Elastic microslip in belt drive: influence of bending, shear and extension

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Abstract

We develop a model of elastic microslip with account for deformations of belt as a Cosserat rod. The general nonlinear problem with friction is divided into two stages: fitting the belt on the pulleys and then deforming the belt by the given pulley rotations. At the second stage we assume that friction forces act in the contact areas providing the displacements of the belt points equal to the displacement of pulley points. This problem is solved for small displacements and rotations using the superposition principle.

1 Introduction

The first study of the belt mechanics from the point of view of shear model was reported in [1]. An independent research of shear microslip concerning general friction modelling can be found in [2]. A combination of the belt shear and extension was presented in [3, 4], however the bending stiffness was neglected there.
In the present paper we address the mechanics of the belt drive taking into account the effect of elastic microslip. We formulate and solve the quasi-static nonlinear problem of the belt deformation and contact interaction of the belt with two equal non-smooth pulleys. The goal of the present study is in particular the analysis of influence of belt deformation and elastic microslip on the transmission ratio depending on the applied load.
The belt is modelled as elastic rod which initial configuration is a circle. In the geometrically nonlinear formulation we take into account bending, transverse shear and extension, and also friction on pulleys. The problem is solved in two stages.
At the first stage we model the fitting of the belt on the pulleys, determine the stress-strain state of the belt and calculate the contact pressure. We overcome difficulties of the nonlinear contact problem using computer mathematics. The arising boundary value problems are solved numerically by the shooting method and by the finite difference method.
At the second stage we consider the problem with prescribed rotations of pulleys and applied torques. We use the equations in variations superposed upon the stress state calculated at the first stage. We derive and solve the linear ODEs which variable

coefficients are determined at the first stage. The state of the belt on pulleys is described by the second-order ODEs, their solution allows determining the contact pressure and friction forces. For the free spans of belt we formulate and solve the sixth-order problems. As a result a general 16th-order system is combined and solved. We apply computer mathematics here as well.

2 Equations in variations

Before the varying we have the nonlinear static problem with tension and shear [5, 6]. (The simpler versions are possible, without shear or without tension, see [7].) We transform the obtained expressions for variables and constants using the coordinate $s \in [-s_1, s_4]$; $l = s_4 + s_1$ is the belt length (see Fig. 1). This coordinate is the arc coordinate in the reference undeformed configuration. The belt form in the reference configuration is circle [5, 6, 7]. The simplified model of elastic microslip in belt drive is presented in [8].
The equations in variations [9] read:

$$\begin{aligned} &\tilde{\mathbf{Q}}' = -\tilde{\mathbf{q}}, \;\; \tilde{\mathbf{M}}' + \mathbf{u}' \times \mathbf{Q} + \mathbf{r} \times \tilde{\mathbf{Q}} = -\tilde{\mathbf{m}}, \\ &\tilde{\mathbf{M}} = \mathbf{A}^{-1} \cdot \boldsymbol{\theta}' + \boldsymbol{\theta} \times \mathbf{M}, \;\; \tilde{\mathbf{Q}} = \mathbf{B}^{-1} \cdot \boldsymbol{\gamma} + \boldsymbol{\theta} \times \mathbf{Q}, \;\; \boldsymbol{\gamma} \equiv \mathbf{u}' - \boldsymbol{\theta} \times \mathbf{r}'. \end{aligned} \tag{1}$$

