The analysis of stress-strain state of a composite plane with interface crack for John's harmonic material

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Abstract

The analytical solutions of nonlinear problems for bi-material plane with an interface crack are obtained. The plane is made by joining of two half-planes from different materials. The plane is subjected to constant nominal (Piola) stresses at infinity. On the crack an external loading is applied. Mechanical properties of half-planes are described by the model of John's harmonic material. This model has allowed to use the methods of complex functions at solution of nonlinear plane-strain problems. The stresses and displacements are expressed through two analytic functions of a complex variable, defined from nonlinear boundary equations on an interface crack. Two problems are solved: a plane with a free interface crack and interface crack loaded uniform pressure. In the second problem the boundary conditions on a crack are depend from the deformation of its coasts. The exact analytical formulas for stresses and displacements are found. Using a global solutions the asymptotic expansions have been constructed for the stresses and displacements in vicinity of a crack tip.

In nonlinear problem of uniaxial extension of a plane with free crack it is established, that the formulas given the crack opening and the stress intensity factors (SIF) near the crack tips completely coincide with the similar formulas derived from the equations of a linear elasticity. The nominal stresses have the root singularity at the tips of a crack; the Cauchy stresses have no singularity.

It was found out, that in the problem of the crack under action of uniform pressure some critical pressures proportional to the shear module are exist and their excess leads to loss of a material stability and large stresses and strains.

1 General relations

In cartesian coordinates (x_1, x_2) the equations of equilibrium and compatibility of plane-strain problems in complex form are [1]

$$(s_{11} + is_{12})'_1 + i(s_{22} - is_{21})'_2 = 0, \tag{1}$$

$$(g_{22} - ig_{12})'_1 + i(g_{11} + ig_{21})'_2 = 0, \tag{2}$$

where s_{ij}, g_{ij} are components of nominal (Piola) stress tensor $\mathbf{S} = s_{\alpha\beta}\,\mathbf{e}_\alpha\mathbf{e}_\beta$ and deformation gradient $\mathbf{G} = g_{\alpha\beta}\,\mathbf{e}_\alpha\mathbf{e}_\beta$. The complex variables of initial and current configurations are $z = x_1 + ix_2$, $\zeta = \xi_1 + i\xi_2$ and the function of nominal stresses is σ. The equations (1), (2) are satisfied identically by substituting of expressions

$$s_{11} + is_{12} = \frac{\partial\sigma}{\partial z} - \frac{\partial\sigma}{\partial\overline{z}}, \quad s_{22} - is_{21} = \frac{\partial\sigma}{\partial z} + \frac{\partial\sigma}{\partial\overline{z}}, \tag{3}$$

$$g_{11} + ig_{21} = \frac{\partial\zeta}{\partial z} + \frac{\partial\zeta}{\partial\overline{z}}, \quad g_{22} - ig_{12} = \frac{\partial\zeta}{\partial z} - \frac{\partial\zeta}{\partial\overline{z}}. \tag{4}$$

Complex functions ζ and σ are defined from the law of elasticity and boundary conditions of the problem.
John's elastic potential (strain energy density) is considered [2]

$$\Phi = 2\mu[F(I) - J], \quad I = \lambda_1 + \lambda_2, \quad J = \lambda_1\lambda_2,$$

$$8\mu bF'(I) = I + \sqrt{I^2 - 16bc},$$

where λ_1, λ_2 are principal stretches; the factors b, c are defined by transition to Hookean law under small deformations: $4\mu b = 1 + \mu/(\lambda + 2\mu), c = 2\mu(1 - 2\mu b)$.
The law of elasticity for plane problem we shall write in complex form [1, 3]

$$\begin{aligned} s_{11} + is_{12} &= 2\mu\left[\frac{2}{I}F'(I)\frac{\partial\zeta}{\partial z} - \frac{\partial\zeta}{\partial z} + \frac{\partial\zeta}{\partial\overline{z}}\right], \\ s_{22} - is_{21} &= 2\mu\left[\frac{2}{I}F'(I)\frac{\partial\zeta}{\partial z} - \frac{\partial\zeta}{\partial z} - \frac{\partial\zeta}{\partial\overline{z}}\right]. \end{aligned} \tag{5}$$

