Statics and harmonic oscillatoins of springs as rods of arbitrary spatial shape

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Abstract

The springs are considered as linearly elastic rods of arbitrary spatial shape with taking into account tension and shear. The equations of rod dynamics are written in direct tensor form. The system of equations in components is derived using the assumption of cross-section symmetry. These equations admit the solution in quadratures. Also they can be solved by the shooting method implemented in the built-in function of computer mathematics packages. As an example, we present the static and dynamic analysis of a conical helical spring. The algorithm of calculation is presented for the forced and free harmonic oscillations.

1 Introduction

The springs applied in mechanical engineering have often forms of rods [1]. The progress in mechanics of deformable solid bodies [2] resolves the previous problems [3]. The calculation of rods is simplified especially by wide-spreading of computer mathematics [4, 5]. The transition to the case of harmonic oscillations has no difficulties.

From the recent studies into the spring dynamics, we note [6]–[9]. In [6] the effect of axial comression on the natural frequencies is studied. The comparison between different dynamic models of helical springs is discussed using dispersion curves in [7]. The barrel and hyperboloidal springs are investigated in the paper [8] using Laplace transform. The effects of rotational inertia, axial, shear and torsional deformation for helical springs of different cross sections are studied in [9].

2 Linear theory of rods

The equations of linear theory of rods are reliably established [2]. For harmonic oscillations with the frequency ω, they have the form:

$$\begin{aligned} \mathbf{Q}' &= -\mathbf{q} - \omega^2 \rho (\mathbf{u} + \boldsymbol{\theta} \times \boldsymbol{\varepsilon}), \\ \mathbf{M}' &= -\mathbf{r}' \times \mathbf{Q} - \mathbf{m} - \omega^2 (\mathbf{I} \cdot \boldsymbol{\theta} + \rho \boldsymbol{\varepsilon} \times \mathbf{u}), \\ \boldsymbol{\theta}' &= \mathbf{A} \cdot \mathbf{M} + \mathbf{C} \cdot \mathbf{Q}, \\ \mathbf{u}' &= \boldsymbol{\theta} \times \mathbf{r}' + \mathbf{B} \cdot \mathbf{Q} + \mathbf{M} \cdot \mathbf{C}. \end{aligned} \tag{1}$$

The rods are considered as material lines whose particles are the elementary bodies with the vectors of displacement $\mathbf{u}$ and rotation $\boldsymbol{\theta}$. The loads are external distributed per unit length forces and moments $\mathbf{q}$, $\mathbf{m}$ and internal (in the cross sections) ones $\mathbf{Q}$, $\mathbf{M}$. Prime indicates derivative with respect to material coordinate s (not necessarily arc coordinate). The tensors of the second rank $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ characterize the elastic compliances of rod including the bending, twisting, tension and shear compliance. Three-dimensional models have to be used to calculate them; for example, the Saint-Venant problem. The inertial rod parameters are the density ρ, the eccentricity vector $\boldsymbol{\varepsilon}$ and the tensor of inertia $\mathbf{I}$. The initial form of rod is given by the position vector dependence $\mathbf{r}(s)$ on the material coordinate.
The first two equations in (1) are the balance of momentum and the balance of moment of momentum. The third and the fourth are the relations of elasticity. These relations connect the strain vectors $\boldsymbol{\kappa} \equiv \boldsymbol{\theta}'$, $\boldsymbol{\gamma} \equiv \mathbf{u}' - \boldsymbol{\theta} \times \mathbf{r}'$ with the force factors $\mathbf{Q}$, $\mathbf{M}$. The entire system (1) can be derived from the principle of virtual work [2].