Here we denote: $(...)'$ is the derivative with respect to coordinate s; $(\tilde{...})$ is the variation of a value; $\mathbf{r}$, $\mathbf{Q}$, $\mathbf{M}$ are the position vector of rod particles, force and moment in the rod before varying, respectively; $\mathbf{A}$, $\mathbf{B}$ are the compliance tensors of the rod (we take physically linear model); the vectors $\mathbf{u} \equiv \tilde{\mathbf{r}}$, $\boldsymbol{\theta}$ describe small displacements and rotations, respectively; and the vectors $\tilde{\mathbf{Q}}$, $\tilde{\mathbf{M}}$ are the force and moment variations, respectively. $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{m}}$ are the variations of external load and moment distributed per unit length, respectively.
In the present paper we restrict ourselves to the plane problem. Therefore the vectors $\mathbf{r}$, $\mathbf{Q}$, $\mathbf{u}$, $\tilde{\mathbf{Q}}$, $\tilde{\mathbf{q}}$ lie in the drawing plane xy, and the vectors $\mathbf{M}$, $\tilde{\mathbf{M}}$, $\boldsymbol{\theta}$, $\tilde{\mathbf{m}}$ have just one component (for example, $\mathbf{M} = M\mathbf{k}$) directed along the z-axis. We take the following expressions for belt compliance tensors: the tensor $\mathbf{A} = A\mathbf{kk}$ determine the bending compliance, the tensor $\mathbf{B} = B_1\mathbf{e}_1\mathbf{e}_1 + B_2\mathbf{e}_2\mathbf{e}_2$ determine the tension and shear compliance. The unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ are rotated with respect to Cartesian axes x, y by an angle φ. The rules of differentiation are $\mathbf{e}_1' = \varphi'\mathbf{e}_2$, $\mathbf{e}_2' = -\varphi'\mathbf{e}_1$.

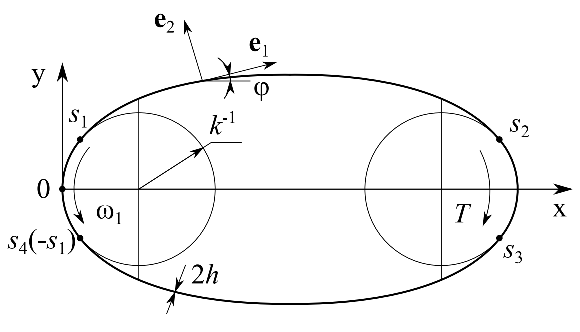

Figure 1: Scheme of belt drive

The term $\boldsymbol{\theta} \times \mathbf{M}$ vanishes for the plane deformation. We simplify and get the following system from (1):

$$\tilde{\mathbf{Q}}' = -\tilde{\mathbf{q}}, \quad \tilde{M}' + \mathbf{k} \cdot \left(\mathbf{u} \times \mathbf{Q} + \mathbf{r}' \times \tilde{\mathbf{Q}} \right) = -\tilde{m},$$
$$\tilde{M} = A^{-1}\theta', \quad \tilde{\mathbf{Q}} = \mathbf{B}^{-1} \cdot \boldsymbol{\gamma} + \theta \mathbf{k} \times \mathbf{Q}, \quad \boldsymbol{\gamma} = \mathbf{u}' - D\theta\mathbf{n},$$
$$D \equiv |\mathbf{r}'| = \sqrt{(1 + B_1 Q_1)^2 + (B_2 Q_2)^2} = \sigma', \quad \mathbf{n} = \mathbf{k} \times \mathbf{t}. \tag{2}$$

We shall write the systems in components specifically for the free segment and contact segment using different bases.

3 Contact segments

In the contact segment we decompose the vectors by the tangent and normal unit vectors. With the rotation of the pulley of radius k^{-1} by an angle ω_1 we have:

$$\mathbf{u} = u_t \mathbf{t} + u_n \mathbf{n} = -(k^{-1}\omega_1 + h\theta)\mathbf{t},$$
$$\mathbf{t} = D^{-1}\mathbf{r}' = t_1 \mathbf{e}_1 + t_2 \mathbf{e}_2; \quad t_1 = D^{-1}(1 + B_1 Q_1), \quad t_2 = D^{-1} B_2 Q_2. \tag{3}$$

The problem is linear, therefore we can take $\omega_1 = 1$. The formulae of differentiation are

$$\mathbf{t}'(s) = D\dot{\mathbf{t}}(\sigma) = -Dk\mathbf{n}, \quad \mathbf{n}'(s) = Dk\mathbf{t},$$
$$\mathbf{u}' = (u_t' + Dku_n)\mathbf{t} + (u_n' - Dku_t)\mathbf{n} = -h\theta'\mathbf{t} + D(\omega_1 + kh\theta)\mathbf{n}. \tag{4}$$

