Symplectic framework, discrete variational approach and Harten's multiresolution scheme in beam dynamics

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Abstract

We consider general symplectic set-up which provides common framework for a class of problems related to the (quasi)classical description of nonlinear beam dynamics: metaplectic structure, Segal-Bargmann representation, orbital theory. All that allows to unify different important and needful accessories of any reasonable dynamical approach: (Melnikov) perturbations, quasiclassics, Floer loops, symplectic/wavelet scales. After that, we consider the applications of discrete wavelet analysis technique (Harten's multiresolution calculus for maps) to the maps which arise as the discretization of continuous invariant nonlinear polynomial problems (Veselov-Marsden's approach). Our main point is a generalization of wavelet analysis which can be applied for both discrete and continuous cases. It provides explicit multiresolution decomposition for solutions of discrete problems which are the correct discretizations of the corresponding continuous cases.

1 Introduction

In this paper, we consider hidden dynamical symmetry as a key generic feature instead of kinematical one presented in the companion paper in this Volume [1]. In Section 2, according to the orbit method in geometric quantization theory background, we construct the symplectic and Poisson structures [2] associated with generalized representations and orbital theory [3], [4] by using metaplectic structure. We apply such an approach to analysis of Melnikov functions in the theory of homoclinic chaos in perturbed Hamiltonian systems [5] in Section 3 and for calculation of the Arnold–Weinstein curves (closed loops) in the Floer variational approach [6] in Section 4. In Sections 5 and 6, we sketched out possible applications of very useful fast wavelet transform technique [4] for analysis of symplectic scale of spaces [7], [8] and for quasiclassical Wigner-Weyl evolution dynamics [2], [3]. This method gives maximally sparse representation of the (differential) operator that allows, at least in principle, to calculate very fast the contributions from each level of resolution of the whole tower of hidden scales. Section 7 is divided in two parts devoted to very useful scheme(s) of discretization. In Subsection 7.1, according to the Marsden-Veselov approach, we consider symplectic and Lagrangian background for the case of discretization of flows by the corresponding maps [9] and in Subsection 7.2, we present the construction of the corresponding solutions by applications of the multi-scale approach of A. Harten [10] based on generalization of multiresolution analysis for the case of maps. "All that jazz" considered here, was applied by the authors to a number of physical problems of beam physics, accelerator physics, plasma physics, and quantum physics. In this short exposition, we constrain ourselves by ideological paradigms only. All details, constructions, and results can be found in [11]-[33].

2 Symplectic Structure, Metaplectic Group, Representations

Let $Sp(n)$ be symplectic group, $Mp(n)$ be its unique two- fold covering – metaplectic group [2], [3]. Let V be a symplectic vector space with symplectic form (,), then $R \oplus V$ is nilpotent Lie algebra - Heisenberg algebra:

$$[R, V] = 0, \quad [v, w] = (v, w) \in R, \quad [V, V] = R.$$

$Sp(V)$ is a group of automorphisms of Heisenberg algebra.
Let N be a group with Lie algebra $R \oplus V$, i.e. Heisenberg group. By Stone– von Neumann theorem Heisenberg group has unique irreducible unitary representation in which $1 \mapsto i$. Let us also consider the projective representation of simplectic group $Sp(V)$: $U_{g_1}U_{g_2} = c(g_1, g_2) \cdot U_{g_1 g_2}$, where c is a map: $Sp(V) \times Sp(V) \to S^1$, i.e. c is S^1-cocycle.
This representation is unitary representation of universal covering, i.e. metaplectic group $Mp(V)$. We give this representation without Stone-von Neumann theorem. Consider a new group $F = N' \bowtie Mp(V)$, $\bowtie$ is semidirect product (we consider instead of $N = R \oplus V$ the $N' = S^1 \times V$, $S^1 = (R/2\pi Z)$). Let V^* be dual to V, $G(V^*)$ be automorphism group of V^*.Then F is subgroup of $G(V^*)$, which consists of elements, which acts on V^* by affine transformations.
This is the key point!
Let $q_1, ..., q_n; p_1, ..., p_n$ be symplectic basis in V, $\alpha = pdq = \sum p_i dq_i$ and $d\alpha$ be symplectic form on V^*. Let M be fixed affine polarization, then for $a \in F$ the map $a \mapsto \Theta_a$ gives unitary representation of G: $\Theta_a : H(M) \to H(M)$.
Explicitly we have for representation of N on H(M):

