A posteriori error estimates for approximate solutions and adaptive algorithms for plane problems of elasticity theory

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Abstract

This work is devoted to functional approach [1],[2],[3] to a posteriori error control in classical [2] and Cosserat elasticity [4]-[5]. The approach yields reliable majorants that are valid for all conforming solutions of problems regardless of methods used for a numerical implementation of a solution process. Estimates include additional auxiliary fields and mesh-independent constants. It is shown that a reasonable and natural choice of conforming finite element approximations in the Hilbert space $H(div)$ for additional variables provides an efficient implementation of the error control. Efficiency of the above technique is shown on one set of numerical examples including consequent mesh adaptations with standard MATLAB tools as in [6].

1 Introduction

Nowadays, the theory of a posteriori error control is well-developed. The amount of the corresponding literature is vast and it is increasing continuously from the end of 1970-s (see, for instance, [3], [7], [8] for a review). Concerning error control for various problems of the elasticity theory, the first "geometrical" method appeared in [9] (much earlier than others, like [10], [11], [12]).
Fundamentals of the functional approach to a posteriori error control, including estimates for various problems of continuum mechanics, have been worked out in detail (see [13], [2], [3] for references). For example, functional a posteriori estimates for linear elasticity were obtained in [14] and [2] using different methodologies. Also in [14] some estimates for plane stress, plane strain and axisymmetric problems are considered.
Functional estimates for problems of Cosserat [15] elasticity have appeared during the last few years. Such media possess a wider range of properties as compared with classical continuous media. A mathematical description of Cosserat media can be found in [16] and [17]. In the last decades, methods for the numerical solution of problems related to the Cosserat continuum began to develop more intensively (see, for example, [18], [19], [20], [21], [22]). On the other hand, there are only few papers

addressed to a posteriori error control for computed approximations – [23], [24], [4], [5], and this work requires further developments related to the evolution of adaptive algorithms.

2 Statement

Majorants for both mathematical models (classical and Cosserat elasticity) have some important features in common. Estimates have the form

$$|||e|||\leq M := D(\tilde{u}, s^*) + R(s^*) + \text{penalty terms}, \qquad e := u - \tilde{u}, \tag{1}$$

where u contains all components of the exact solution, $\tilde{u}$ represents approximations of these components, e is the corresponding error formed by deviations from exact values, s^* is a set of auxiliary variables, and $|||...|||$ denotes the energy norm of the error. Term D represents errors in constitutive relations, R is a residual term including mesh-independent constants. The estimate (1) may contain optional penalty terms that violate the symmetry condition in a weak form. The right-hand side of (1) depends only on the known data – approximate solution, constants, positive parameters, additional variables, and it can be calculated explicitly. This estimate is exact in the sense that the equality is possible to be achieved with a proper setting of parameters and variables. For instance, estimates for problems in classical and Cosserat elasticity have the form (1) – see [2], [4] and [24] for details. All auxiliary fields can be constructed on a basis of finite elements suitable for space $H(div)$ – the Hilbert space of square summable vector-functions with square summable divergence.

A correct choice of one or more free variables in functional-type error estimates (majorants) allows obtaining accurate guaranteed upper error estimates. The functional approach does not impose significant additional restrictions (for example, satisfaction of equilibrium equations) on free variables. Any functional-type error estimate is universal – it is applicable to an arbitrary approximate solution from the corresponding energy space. It remains valid regardless of the approach used for calculating that solution, thus allows to take into account various error sources. Functional estimate (1) includes constants that depend only on domain geometry and not on the mesh for Finite Element Method (FEM). In addition to the global error estimation, the functional majorant can be used as an indicator of the local error distribution, considering the contributions to the global error on each finite element.