Now we rewrite the system (2) for the plane deformation in components. The balance equations take the form:

$$\tilde{Q}_t' + Dk\tilde{Q}_n = -\tilde{q}_t, \quad \tilde{Q}_n' - Dk\tilde{Q}_t = -\tilde{q}_n,$$
$$\tilde{M}' + (u_t' + Dku_n)Q_n - (u_n' - Dku_t)Q_t + D\tilde{Q}_n = -\tilde{m}. \tag{5}$$

In the elasticity relations we express the stiffness tensor in the following form:

$$\mathbf{B}^{-1} = B_1^{-1}\mathbf{e}_1\mathbf{e}_1 + B_2^{-1}\mathbf{e}_2\mathbf{e}_2 = \mathbf{b} = b_t\mathbf{tt} + b_n\mathbf{nn} + b_{tn}(\mathbf{tn} + \mathbf{nt}),$$
$$b_t = B_1^{-1}t_1^2 + B_2^{-1}t_2^2, \quad b_n = B_1^{-1}t_2^2 + B_2^{-1}t_1^2, \quad b_{tn} = t_2 t_1(-B_1^{-1} + B_2^{-1}). \tag{6}$$

We take into account the fact that $n_1 = -t_2$, $n_2 = t_1$. For the displacements, strains and forces we have

$$\mathbf{u} = -(k^{-1}\omega_1 + h\theta)\mathbf{t}, \quad \boldsymbol{\gamma} = -h\theta'\mathbf{t} + D\mathbf{n}(\omega_1 - (1 - kh)\theta),$$
$$\tilde{\mathbf{Q}} = \mathbf{b} \cdot \boldsymbol{\gamma} + \theta(Q_t\mathbf{n} - Q_n\mathbf{t}),$$
$$\tilde{Q}_t = b_t\gamma_t + b_{tn}\gamma_n - \theta Q_n = -b_t h\theta' + b_{tn}D\omega_1 - (b_{tn}D(1 - kh) + Q_n)\theta,$$
$$\tilde{Q}_n = b_{tn}\gamma_t + b_n\gamma_n + \theta Q_t =$$
$$= -b_{tn}h\theta' + b_n D\omega_1 - (b_n D(1 - kh) - Q_t)\theta. \tag{7}$$

We write the relation between the force and moment loads in the form:

$$\tilde{m} = h\tilde{q}_t \Rightarrow \tilde{M}' - h\theta' Q_n - D(\omega_1 + kh\theta)Q_t + D\tilde{Q}_n = h\left(\tilde{Q}_t' + Dk\tilde{Q}_n\right). \tag{8}$$

Then we substitute the variations of the force factors and derive the ODE for $\theta(s)$. Let us rewrite this equation as follows:

$$\begin{aligned}
&L[\theta] \equiv c_0\theta'' + c_1\theta' + c_2\theta = c_\omega\omega_1,\\
&c_0 \equiv A^{-1} + h^2 b_t, \quad c_1 \equiv h^2 b_t',\\
&c_2 \equiv -b_n D^2(1-kh)^2 + DQ_t(1-2kh) + h(1-kh)(b_{tn}D)' + hQ_n',\\
&c_\omega \equiv DQ_t - b_n D^2(1-kh) + h(b_{tn}D)'.
\end{aligned} \tag{9}$$

Here we have the second-order ODE with variable coefficients in the operator L. The solution is determined by the values at the ends $\theta(-s_1) \equiv \theta_4$, $\theta(s_1) \equiv \theta_1$, it depends on them linearly (according to the law of superposition):

$$\begin{aligned}
&\theta(s) = \omega_1\Theta_\omega(s) + \theta_q\Theta_1(s) + \theta_4\Theta_4(s);\\
&L[\Theta_\omega] = c_\omega, \quad L[\Theta_1] = L[\Theta_4] = 0,\\
&s = -s_1: \quad \Theta_\omega = \Theta_1 = 0, \quad \Theta_4 = 1,\\
&s = s_1: \quad \Theta_\omega = \Theta_4 = 0, \quad \Theta_1 = 1.
\end{aligned} \tag{10}$$