$$(\Theta_q f)^*(x) = e^{-iqx} f(x), \quad \Theta_p f(x) = f(x - p)$$

The representation of N on H(M) is irreducible. Let A_q, A_p be infinitesimal operators of this representation

$$A_q = \lim_{t \to 0} \frac{1}{t}[\Theta_{-tq} - I], \quad A_p = \lim_{t \to 0} \frac{1}{t}[\Theta_{-tp} - I],$$

$$\text{then} \qquad A_q f(x) = i(qx) f(x), \quad A_p f(x) = \sum p_j \frac{\partial f}{\partial x_j}(x)$$

Now we give the representation of infinitesimal basic elements. Lie algebra of the group F is the algebra of all (nonhomogeneous) quadratic polynomials of (p,q) relatively Poisson bracket (PB). The basis of this algebra consists of elements $1, q_1, ..., q_n$, $p_1, ..., p_n$, $q_i q_j, q_i p_j$, $p_i p_j$, $\quad i, j = 1, ..., n, \quad i \leq j$,

$$PB\ is \quad \{f, g\} = \sum \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \quad \text{and} \quad \{1, g\} = 0 \quad for \text{ all } g,$$

$$\{p_i, q_j\} = \delta_{ij}, \quad \{p_i q_j, q_k\} = \delta_{ik} q_j, \quad \{p_i q_j, p_k\} = -\delta_{jk} p_i, \quad \{p_i p_j, p_k\} = 0,$$
$$\{p_i p_j, q_k\} = \delta_{ik} p_j + \delta_{jk} p_i, \quad \{q_i q_j, q_k\} = 0, \{q_i q_j, p_k\} = -\delta_{ik} q_j - \delta_{jk} q_i$$

so, we have the representation of basic elements $f \mapsto A_f : 1 \mapsto i, q_k \mapsto i x_k$,

$$p_l \mapsto \frac{\delta}{\delta x^l}, p_i q_j \mapsto x^i \frac{\partial}{\partial x^j} + \frac{1}{2}\delta_{ij}, \qquad p_k p_l \mapsto \frac{1}{i}\frac{\partial^k}{\partial x^k \partial x^l}, q_k q_l \mapsto i x^k x^l$$

All that provides the structure of the Poisson manifolds for representation of any (nilpotent) algebra or in other words to some sort of continuous wavelet transform.
The Segal-Bargman Representation. Let $z = 1/\sqrt{2} \cdot (p - iq), \quad \bar{z} = 1/\sqrt{2} \cdot (p + iq), \quad p = (p_1, ..., p_n), \quad F_n$ is the space of holomorphic functions of n complex variables with $(f, f) < \infty$, where

$$(f, g) = (2\pi)^{-n} \int f(z)\overline{g(z)} e^{-|z|^2} dp dq$$

Consider a map $U : H \to F_n$, where H is with real polarization, F_n is with complex polarization, then we have

$$(U\Psi)(a) = \int A(a, q)\Psi(q) dq, \qquad \text{where} \quad A(a, q) = \pi^{-n/4} e^{-1/2(a^2+q^2)+\sqrt{2}aq}$$

i.e. the Bargmann formula produce wavelets.We also have the representation of Heisenberg algebra on F_n :

$$U\frac{\partial}{\partial q_j}U^{-1} = \frac{1}{\sqrt{2}}\left(z_j - \frac{\partial}{\partial z_j}\right), \qquad U q_j U^{-1} = -\frac{i}{\sqrt{2}}\left(z_j + \frac{\partial}{\partial z_j}\right)$$

and also : $\omega = d\beta = dp \wedge dq$, where $\beta = i\bar{z}dz$.
Orbital Theory for Representations. Let coadjoint action be $< g \cdot f, Y > = < f, Ad(g)^{-1} Y >$, where $<, >$ is pairing $g \in G, \quad f \in g^*, \quad Y \in \mathcal{G}$. The orbit is $\mathcal{O}_f = G \cdot f \equiv G/G(f)$. Also, let A=A(M) be algebra of functions, V(M) is A-module of vector fields, A^p is A-module of p-forms. Vector fields on orbit is