3 Integration of static equations in quadratures

In statics ($\omega = 0$) the equations (1) are integrated in quadratures [3] as follows:

$$\begin{aligned} \mathbf{Q} &= -\int_0^s \mathbf{q} ds + \mathbf{Q}_0,\ \mathbf{M} = \int_0^s (\mathbf{Q} \times \mathbf{r}' - \mathbf{m})\, ds + \mathbf{M}_0, \\ \boldsymbol{\theta} &= \int_0^s (\mathbf{A} \cdot \mathbf{M} + \mathbf{C} \cdot \mathbf{Q})\, ds + \boldsymbol{\theta}_0,\ \mathbf{u} = \int_0^s (\boldsymbol{\theta} \times \mathbf{r}' + \mathbf{B} \cdot \mathbf{Q} + \mathbf{M} \cdot \mathbf{C})\, ds + \mathbf{u}_0. \end{aligned} \tag{2}$$

Here four vector constants arise, and they are determined by using the boundary conditions. If the rod is fixed at the tip $s = 0$ and loaded at the other tip $s = L$, then from the first two equations we determine $\mathbf{Q}_0$, $\mathbf{M}_0$ right away, and in another equations we have $\boldsymbol{\theta}_0 = 0$, $\mathbf{u}_0 = 0$ - it is the statically determined problem. With the arbitrary given initial form $\mathbf{r}(s)$, the integrals (2) are calculated in components easily by means of computer mathematics. We can use the integration by parts:

$$\int \mathbf{Q} \times \mathbf{r}' ds = \mathbf{Q} \times \mathbf{r} + \int \mathbf{q} \times \mathbf{r} ds,\ \int \boldsymbol{\theta} \times \mathbf{r}' ds = \boldsymbol{\theta} \times \mathbf{r} - \int (\mathbf{A} \cdot \mathbf{M} + \mathbf{C} \cdot \mathbf{Q}) \times \mathbf{r} ds$$

In the case of statically indetermined problem, we need to construct and solve the linear algebraic system following from the boundary condition in order to determine the vector constants $\mathbf{Q}_0$, $\mathbf{M}_0$, $\boldsymbol{\theta}_0$, $\mathbf{u}_0$.
The case of a closed rod requires a special consideration. Instead of boundary conditions we have the conditions of periodicity: $\mathbf{Q}(0) = \mathbf{Q}(L)$ etc. For the first two equalities in (1), we obtain the identities, namely the balances of forces and moments. The constants $\mathbf{Q}_0$, $\mathbf{M}_0$ are indeterminate herewith. The least equalities give the following conditions of the uniqueness of rotations and displacements:

$$\begin{aligned} \oint (\mathbf{A} \cdot \mathbf{M} + \mathbf{C} \cdot \mathbf{Q})\, ds = 0, \\ \oint [\mathbf{r} \times (\mathbf{A} \cdot \mathbf{M} + \mathbf{C} \cdot \mathbf{Q}) + \mathbf{B} \cdot \mathbf{Q} + \mathbf{M} \cdot \mathbf{C}]\, ds = 0. \end{aligned} \tag{3}$$

This is the linear algebraic system for the unknowns $\mathbf{Q}_0$, $\mathbf{M}_0$.
Let us clarify the form of compliances. If the section has two axes of symmetry, then they are

$$\begin{aligned}\mathbf{A} &= (A_\perp \mathbf{E} + (A_\tau - A_\perp)\,\boldsymbol{\tau\tau})\,|\mathbf{r}'|\,,\ \boldsymbol{\tau} = \mathbf{r}'/|\mathbf{r}'|,\\ \mathbf{B} &= (B_\perp \mathbf{E} + (B_\tau - B_\perp)\,\boldsymbol{\tau\tau})\,|\mathbf{r}'|\,,\ \mathbf{C} = 0 \end{aligned} \tag{4}$$

We denoted: $A_\perp$, A_τ are the bending and twisting compliances, B_τ, $B_\perp$ are the tension and shear compliances. $\mathbf{E}$ is the identity tensor, $\boldsymbol{\tau}$ is the tangent vector. The factor $|\mathbf{r}'|$ is here because the length of the element is $|\mathbf{r}'|\,ds$. For the thin rods one usually assumes the absence of tension and shear $\mathbf{B} = 0$ (hence $\mathbf{C} = 0$).

