident wave are accompanied by considerable internal efforts that are developed in all movable clamps and also by great amplitudes of the plate elevation, the knowledge of the boundaries and location of stopping and passing bands is of practical value.

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M. G. Zhuchkova, V.O., Bolshoj pr., 61, St. Petersburg, 199178, Russia