Substituting (3), (4) into (5), we obtain the equations for functions $\sigma(z, \overline{z})$ and $\zeta(z, \overline{z})$

$$\begin{aligned} \frac{\partial\sigma}{\partial z} + 2\mu\frac{\partial\zeta}{\partial z} &= 4\mu\frac{1}{I}F'(I)\frac{\partial\zeta}{\partial z}, \\ \frac{\partial\sigma}{\partial\overline{z}} + 2\mu\frac{\partial\zeta}{\partial\overline{z}} &= 0. \end{aligned} \tag{6}$$

The solution of equations (6) is given by [4, 3]

$$\begin{aligned} \zeta &= b\varphi(z) + \overline{\psi(z)} + \frac{cz}{\overline{\varphi'(z)}}, \\ \sigma &= (1 - 2\mu b)\varphi(z) - 2\mu\overline{\psi(z)} - 2\mu\frac{cz}{\overline{\varphi'(z)}}, \end{aligned} \tag{7}$$

where $\varphi(z), \psi(z)$ are analytic functions of z. From (7) it follows $\sigma + 2\mu\zeta = \varphi(z)$.
For to simplify the boundary equations for functions $\varphi(z), \psi(z)$ we shall introduce an auxiliary function

$$\Omega(z) = \frac{c}{\overline{\varphi'}(z)} + \overline{\psi'}(z) - c\frac{\overline{\varphi''}(z)}{\overline{\varphi'^2}(z)}.$$

The formulas for nominal stresses becomes:

$$s_{11} + is_{12} = (1 - 2\mu b)\varphi'(z) - 2\mu \left(\frac{2c}{\overline{\varphi'(z)}} - \Omega(\overline{z}) + c\frac{(z-\overline{z})\overline{\varphi''(z)}}{\overline{\varphi'^2(z)}} \right),$$
$$s_{22} - is_{21} = (1 - 2\mu b)\varphi'(z) - 2\mu \left(\Omega(\overline{z}) - c\frac{(z-\overline{z})\overline{\varphi''(z)}}{\overline{\varphi'^2(z)}} \right), \quad (8)$$

In addition, the stresses and strains satisfy to the equalities

$$(s_{11} + is_{12}) + 2\mu(g_{22} - ig_{12}) = \varphi'(z),$$
$$(s_{22} - is_{21}) + 2\mu(g_{11} + ig_{21}) = \varphi'(z).$$

Consider the Cauchy's stress tensor $\mathbf{T} = t_{\alpha\beta}\,\mathbf{e}_\alpha\mathbf{e}_\beta$, from formula $\mathbf{S} = \mathbf{G}^{-1} \cdot J\mathbf{T}$ we obtain

$$\varkappa_1(t_{11} + it_{12}) = s_{11} + is_{12},$$
$$\varkappa_2(t_{22} - it_{21}) = s_{22} - is_{21}, \quad (9)$$

where $\varkappa_k = |\mathbf{e}_k \cdot J\mathbf{G}^{-1}|$ are the multiples of the areal change, $J = \det \mathbf{G}$.

2 The problem of the interface crack

A bi-material plane with an interface crack is considered. The crack is located in the interval $[-a, a]$ of interface line (Fig. 1). Nominal stresses are set at infinity $s_{ij} \to s_{ij}^\infty$ (for each half-plane). The surfaces of the crack are free from stresses

$$(s_{22} - is_{21})^+ = 0, \quad (s_{22} - is_{21})^- = 0, \quad |x_1| < a. \quad (10)$$

The stresses (8) substitute in equations (10)

$$[(1 - 2\mu_2 b_2){\varphi'}_2(z) - 2\mu_2\Omega_2(\overline{z})]^+ = 0,$$
$$[(1 - 2\mu_1 b_1){\varphi'}_1(z) - 2\mu_1\Omega_1(\overline{z})]^+ = 0. \quad (11)$$

Here, we introduce the functions $h(z)$ and $r(z)$ which are analytic in all plane, excluding the interface. It allows to simplify the statement of the boundary problems and their solution. In upper half-plane S_2

$$h(z) = (1 - 2\mu_2 b_2){\varphi'}_2(z) + 2\mu_1\Omega_1(z),$$
$$r(z) = b_2{\varphi'}_2(z) - \Omega_1(z). \quad (12)$$

Complex potentials we shall express through functions (12)

$$\varphi'_2(z) = \frac{(h + 2\mu_1 r)(z)}{1 + 2(\mu_1 - \mu_2)b_2}, \quad (13)$$