Modern adaptive algorithms for finite element methods consist of four main steps: solve, estimate, mark and refine. Concerning (1) the procedure can be specified as follows:

1. **solve** means *compute* $\tilde{u}$ *on a current finite element mesh*;
2. **estimate** means *compute* (1) *from individual loads to elements*;
3. **mark** means *mark elements of a mesh with large local errors by some marking strategy*;
4. **refine** means *divide marked elements and locally refine a mesh.*

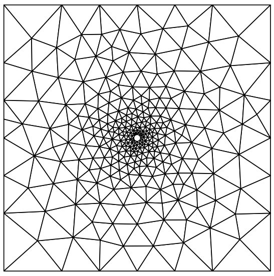
3 Numerical results

One of the efficient ways to compute functional-type a posteriori error estimates for plane problems is to use mixed-FEM approximations. For example, these may be Raviart-Thomas [25] or Arnold-Boffi-Falk [26] approximations. For computation of approximate solutions, the commercial software can be used. Below, we present some recent results as an illustration.

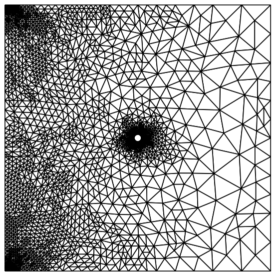
Table 3: Comparison of results for uniform and adaptive mesh refinements, where the lowest-order Raviart-Thomas approximation is used for the implementation of the majorant M from (1)

Uniform refinement (classical elasticity)					
MESH	1	2	3	4	5
NODES	295	1147	4522	17956	71560
ELEMENTS	557	2228	8912	35648	142592
RELATIVE ERROR, %	10.1	6.6	4.2	2.6	1.6
Reference indicator (classical elasticity)					
MESH	1	2	3	7	20
NODES	295	353	423	765	2050
ELEMENTS	557	664	793	1428	3906
RELATIVE ERROR, %	10.1	6.9	4.9	2.6	1.6
Majorant-based indicator (classical elasticity)					
MESH	1	2	4	5	7
NODES	295	323	536	876	2955
ELEMENTS	557	606	1002	1648	5693
RELATIVE ERROR, %	10.1	7.1	3.7	2.7	1.4
$I_{eff} = M/\|\|\|e\|\|\|$	1.2	1.2	1.3	1.3	1.2
Reference indicator (Cosserat elasticity)					
MESH	1	2	3	7	22
NODES	295	348	410	720	2114
ELEMENTS	557	652	764	1334	3996
RELATIVE ERROR, %	12.0	9.8	8.0	5.1	3.0

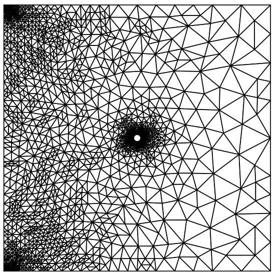
Example (square domain with a hole). Let us consider one canonical example (see figure 1), where the results of adaptations and error estimation are compared for classical elasticity and Cosserat elasticity. Geometry and material properties for this example are taken from [19]: square side is 16.2 mm, hole radius is 0.216 mm, traditional elastic constants are λ= 0.11538e10 N/m^2 and μ= 0.76923e9 N/m^2, additional parameters of a microstructure are B= 0.31762e2 N and μ_c= 0.25638e11 N/m^2, size of particles is 0.2 mm (we note that the radius of the hole is close to the size of particles of a microstructure). The left edge of the square is clamped and a tensile loading is equal to 1 MPa (applied to the opposite edge).
Results collected in table 3 are mostly devoted to the classical elasticity. We start from the case where no adaptation is applied (uniform refinements). This part of



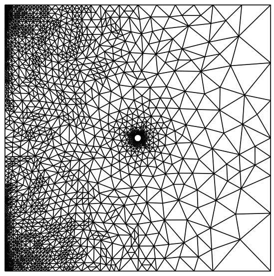
(a) 295 nodes



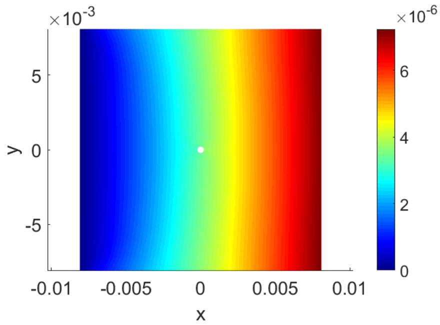
(b) 2955 nodes (majorant)