$$\sigma(\mathcal{O}, X)_f(\phi) = \frac{d}{dt}(\phi(\exp tXf))\Big|_{t=0}$$

where $\phi \in A(\mathcal{O}), \quad f \in \mathcal{O}$. Then $\mathcal{O}_f$ are homogeneous symplectic manifolds with 2-form $\Omega(\sigma(\mathcal{O}, X)_f, \sigma(\mathcal{O}, Y)_f) = < f, [X, Y] >$, and $d\Omega = 0$. PB on $\mathcal{O}$ have the next form $\{\Psi_1, \Psi_2\} = p(\Psi_1)\Psi_2$ where p is $A^1(\mathcal{O}) \to V(\mathcal{O})$ with definition $\Omega(p(\alpha), X) = i(X)\alpha$. Here $\Psi_1, \Psi_2 \in A(\mathcal{O})$ and $A(\mathcal{O})$ is Lie algebra with bracket $\{,\}$. Now let N be a Heisenberg group. Consider adjoint and coadjoint representations in some particular case. $N = (z, t) \in C \times R, z = p + iq$; compositions in N are $(z, t) \cdot (z', t') =$

$(z+z', t+t'+B(z,z'))$, where $B(z,z') = pq' - qp'$. Inverse element is $(-t,-z)$. Lie algebra n of N is $(\zeta,\tau) \in C \times R$ with bracket $[(\zeta,\tau),(\zeta',\tau')] = (0, B(\zeta,\zeta'))$. Center is $\tilde{z} \in n$ and generated by (0,1); Z is a subgroup $\exp \tilde{z}$. Adjoint representation N on n is given by formula $Ad(z,t)(\zeta,\tau) = (\zeta, \tau + B(z,\zeta))$ Coadjoint: for $f \in n^*$, $g = (z,t)$, $(g \cdot f)(\zeta,\zeta) = f(\zeta,\tau) - B(z,\zeta)f(0,1)$ then orbits for which $f|_{\tilde{z}} \neq 0$ are plane in n^* given by equation $f(0,1) = \mu$. If $X = (\zeta,0)$, $Y = (\zeta',0)$, $X,Y \in n$ then symplectic structure is

$$\Omega(\sigma(\mathcal{O},X)_f, \sigma(\mathcal{O},Y)_f) =< f,[X,Y] >= f(0,B(\zeta,\zeta'))\mu B(\zeta,\zeta')$$

Also we have for orbit $\mathcal{O}_\mu = N/Z$ and $\mathcal{O}_\mu$ is Hamiltonian G-space.
According to this approach, we can construct, in principle, many generalized "symplectic wavelet constructions" with corresponding symplectic or Poisson structure on it by using methods of geometric quantization theory Very useful particular spline–wavelet basis with uniform exponential control on stratified and nilpotent Lie groups was considered in [4].

3 Invariant Perturbations: Melnikov Functions

We give now some point of applications of wavelet methods [11]-[33] to Melnikov approach in the theory of homoclinic chaos in perturbed Hamiltonian systems [5].
In explicit Hamiltonian form we have:

$$\dot{x} = J \cdot \nabla H(x) + \varepsilon g(x,\Theta), \quad \dot{\Theta} \;=\; \omega, \quad (x,\Theta) \in R^n \times T^m,$$

for $\varepsilon = 0$ we have:

$$\dot{x} = J \cdot \nabla H(x), \quad \dot{\Theta} = \omega \tag{1}$$

For $\varepsilon = 0$ we have homoclinic orbit $\bar{x}_0(t)$ to the hyperbolic fixed point x_0. For $\varepsilon \neq 0$ we have normally hyperbolic invariant torus T_ε and condition on transversally intersection of stable and unstable manifolds $W^s(T_\varepsilon)$ and $W^u(T_\varepsilon)$ in terms of Melnikov functions $M(\Theta)$ for $\bar{x}_0(t)$:

$$M(\Theta) = \int_{-\infty}^{\infty} \nabla H(\bar{x}_0(t)) \wedge g(\bar{x}_0(t), \omega t + \Theta) dt$$

This condition has the next form:

$$M(\Theta_0) = 0, \qquad \sum_{j=1}^{2} \omega_j \frac{\partial}{\partial \Theta_j} M(\Theta_0) \neq 0$$