However the values θ_1, θ_4 are yet unknown; they will depend linearly on the angles of pulley rotation ω_1, ω_2.
We solve this boundary value problem and find the bending moment. At the ends of the segment they are

$$\begin{aligned}
&M_4^+ = M_{4\omega}^+ + M_{44}^+\theta_4 + M_{41}^+\theta_1,\\
&M_1^- = M_{1\omega}^- + M_{14}^-\theta_4 + M_{11}^-\theta_1.
\end{aligned} \tag{11}$$

The superscripts $\pm$ denote the shift of the considered point; s_1^- is on the pulley, s_1^+ is on the free span.
We consider the second pulley with the segment $[s_2, s_3]$, unknowns θ_2, θ_3, and angle of rotation ω_2 in the same manner. However, the coefficients in the operator L are different. Therefore we distinguish the operators L^1, L^2. We make the change transiting to L^2:

$$Q_t^{(2)}(s) = -Q_t^{(1)}(l/2 - s), \quad Q_n^{(2)}(s) = Q_n^{(1)}(l/2 - s). \tag{12}$$

Here we account for symmetry of the state before varying. Similar to (11) we find

$$\begin{aligned}
&M_2^+ = M_{2\omega}^+ + M_{22}^+\theta_2 + M_{23}^+\theta_3,\\
&M_3^- = M_{3\omega}^- + M_{32}^-\theta_2 + M_{33}^-\theta_3.
\end{aligned} \tag{13}$$

The coefficients of the expressions (11) and (13) will be present in the matrix of the resolving linear algebraic system of equations for four unknowns θ_i below.

4 Free segments (belt spans)

We have the following system in these segments with the absence of distributed loads

$$\begin{aligned}
&\tilde{\mathbf{Q}}' = 0, \quad \tilde{\mathbf{M}}' + \mathbf{u}' \times \mathbf{Q} + \mathbf{r}' \times \tilde{\mathbf{Q}} = 0,\\
&\boldsymbol{\theta}' = \mathbf{A} \cdot (\tilde{\mathbf{M}} - \boldsymbol{\theta} \times \mathbf{M}), \quad \mathbf{u}' = \mathbf{B} \cdot (\tilde{\mathbf{Q}} - \boldsymbol{\theta} \times \mathbf{Q}) + \boldsymbol{\theta} \times \mathbf{r}'.
\end{aligned} \tag{14}$$

Consider the projections onto Cartesian axes. We integrate the first two equations as follows:

$$\tilde{\mathbf{Q}} = \text{const}, \quad \tilde{M} + u_x' Q_y - u_y' Q_x + x\tilde{Q}_y - y\tilde{Q}_x = \tilde{M}_* = \text{const}. \tag{15}$$

The third and the fourth equations of (14) become

$$\theta' = A\tilde{M}, \quad u_x' = B_x(\tilde{Q}_x + \theta Q_y) + B_{xy}(\tilde{Q}_y - \theta Q_x) - \theta y',$$
$$u_y' = B_y(\tilde{Q}_y - \theta Q_x) + B_{xy}(\tilde{Q}_x + \theta Q_y) + \theta x'. \tag{16}$$

For the six unknowns $\tilde{Q}_x, \ \tilde{Q}_y, \ \tilde{M}_*, \ \theta, \ u_x, \ u_y$ we derive the linear homogeneous system of ODE (the first three unknowns are constants, their derivatives equal zero):

$$Y' = F(s, Y), \quad Y \equiv \left(\tilde{Q}_x \ \ \tilde{Q}_y \ \ \tilde{M}_* \ \ \theta \ \ u_x \ \ u_y\right)^T. \tag{17}$$