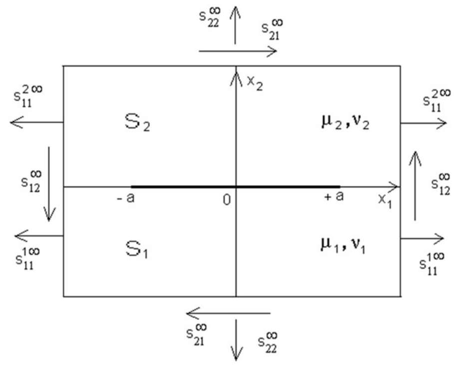


Figure 1: Bi-material plane with interface crack.

$$\Omega_1(z) = \frac{[b_2 h - (1 - 2\mu_2 b_2) r](z)}{1 + 2(\mu_1 - \mu_2) b_2}.$$

Formulas for the lower half-plane S_1 we obtain by cyclic changing of the indexes. The boundary conditions (11) in functions (12) take the form

$$h^+(x_1) - h^-(x_1) = 0, \quad r^+(x_1) + \delta r^-(x_1) = 0, \tag{14}$$

$$\delta = \frac{\mu_2(1 - 2\mu_1 b_1)}{\mu_1(1 - 2\mu_2 b_2)} \cdot \frac{1 + 2(\mu_1 - \mu_2) b_2}{1 + 2(\mu_2 - \mu_1) b_1}.$$

The solutions of equations (14) are

$$h(z) = h(\infty), \quad (r - Dh)(z) = AX(z)(z - 2ia\beta),$$

where

$$X(z) = \frac{1}{\sqrt{z^2 - a^2}} \left(\frac{z + a}{z - a} \right)^{i\beta}, \quad \beta = \frac{\ln \delta}{2\pi},$$

$$A = \frac{1}{2} \frac{(1 + 2b_2(\mu_1 - \mu_2))(1 + 2b_1(\mu_2 - \mu_1))}{\mu_1(1 - 2\mu_1 b_1) + \mu_2(1 - 2\mu_2 b_2)} (s_{22}^\infty - i s_{21}^\infty),$$

$$D = -\frac{1}{2} \frac{1 - 2\mu_1 b_1 - 2\mu_2 b_2}{\mu_1(1 - 2\mu_1 b_1) + \mu_2(1 - 2\mu_2 b_2)}.$$

The SIF of nominal stresses in vicinities of the crack ends are calculated under the formulas similar to that are used in linear elasticity [5]

$$K^\pm = \sqrt{2\pi} \lim_{r \to \pm 1 \pm 0} \left[(\pm\tau - 1)^{0.5 \pm i\beta} (s_{22} - i s_{21})(\tau) \right] = \tag{15}$$

$$= \pm\sqrt{\pi}(1 \mp 2i\beta) 2^{i\beta} (s_{22}^\infty - i s_{21}^\infty),$$

where $\tau = x_1/a$ is dimensionless variable on an interface. The same SIF are obtained in the linear problem of interface crack [5].
The displacements of the crack surfaces we shall find under the formula

$$g_{11} + i g_{21} = 1 + u_1' + i u_2' = \frac{1}{2\mu} \varphi'(t),$$

where u_1 and u_2 are the components of displacements. The disclosing of a crack for uniaxial extension of plane by the stresses s_{22}^{∞} is given by

$$\Delta u_2(z) = \frac{s_{22}^{\infty}}{2\mu_1\mu_2}\sqrt{(\mu_1+3\mu_2-4\nu_1\mu_2)(\mu_2+3\mu_1-4\nu_2\mu_1)}\sqrt{a^2-z^2}\left(\frac{a+z}{a-z}\right)^{i\beta}.$$

The similar formula is obtained in a linear problem [5].
Let's assume $z = a + re^{i\theta}$ and construct the asymptotic expansions of nominal stresses in a vicinity of a right tip of crack at $r \to 0$

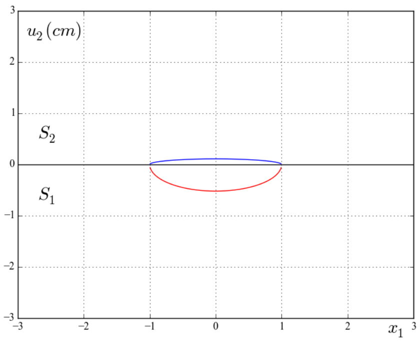
$$s_{11} + is_{12} = A_1 + B_1(re^{-i\theta})^{-i\beta-0.5} + O(\sqrt{r}),$$