According to the approach of Birkhoff-Smale-Wiggins we determine the region(s) in parameter space in which we can observe the chaotic-like behaviour [13].
If we cannot solve equations (1) explicitly in time, then we use our wavelet approach [11]-[33] for the computations of homoclinic (heteroclinic) loops as the solution of

system (1) in general multiscale basis. For analysis of quasiperiodic Melnikov functions

$$M^{m/n}(t_0) = \int_0^{mT} DH(x_\alpha(t)) \wedge g(x_\alpha(t), t + t_0) dt$$

we used periodization of our wavelet construction.
We also used symplectic Melnikov function approach in which we have:

$$\begin{aligned} M_i(z) &= \lim_{j \to \infty} \int_{-T_j^*}^{T_j} \{h_i, \hat{h}\}_{\Psi(t,z)} dt \\ d_i(z, \varepsilon) &= h_i(z_\varepsilon^u) - h_i(z_\varepsilon^s) = \varepsilon M_i(z) + O(\varepsilon^2) \end{aligned}$$

where $\{,\}$ is the Poisson bracket, $d_i(z, \varepsilon)$ is the Melnikov distance. So, we need symplectic invariant expression for Poisson brackets. The computations can be done according to invariant calculation of Poisson brackets considered here.

4 Floer Approach for Closed Loops

Now we consider the generalization of wavelet variational approach to the symplectic invariant calculations of closed loops in Hamiltonian systems [6]. As we demonstrated in [13] we have the parametrization of our solution by some reduced algebraical problem, but in contrast to the standard cases, where the solution is parametrized by construction based on scalar refinement equation, in symplectic case we have parametrization of the solution by matrix problems – Quadratic Mirror Filters equations. Now we consider a different approach.
Let(M, ω) be a compact symplectic manifold of dimension $2n$, ω is a closed 2-form (nondegenerate) on M which induces an isomorphism $T^*M \to TM$. Thus every smooth time-dependent Hamiltonian $H : \mathbf{R} \times M \to \mathbf{R}$ corresponds to a time-dependent Hamiltonian vector field $X_H : \mathbf{R} \times M \to TM$ defined by

$$\omega(X_H(t, x), \xi) = -d_x H(t, x)\xi \tag{2}$$

for $\xi \in T_x M$. Let H (and X_H) is periodic in time: $H(t + T, x) = H(t, x)$ and consider corresponding Hamiltonian differential equation on M:

$$\dot{x}(t) = X_H(t, x(t)) \tag{3}$$

The solutions $x(t)$ of (3) determine a 1-parameter family of diffeomorphisms $\psi_t \in \mathrm{Diff}(M)$ satisfying $\psi_t(x(0)) = x(t)$. These diffeomorphisms are symplectic: $\omega = \psi_t^* \omega$. Let $L = L_T M$ be the space of contractible loops in M which are represented by smooth curves $\gamma : \mathbf{R} \to M$ satisfying $\gamma(t + T) = \gamma(t)$. Then the contractible T-periodic solutions of (3) can be characterized as the critical points of the functional $S = S_T : L \to \mathbf{R}$:

$$S_T(\gamma) = -\int_D u^*\omega + \int_0^T H(t, \gamma(t)) \mathrm{d}t, \tag{4}$$

where $D \subset \mathbf{C}$ be a closed unit disc and $u : D \to M$ is a smooth function, which on boundary agrees with γ, i.e. $u(\exp\{2\pi i\Theta\}) = \gamma(\Theta T)$. Because $[\omega]$, the cohomology class of ω, vanishes then $S_T(\gamma)$ is independent of choice of u. Tangent space $T_\gamma L$ is the space of vector fields $\xi \in C^\infty(\gamma^* TM)$ along γ satisfying $\xi(t+T) = \xi(t)$. Then we have for the 1-form $\mathrm{d}f : TL \to \mathbf{R}$

$$\mathrm{d}S_T(\gamma)\xi = \int_0^T (\omega(\dot\gamma, \xi) + \mathrm{d}H(t,\gamma)\xi)\mathrm{d}t \tag{5}$$

and the critical points of S are contractible loops in L which satisfy the Hamiltonian equation (3). Thus the critical points are precisely the required T-periodic solution of (3).
To describe the gradient of S we choose a on almost complex structure on M which is compatible with ω. This is an endomorphism $J \in C^\infty(\mathrm{End}(TM))$ satisfying $J^2 = -I$ such that

$$g(\xi,\eta) = \omega(\xi, J(x)\eta), \qquad \xi, \eta \in T_x M \tag{6}$$

defines a Riemannian metric on M. The Hamiltonian vector field is then represented by $X_H(t,x) = J(x)\nabla H(t,x)$, where ∇ denotes the gradient w.r.t. the x-variable using the metric (6). Moreover the gradient of S w.r.t. the induced metric on L is given by

$$\mathrm{grad}S(\gamma) = J(\gamma)\dot\gamma + \nabla H(t,\gamma), \qquad \gamma \in L \tag{7}$$