4 Use of computer mathematics

We think about another approach to be more efficient. We use the numerical integration of the system of ordinary differential equations (ODE) of 12th order by means of the computer mathematics [5]. This requires to write the equations (1) in components. Projecting (1) into the Cartesian axes x, y, z, we have:

$$\begin{aligned} Q_x{}' = -q_x,\ Q_y{}'...,\ M_x{}' = Q_y z' - Q_z y' - m_x,\ M_y{}'...,\\ \theta_x{}' = A_\perp\,|\mathbf{r}'|\,M_x + (A_\tau - A_\perp)x'|\mathbf{r}'|^{-1}(x'M_x + y'M_y + z'M_z),\ \theta_y{}'...,\\ u_x{}' = \theta_y z' - \theta_z y' + B_\perp\,|\mathbf{r}'|\,Q_x + (B_\tau - B_\perp)x'|\mathbf{r}'|^{-1}(x'Q_x + y'Q_y + z'Q_z),\ u_y{}'... \end{aligned} \tag{5}$$

Then we rewrite it to the matrix form:

$$\begin{aligned} Y' = F(s,Y);\ Y = (Q_x\,Q_y\,...\,u_z)^T,\ F_0 = -q_x(s),\ ...,\\ F_{11} = Y_6 y' - Y_7 x' + B_\perp m(s) Y_2 + \frac{B_\tau - B_\perp}{m(s)} z'(x'Y_0 + y'Y_1 + z'Y_2);\\ m(s) \equiv \sqrt{x'^2 + y'^2 + z'^2}. \end{aligned} \tag{6}$$

With the given boundary conditions (six at both the ends), the system is solved in Mathcad by the shooting method with the built-in functions sbval—Rkadapt [4]. The same results are obtained by the finite difference method and the standard solver of algebraic systems in Wofram Mathematica.
As a benchmark example we consider the steel conical spring with the following equations:

$$x(s) = \rho(s)\cos s,\ y(s) = \rho(s)\sin s,\ z(s) = z_1 s;\ \rho(s) = \rho_0 + \rho_1 s \tag{7}$$

Here we have three constants; with $\rho_1 = 0$ the curve is the helix. The end $s = 0$ is fixed, hence displacement and rotation are zero: $\mathbf{u} = 0, \boldsymbol{\theta} = 0$. The free end $s = 10\pi$ (5 coils) is loaded by the force $\mathbf{Q}$ (the moment is absent). We show the initial state of the spring and two deformed states: with $\mathbf{Q} = -P_1\mathbf{k}$ (along z-axis) (Fig. 1a) and $\mathbf{Q} = P_2\,\mathbf{i}$ (along x-axis) (Fig. 1b). The geometric parameters are: the section radius is $r = 1$ cm, the parameters of the curve are $\rho_0 = 10$ cm, $\rho_1 = -1.5$ cm, $z_1 = 2$ cm. The rod compliances are $A_\perp = 4/E\pi r^4$, $A_\tau = 2/\mu\pi r^4$, $B_\tau = 1/E\pi r^2$ $B_\perp = 7/6\mu\pi r^2$ (E is the Young's modulus, μ is the shear modulus and 6/7 is the