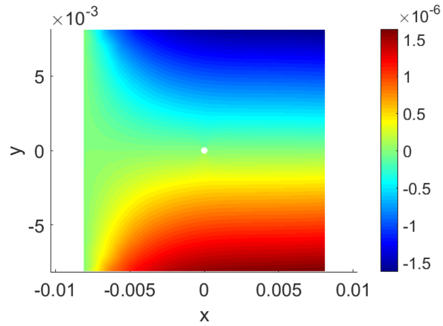
(c) 2050 nodes (reference 1)



(d) 2114 nodes (reference 2)



(e) u_x



(f) u_y

Figure 1: (a) – initial mesh, (b) – result of adaptation by majorant-based indicator, (c) – result of adaptation by the reference indicator for classical elasticity, (d) – result of adaptation by the reference indicator for Cosserat elasticity, (e) and (f) – components of the solution u for classical elasticity (displacements)

the table shows how the solution process proceeds without any a posteriori error estimation. Values collected in lines of the table 3 are as follows: mesh numbers and the corresponding amounts of nodes and elements, and the relative error computed with the so-called *reference solution* – an approximate solution obtained on a fine mesh. Note that the reference solution is required only for numerical experiments on validation and comparison of different approaches. In engineering practise it is too time-consuming to compute it. For example, the reference mesh for the mesh 5 of the uniform refinement consists of 1141792 nodes and 2281472 elements. The reference solution is also used for the construction of the reference indicator based on the energy norm of the difference between solutions on coarse and fine meshes. It is necessary for getting a final mesh (target) for comparisons, but this approach yields a large amount of steps of consequent mesh adaptations to achieve a good result.

The third block of the results is devoted to error estimation by the functional-type error majorant from [2]. The ratio between the error majorant M and the error $|||e|||$ is used as the main quality measure. This parameter is usually called *the efficiency index* – it is denoted by I_{eff}. In the last part of the table, results for similar reference indicator for the Cosserat elasticity are collected. Figure 1 includes following subplots: the initial mesh (a), the mesh 7 for the majorant-based indicator (b), the mesh 20 for the reference indicator for classical elasticity (c), the mesh 22 for the reference indicator for Cosserat elasticity (d), and the classical solution (e-f).

Results show that the functional approach provides reliable guaranteed upper bounds of the energy norm of the error with stable efficiency. Number of nodes required to reach the 98%-level of the accuracy with uniform refinements is 24-times larger than for the adaptive algorithm with the majorant M from (1). Thus, such approach saves a lot of computational resources to get an approximate solution of a good quality. After comparison of (b), (c) and (d), we make the conclusion that, for considered parameters, geometry and loading, both reference indicators and the majorant come to similar adaptive meshing.

In addition, table 4 illustrates the behavior of error estimation for several steps with uniform mesh refinements for the simplest Arnold-Boffi-Falk approximation. From these results for Cosserat elasticity we conclude that the efficiency index of estimates remains stable and overestimation of the true error is moderate and acceptable.

Table 4: Results for the lowest order Arnold-Boffi-Falk approximation for nested meshes [5]

MESH	1	2	3	4
D.O.F.[1]	504	1872	7200	28224
RELATIVE ERROR, %	15.8	11.1	7.3	4.0
I_{eff}	1.2	1.2	1.2	1.3

[1] number of degrees of freedom (111744 for the reference mesh)

Conclusions

The main conclusions are:

- The functional approach is reliable and estimates are guaranteed upper bounds of errors. This property is known from the theory and it is confirmed practically in the process of execution of adaptive algorithms.
- For the considered classes of problems, $H(div)$-conforming approximations as Raviart-Thomas or Arnold-Boffi-Falk yield good results from the viewpoint of a stability of the efficiency index and a moderate overestimation of the true error.

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