Studying the critical points of S is confronted with the well-known difficulty that the variational integral is neither bounded from below nor from above. Moreover, at every possible critical point the Hessian of f has an infinite dimensional positive and an infinite dimensional negative subspaces, so the standard Morse theory is not applicable. The additional problem is that the gradient vector field on the loop space L

$$\frac{\mathrm{d}}{\mathrm{d}s}\gamma = -\mathrm{grad}f(\gamma) \tag{8}$$

does not define a well posed Cauchy problem. Fortunately, Floer [6] found a way to analyse the space $\mathcal{M}$ of bounded solutions consisting of the critical points together with their connecting orbits. He used a combination of variational approach and Gromov's elliptic technique. A gradient flow line of f is a smooth solution $u : \mathbf{R} \to M$ of the partial differential equation

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H(t,u) = 0, \tag{9}$$

which satisfies $u(s, t+T) = u(s,t)$. The key point is to consider (9) not as the flow on the loop space but as an elliptic boundary value problem. It should be noted that (9) is a generalization of equation for Gromov's pseudoholomorphic curves (correspond to the case $\nabla H = 0$ in (9)). Let $\mathcal{M}_T = \mathcal{M}_T(H,J)$ the space of bounded solutions of (9), i.e. the space of smooth functions $u : \mathbf{C}/iT\mathbf{Z} \to M$, which are contractible, solve equation (9) and have finite energy flow:

$$\Phi_T(u) = \frac{1}{2}\int_{-\infty}^{\infty}\int_0^T \left(|\frac{\partial u}{\partial s}|^2 + |\frac{\partial u}{\partial t} - X_H(t,u)|^2 \right) \mathrm{d}t\mathrm{d}s \quad < \infty. \tag{10}$$

For every $u \in M_T$ there exists a pair x, y of contractible T-periodic solutions of (3), such that u is a connecting orbit from y to x:

$$\lim_{s\to-\infty} u(s,t) = y(t), \qquad \lim_{s\to+\infty} = x(t) \tag{11}$$

So, our approach, which we may apply as on the level of standard boundary problem (9) as on the level of variational approach (10), together with FWT representation for all involved operators (in our case, J and ∇) can provide economically computable multiscale representation for Hamiltonian closed loops.

5 Quasiclassical Evolution

Let us consider classical and quantum dynamics in phase space $\Omega = R^{2m}$ with coordinates (x, ξ) and generated by Hamiltonian $\mathcal{H}(x,\xi) \in C^\infty(\Omega; R)$. If $\Phi_t^{\mathcal{H}} : \Omega \longrightarrow \Omega$ is (classical) flow then time evolution of any bounded classical observable or symbol $b(x,\xi) \in C^\infty(\Omega, R)$ is given by $b_t(x,\xi) = b(\Phi_t^{\mathcal{H}}(x,\xi))$. Let $H = Op^W(\mathcal{H})$ and $B = Op^W(b)$ are the self-adjoint operators or quantum observables in $L^2(R^n)$, representing the Weyl quantization of the symbols $\mathcal{H}, b$ [2], [3]:

$$(Bu)(x) = \frac{1}{(2\pi\hbar)^n}\int_{R^{2n}} b\left(\frac{x+y}{2}, \xi\right) \cdot e^{i<(x-y),\xi>/\hbar} u(y)\mathrm{d}y\mathrm{d}\xi,$$

where $u \in S(R^n)$ and $B_t = e^{iHt/\hbar} B e^{-iHt/\hbar}$ be the Heisenberg observable or quantum evolution of the observable B under unitary group generated by H. B_t solves the Heisenberg equation of motion

$$\dot{B}_t = \frac{i}{\hbar}[H, B_t].$$

Let $b_t(x,\xi;\hbar)$ is a symbol of B_t then we have the following equation for it