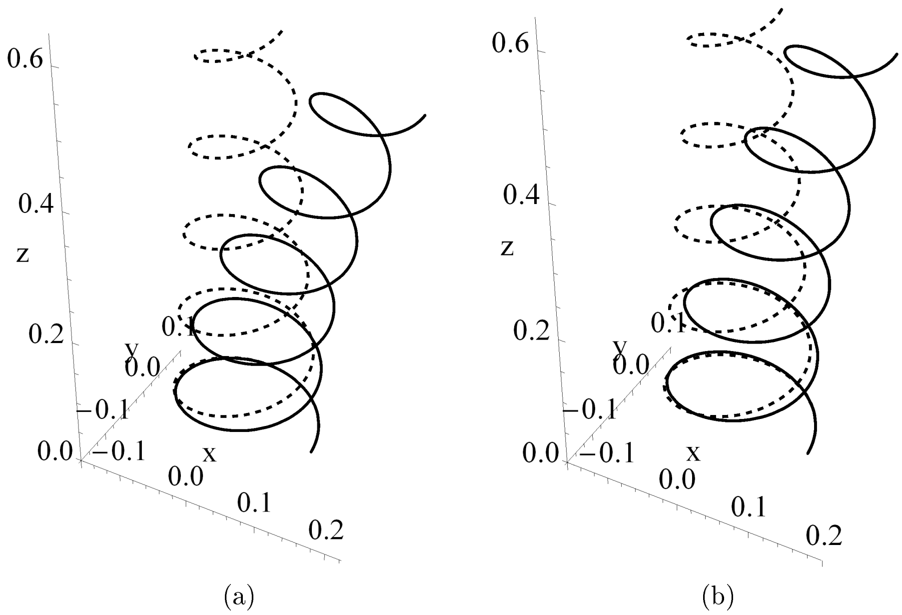


Figure 1: Configurations of spring

shear coefficient). In Fig. 1a the load is $P_1 = 5000$ N, in Fig. 1b $P_2 = 500$ N. Cartesian coordinates in the initial state correspond to the formulas (7), and in the actual state they correspond to $x + u_x$, $y + u_y$, $z + u_z$.

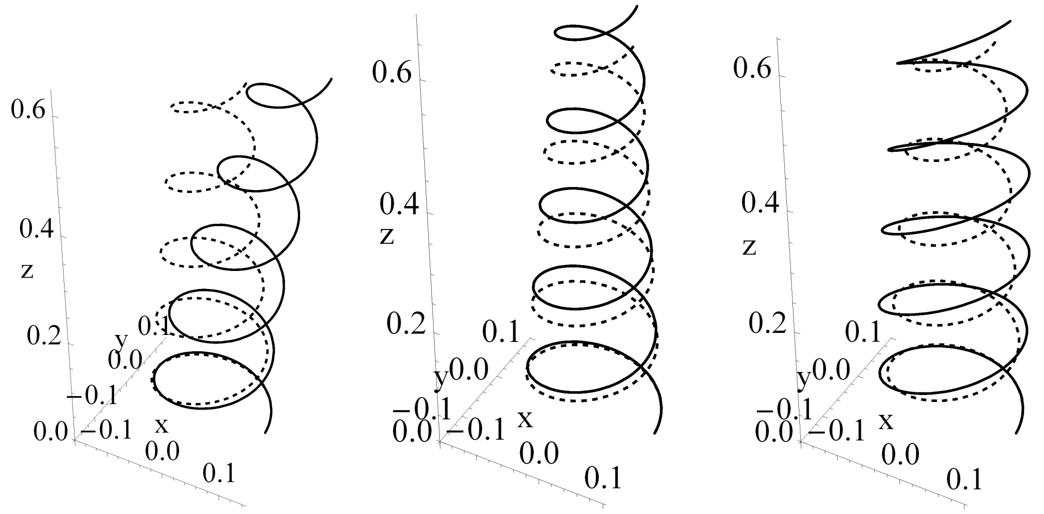
In Fig. 1 the deformations are large, because the loads are so. But the problem is linear, and hence the displacements are proportional to the loading parameters. In numerical integrating we can restrict ourselves with the small loads, avoiding the problems with convergence of the shooting method.

We can use the set of points instead of the analytic dependences as in (7) to define the initial shape. Using the set of points we apply the built-in functions for the regression (regress—interp in Mathcad). This approach is realized in [5] for the plane nonlinearly elastic springs.

5 Harmonic oscillations

The numerical algorithm discussed above can be extended easy to the case of forced harmonic oscillations. For this purpose we need to add the inertial loads (proportional to ω^2) [10]. In the case of the free oscillations we have the eigenvalue problem, for obtaining the solution of which we include in the system one more equation: $(\omega^2)' = 0$. We can prescribe an additional boundary condition because the modes of oscillations are determined to within a constant factor.

As a benchmark example we analyse the free oscillations of the spring considered above in Sect. 4. Eccentricity and inertia of rotation are not taken into account. The resulting eigenmodes are in Fig. 2.



(a) (b) (c)

Figure 2: Eigenmodes of spring

6 Conclusion

One-dimensional rod model was applied to statical and dynamical analysis of springs of arbitrary spatial shape. The linear equations in the direct tensor form and in components were presented. The solution in quadratures and the solution by means of computer mathematics were proposed. The first approach is helpful in statically determined problems. The second approach uses the numerical integration methods such as the shooting method and the finite difference method. A benchmark example of conical spring was considered. Configuration of spring under the static loading and the first three eigenmodes were shown.

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