$$s_{22} - is_{21} = A_2 + B_2(re^{-i\theta})^{-i\beta-0.5} + O(\sqrt{r}),$$

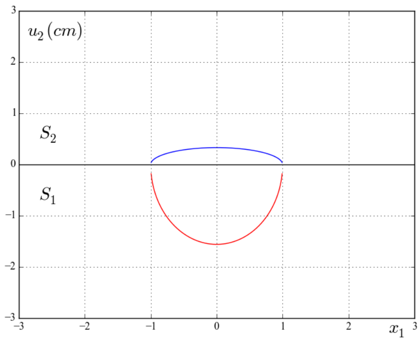
where A, B are const. The nominal stresses have singularity $1/\sqrt{r}$.
The multiples of the areal change $\varkappa_1$ and $\varkappa_2$ have a singularity $1/\sqrt{r}$ at $r \to 0$, hence the Cauchy stresses (9) have no singularities on the ends of the crack

$$t_{11} + it_{12} = C_1 + D_1\sqrt{r} + O(r), \quad t_{22} - it_{21} = C_2 + D_2\sqrt{r} + O(r).$$

The calculations of displacements of crack surfaces are performed. The material parameters are considered: $\mu_1 = 1$ MPa, $\mu_2 = 5$ MPa, $\nu_1 = 0.48$, $\nu_2 = 0.45$. One-axis stretching along x_2-axis is taken as external loading: $s_{22}^{\infty} = 0.1$ MPa (a) and $s_{22}^{\infty} = 0.3$ MPa (b). The results of calculations are presented on Fig. 2.



(a)



(b)

Figure 2: The displacements of crack surfaces.

3 Interface crack loaded by pressure

The nonlinear plane-strain problem of a bi-material plane with an interface crack loading pressure is examined. Feature of a problem is dependence of boundary conditions on deformation of coasts. It was found out, that there are some critical pressures proportional to the shear module which excess conducts to the lost of stability material. The boundary conditions on a crack are reduced to [4]

$$s_{22} - is_{21} = -p\,(g_{11} + ig_{21}), \tag{16}$$

where $p > 0$ is value of pressure. Using formulas (8), (16) we obtain

$$s_{22} - is_{21} = -\frac{p}{2\mu - p}\varphi'(z), \quad g_{11} + ig_{21} = \frac{p}{2\mu - p}\varphi'(z). \tag{17}$$

It is visible in (17), that stresses and strains tends to ∞ when $p \to 2\mu$. Function $\varphi'(z) \neq 0$ if $p = 2\mu$. The boundary conditions for nominal stresses on the coats of crack are

$$[s_{22} - is_{21}]^+ = -\frac{p}{2\mu_2 - p}\varphi'_2(z), \quad [s_{22} - is_{21}]^- = \frac{p}{2\mu_1 - p}\varphi'_1(z). \tag{18}$$

The sum and the difference of the equations (18), written through functions h and r, gives us the equations

$$[h - pr]^+(t) - [h - pr]^- = 0,$$

$$r^+(t) + \delta(p)\, r^-(t) = f(p), \quad t \in (-a, a), \tag{19}$$

$$\delta(p) = \frac{2\mu_2 - p}{2\mu_1 - p}\,\frac{1 + 2(\mu_1 - \mu_2)b_2}{1 + 2(\mu_2 - \mu_1)b_1}\,\frac{1 - (2\mu_1 - p)b_1}{1 - (2\mu_2 - p)b_2},$$

$$f(p) = -\frac{1 - (2\mu_1 - p)b_1 - (2\mu_2 - p)b_2}{(2\mu_1 - p)[1 + 2(\mu_2 - \mu_1)b_1][1 - (2\mu_2 - p)b_2]}(h + pr)(\infty).$$

The factor $\delta(p)$ in equation (19) changes a sign depending on value of pressure p. If $\mu_1 \leq \mu_2$, then at $p < 2\mu_1$ and $p > 2\mu_2$ a factor δ will be positive, and at $2\mu_1 < p < 2\mu_2$ — negative. The form of the solution of the equation (19) depends on a sign on this parameter. Further we shall consider separately three cases. Cases $\delta = \infty$ and $\delta = 0$ to which correspond critical pressure $p_1 = 2\mu_1$ and $p_2 = 2\mu_2$, accordingly, we exclude from consideration.