$$\dot{b}_t = \{\mathcal{H}, b_t\}_M, \tag{12}$$

with initial condition $b_0(x,\xi,\hbar) = b(x,\xi)$. Here $\{f,g\}_M(x,\xi)$ is the Moyal brackets of the observables $f, g \in C^\infty(R^{2n})$, $\{f,g\}_M(x,\xi) = f\sharp g - g\sharp f$, where $f\sharp g$ is the symbol of the operator product and is presented by the composition of the symbols f, g

$$(f\sharp g)(x,\xi) = \frac{1}{(2\pi\hbar)^{n/2}}\int_{R^{4n}} e^{-i<r,\rho>/\hbar + i<\omega,\tau>/\hbar} \cdot f(x+\omega, \rho+\xi) \cdot g(x+r, \tau+\xi)\mathrm{d}\rho\mathrm{d}\tau\mathrm{d}r\mathrm{d}\omega.$$

For our problems it is useful that $\{f,g\}_M$ admits the formal expansion in powers of $\hbar$:

$$\{f,g\}_M(x,\xi) \sim \{f,g\} + 2^{-j} \sum_{|\alpha+\beta|=j\geq 1} (-1)^{|\beta|} \cdot (\partial^\alpha_\xi f D^\beta_x g) \cdot (\partial^\beta_\xi g D^\alpha_x f),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D_x = -i\hbar\partial_x$. So, evolution (12) for symbol $b_t(x,\xi;\hbar)$ is

$$\dot{b}_t = \{\mathcal{H}, b_t\} + \frac{1}{2^j} \sum_{|\alpha|+\beta|=j\geq 1} (-1)^{|\beta|} \cdot \hbar^j (\partial^\alpha_\xi \mathcal{H} D^\beta_x b_t) \cdot (\partial^\beta_\xi b_t D^\alpha_x \mathcal{H}). \tag{13}$$

At $\hbar = 0$ this equation transforms to classical Liouville equation

$$\dot{b}_t = \{\mathcal{H}, b_t\}. \tag{14}$$

Equation (13) plays a key role in many quantum (semiclassical) problem. We note only the problem of relation between quantum and classical evolutions or how long the evolution of the quantum observables is determined by the corresponding classical one [3]. Our approach to solution of systems (13), (14) is based on our technique from [11]-[33] and very useful FWT [4] parametrization for general (pseudo)differential operators.

6 Symplectic Hilbert Scales via Wavelets

We can solve many important dynamical problems such that KAM perturbations, spread of energy to higher modes, weak turbulence, growths of solutions of Hamiltonian equations in case if we consider scales of underlying functional spaces instead of one functional space. For Hamiltonian system and their perturbations for which we need take into account underlying symplectic structure we need to consider symplectic scales of spaces. So, if $\dot{u}(t) = J\nabla K(u(t))$ is Hamiltonian equation we need wavelet description of symplectic or quasicomplex structure on the level of functional spaces. It is very important that according to [5] Hilbert basis is in the same time a Darboux basis to corresponding symplectic structure. We need to provide Hilbert scale $\{Z_s\}$ with symplectic structure [5], [7]. All what we need is the following. J is a linear operator, $J: Z_\infty \to Z_\infty$, $J(Z_\infty) = Z_\infty$, where $Z_\infty = \cap Z_s$. J determines an isomorphism of scale $\{Z_s\}$ of order $d_J \geq 0$. The operator J with domain of definition Z_∞ is antisymmetric in Z: $< Jz_1, z_2 >_Z = - < z_1, Jz_2 >_Z, z_1, z_2 \in Z_\infty$. Then the triple

$$\{Z, \{Z_s | s \in R\}, \quad \alpha = < \bar{J}dz, dz >\}$$

is symplectic Hilbert scale. So, we may consider any dynamical Hamiltonian problem on functional level. As an example, for KdV equation we have

$$Z_s = \{u(x) \in H^s(T^1) | \int_0^{2\pi} u(x)\mathrm{d}x = 0\}, \ s \in R, \quad J = \partial/\partial x,$$

J is isomorphism of the scale of order one, $\bar{J} = -(J)^{-1}$ is isomorphism of order -1. According to [8] general functional spaces and scales of spaces such as Holder–Zygmund, Triebel–Lizorkin and Sobolev can be characterized through wavelet coefficients or wavelet transforms. As a rule, the faster the wavelet coefficients decay, the more the analyzed function is regular [8]. Most important for us example is the scale of Sobolev spaces. Let $H_k(R^n)$ is the Hilbert space of all distributions with finite norm

$$\|s\|^2_{H_k(R^n)} = \int \mathrm{d}\xi (1+|\xi|^2)^{k/2} |\hat{s}(\xi)|^2.$$

Let us consider wavelet transform

$$W_g f(b,a) = \int_{R^n} \mathrm{d}x \frac{1}{a^n} \bar{g}\left(\frac{x-b}{a}\right) f(x),$$

$b \in R^n, \quad a > 0$, w.r.t. analyzing wavelet g, which is strictly admissible, i.e.

$$C_{g,g} = \int_0^\infty \frac{\mathrm{d}a}{a} |\hat{\bar{g}}(ak)|^2 < \infty.$$

Then there is a $c \geq 1$ such that

$$c^{-1}\|s\|^2_{H_k(R^n)} \leq \int_{H^n} \frac{\mathrm{d}b\mathrm{d}a}{a}(1+a^{-2\gamma})|W_g s(b,a)|^2 \leq c\|s\|^2_{H_k(R^n)}.$$

This shows that localization of the wavelet coefficients at small scale is linked to local regularity.
So, we need representation for differential operator (J in our case) in localized multiscale wavelet basis. The problem can be soved by the same FWT approach, mentioned above.

7 Maps and Multiresolution

7.1 Veselov-Marsden Discretization

Discrete variational principles lead to evolution dynamics analogous to the Euler-Lagrange equations [9]. Let Q be a configuration space, then a discrete Lagrangian is a map $L : Q \times Q \to \mathbf{R}$. usually L is obtained by approximating the given Lagrangian. For $N \in N_+$ the action sum is the map $S : Q^{N+1} \to \mathbf{R}$ defined by

$$S = \sum_{k=0}^{N-1} L(q_{k+1}, q_k), \tag{15}$$

where $q_k \in Q$, $k \geq 0$. The action sum is the discrete analog of the action integral in continuous case. Extremizing S over $q_1, ..., q_{N-1}$ with fixing q_0, q_N we have the discrete Euler-Lagrange equations (DEL):

$$D_2 L(q_{k+1}, q_k) + D_1(q_k, q_{q-1}) = 0, \tag{16}$$

for $k = 1, ..., N-1$.
Let

$$\Phi : Q \times Q \to Q \times Q \tag{17}$$

and

$$\Phi(q_k, q_{k-1}) = (q_{k+1}, q_k) \tag{18}$$

is a discrete function (map), then we have for DEL:

$$D_2L \circ \Phi + D_1L = 0 \tag{19}$$

or in coordinates q^i on Q we have DEL:

$$\frac{\partial L}{\partial q_k^i} \circ \Phi(q_{k+1}, q_k) + \frac{\partial L}{\partial q_{k+1}^i}(q_{k+1}, q_k) = 0. \tag{20}$$

It is very important that the map Φ exactly preserves the symplectic form ω:

$$\omega = \frac{\partial^2 L}{\partial q_k^i \partial q_{k+1}^j}(q_{k+1}, q_k)\mathrm{d}q_k^i \wedge \mathrm{d}q_{k+1}^j \tag{21}$$

7.2 Generalized Discrete (Data) Multiresolution

Our approach to solutions of equations (20) is based on applications of general and very efficient methods developed by A. Harten [10], who produced a "General Framework" for multiresolution representation of discrete data. It is based on consideration of basic operators, decimation and prediction, which connect adjacent resolution levels. These operators are constructed from two basic blocks: the discretization and reconstruction operators. The former obtains discrete information from a given continuous functions (flows), and the latter produces an approximation to those functions, from discrete values, in the same function space to which the original function belongs. A "new scale" is defined as the information on a given resolution level which cannot be predicted from discrete information at lower levels. If the discretization and reconstruction are local operators, the concept of "new scale" is also local. The scale coefficients are directly related to the prediction errors, and thus to the reconstruction procedure. If scale coefficients are small at a certain location on a given scale, it means that the reconstruction procedure on that scale gives a proper approximation of the original function at that particular location. This approach may be considered as some generalization of standard wavelet analysis approach. It allows to consider multiresolution decomposition when usual approach is impossible or not efficient (δ-functions case, e.g.).
Let F be a linear space of mappings

$$F \subset \{f | f : X \to Y\}, \tag{22}$$

where X, Y are linear spaces. Let also D_k be a linear operator

$$D_k : f \to \{v^k\}, \quad v^k = D_k f, \quad v^k = \{v_i^k\}, \quad v_i^k \in Y. \tag{23}$$