At $p < 2\mu_1$ and $p > 2\mu_2$ we have $\delta > 0$, in this case the solution to equation (19) holomorphic at infinity is

$$r(z) = r(\infty) + B\,[1 - (z - 2i\beta a)\, X(z)], \tag{20}$$

$$B = -\frac{[1 + 2(\mu_2 - \mu_1)b_1 + 2(\mu_1 - \mu_2)b_2]\, p}{(2\mu_1 - p)[1 - (2\mu_1 - p)b_1] + (2\mu_2 - p)[1 - (2\mu_2 - p)b_2]}.$$

At segment $2\mu_1 < p < 2\mu_2$ factor $\delta < 0$. In this case the solution of equation (19) is given by the formula

$$r(z) = r(\infty) + B\,[1 - X_*(z)]. \tag{21}$$

$$X_*(z) = \left(\frac{z - a}{z + a}\right)^{i\beta}, \quad \beta = \frac{\ln|\,\delta|}{2\pi}.$$

The solution (21) remains limited near to the tips of a crack and anywhere does not zero. In the formula (21) it is supposed, that $\delta \neq -1$, the case $\delta = -1$ has been exclude. Value of parameter $\delta = -1$ is special, the solution (20) tends to infinity, corresponding critical pressure p is a root of the equation $1 + \delta = 0$.

Thus, during research and solution of the equation (19) are revealed three special values of parameter δ, namely, $\delta = \infty$, $\delta = 0$ and $\delta = -1$ to which there correspond critical values of pressure: $p_1 = 2\mu_1$, $p_2 = 2\mu_2$ and $p_* \in (2\mu_1, 2\mu_2)$. The analysis shows, that at approach of pressure to these critical values the maximal displacements of coasts of a crack, and also stress intensity factors tends to infinity. The SIF for the right and left end of a crack we shall define under formulas (15). Let's consider a case when pressure on a crack satisfies to conditions $p < 2\mu_1$ or $p > 2\mu_2$. In the equation (19) parameter $\delta > 0$ and its solution is (20). The SIF are

$$K^+ = -A\sqrt{\pi}(1 - 2i\beta)2^{i\beta}, \quad K^- = +A\sqrt{\pi}(1 + 2i\beta)2^{-i\beta},$$

where

$$A = B\,\frac{(h + pr)(\infty)}{[1 + 2(\mu_1 - \mu_2)b_2][1 + 2(\mu_2 - \mu_1)b_1]}.$$

For negative values of parameter δ in the equation (19), when pressure varies within the limits of $2\mu_1 < p < 2\mu_2$, the SIF are calculated under formulas

$$K^+ = -A\sqrt{2\pi}\,2^{-i\beta}, \quad K^- = -A\sqrt{2\pi}\,2^{i\beta}.$$

The displacements of the coasts of crack it is convenient to define by the second formula (17)

$$g_{11} + ig_{21} = 1 + u_1' + iu_2' = \frac{1}{2\mu - p}\,\varphi'(t),$$

after replacement of complex functions with expressions (14). The displacements on normal to a crack are represented greatest interest

$$u_2^+ = -\frac{a\sqrt{\delta}(2\mu_1 - p)\sqrt{1 - \xi^2}}{(2\mu_2 - p)[1 + 2(\mu_1 - \mu_2)b_2]} B\cos\left[\beta\ln\frac{1 + \xi}{1 - \xi}\right], \quad \xi \in (-1, 1),$$

$$u_2^- = \frac{a(2\mu_2 - p)\sqrt{1 - \xi^2}}{\sqrt{\delta}(2\mu_1 - p)[1 + 2(\mu_2 - \mu_1)b_1]} B\cos\left[\beta\ln\frac{1 + \xi}{1 - \xi}\right], \quad \xi \in (-1, 1).$$

4 Conclusion

For model of John's harmonic material the problem of interface crack in bi-material plane is solved. The cases of a free crack and a crack loaded by uniform pressure are studied. The exact analytical formulas are found for the nominal (Piola) stresses, Cauchy stresses and the displacements. On the base of the common solution the asymptotic expansions of the listed functions are constructed in a vicinity of the ends of a crack. The nominal stresses have a root singularity, Cauchy stresses have no singularity. The stresses and displacements have an oscillation in the vicinity of the tips of the crack. The stress intensity factors for nominal stresses are received. Unlike a linear problem, where SIF have real physical sense, (the speed of liberated energy of deformation at development of a crack), here SIF are entered formally. The question about SIF in nonlinear problems is not studied and requires special research. The formulas of disclosing crack (jumps of displacements) are obtained. It is interesting, that formulas for SIF and disclosing of a free crack completely coincide with the results of similar linear problem.

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