This sequence corresponds to k level discretization of X. Let

$$D_k(F) = V^k = \mathrm{span}\{\eta_i^k\} \tag{24}$$

and the coordinates of $v^k \in V^k$ in this basis are $\hat{v}^k = \{\hat{v}_i^k\}$, $\hat{v}^k \in S^k$:

$$v^k = \sum_i \hat{v}_i^k \eta_i^k, \tag{25}$$

D_k is a discretization operator. Main goal is to design a multiresolution scheme (MR) [10] that applies to all sequences $s \in S^L$, but corresponds for those sequences $\hat{v}^L \in S^L$, which are obtained by the discretization (22).
Since D_k maps F onto V^k then for any $v^k \subset V^k$ there is at least one f in F such that $D_k f = v^k$. Such correspondence from $f \in F$ to $v^k \in V^k$ is reconstruction and the corresponding operator is the reconstruction operator R_k:

$$R_k : V_k \to F, \qquad D_k R_k = I_k, \tag{26}$$

where I_k is the identity operator in V^k (R^k is right inverse of D^k in V^k).
Given a sequence of discretization $\{D_k\}$ and sequence of the corresponding reconstruction operators $\{R_k\}$, we define the operators D_k^{k-1} and P_{k-1}^k

$$\begin{aligned} D_k^{k-1} &= D_{k-1}R_k : V_k \to V_{k-1} \\ P_{k-1}^k &= D_k R_{k-1} : V_{k-1} \to V_k \end{aligned} \tag{27}$$

If the set D_k in nested [10], then

$$D_k^{k-1} P_{k-1}^k = I_{k-1} \tag{28}$$

and we have for any $f \in F$ and any $p \in F$ for which the reconstruction R_{k-1} is exact:

$$\begin{aligned} D_k^{k-1}(D_k f) &= D_{k-1} f \\ P_{k-1}^k(D_{k-1} p) &= D_k p \end{aligned} \tag{29}$$

Let us consider any $v^L \in V^L$, Then there is $f \in F$ such that

$$v^L = D_L f, \tag{30}$$

and it follows from (29) that the process of successive decimation [10]

$$v^{k-1} = D_k^{k-1} v^k, \qquad k = L, ..., 1 \tag{31}$$

yields for all k

$$v^k = D_k f \tag{32}$$

Thus the problem of prediction, which is associated with the corresponding MR scheme, can be stated as a problem of approximation: knowing $D_{k-1} f$, $f \in F$, find a "good approximation" for $D_k f$. It is very important that each space V^L has a multiresolution basis

$$\bar{B}_M = \{\bar{\phi}_i^{0,L}\}_i, \{\{\bar{\psi}_j^{k,L}\}_j\}_{k=1}^L \tag{33}$$

and that any $v^L \in V^L$ can be written as

$$v^L = \sum_i \hat{v}_i^0 \bar{\phi}_i^{0,L} + \sum_{k=1}^{L} \sum_j d_j^k \bar{\psi}_j^{k,L}, \tag{34}$$

where $\{d_j^k\}$ are the k scale coefficients of the associated MR, $\{\hat{v}_i^0\}$ is defined by (25) with $k = 0$. If $\{D_k\}$ is a nested sequence of discretization [10] and $\{R_k\}$ is any corresponding sequence of linear reconstruction operators, then we have from (34) for $v^L = D_L f$ applying R_L:

$$R_L D_L f = \sum_i \hat{f}_i^0 \phi_i^{0,L} + \sum_{k=1}^{L} \sum_j d_j^k \psi_j^{k,L}, \tag{35}$$

where

$$\phi_i^{0,L} = R_L \bar{\phi}_i^{0,L} \in F, \quad \psi_j^{k,L} = R_L \bar{\psi}_j^{k,L} \in F, \quad D_0 f = \sum \hat{f}_i^0 \eta_i^0. \tag{36}$$

When $L \to \infty$ we have sufficient conditions which ensure that the limiting process $L \to \infty$ in (35, 36) yields a multiresolution basis for F. Then, according to (33), (34) we have very useful representation for solutions of equations (20) or for different maps construction in the form which are a counterparts for discrete (difference) cases of our constructions in [11]–[33].

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