The boundary conditions are the prescribed displacements and rotations:

$$s = s_1: \quad \theta = \theta_1, \quad \mathbf{u} = -(k^{-1}\omega_1 + h\theta_1)\mathbf{t}_1,$$
$$s = s_2: \quad \theta = \theta_2, \quad \mathbf{u} = -(k^{-1}\omega_2 + h\theta_2)\mathbf{t}_2,$$
$$s = s_3: \quad \theta = \theta_3, \quad \mathbf{u} = -(k^{-1}\omega_2 + h\theta_3)\mathbf{t}_3,$$
$$s = s_4: \quad \theta = \theta_4, \quad \mathbf{u} = -(k^{-1}\omega_1 + h\theta_4)\mathbf{t}_4. \tag{18}$$

Here the tangent unit vectors $\mathbf{t}_1, ..., \mathbf{t}_4$ are determined by the formula (3). At the four contact area boundaries their projections onto the axes $x, \ y$ are equal by magnitude, but different by signs:

$$t_{1x} = t_1 \cos\varphi_1 - t_2 \sin\varphi_1 = t_{2x} = -t_{3x} = -t_{4x},$$
$$t_{1y} = t_1 \cos\varphi_1 + t_2 \cos\varphi_1 = -t_{2y} = -t_{3y} = t_{4y}. \tag{19}$$

We solve the boundary value problem for the ODE system (17) with the given values of θ at the ends (the displacements are given as well). We repeat this for both spans and determine the bending moments:

$$M_1^+ = M_{1\omega}^+ + M_{11}^+\theta_1 + M_{12}^+\theta_2,$$
$$M_2^- = M_{2\omega}^- + M_{21}^-\theta_1 + M_{22}^-\theta_2,$$
$$M_3^+ = M_{3\omega}^+ + M_{33}^+\theta_3 + M_{34}^+\theta_4,$$
$$M_4^- = M_{4\omega}^- + M_{43}^-\theta_3 + M_{44}^-\theta_4. \tag{20}$$

5 Calculation of the whole belt

In the expressions (11), (13) and (20) we have the set of 16 coefficients and 8 free terms. Using this set we take into account the obvious condition of moment continuity at the points s_i and form the resolving fourth-order linear algebraic system for the unknowns θ_i:

$$\underline{\mathrm{M}\Theta = \Omega}, \Theta = (\theta_1 \ \ \theta_2 \ \ \theta_3 \ \ \theta_4)^T, \quad \Omega_i = M_{i\omega}^- - M_{i\omega}^+,$$
$$\mathrm{M} = \begin{pmatrix} M_{11}^+ - M_{11}^- & M_{12}^+ & 0 & -M_{14}^- \\ -M_{21}^- & M_{22}^+ - M_{22}^- & M_{23}^+ & 0 \\ 0 & -M_{32}^- & M_{33}^+ - M_{33}^- & M_{34}^+ \\ M_{41}^+ & 0 & -M_{43}^- & M_{44}^+ - M_{44}^- \end{pmatrix}. \tag{21}$$

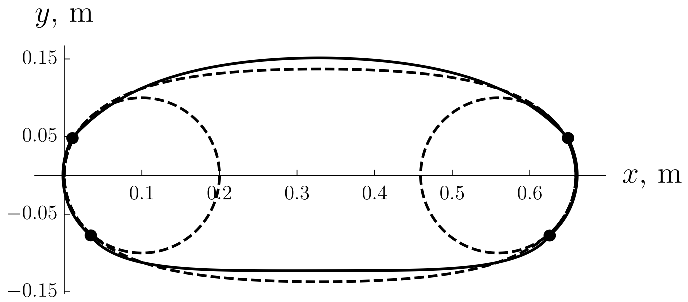

Figure 2: Form of belt. *Dashed line* denotes intermediate state and pulleys

To construct the matrix M it is sufficient to solve the problem for the whole belt four times, each time we take just one non-zero angle of rotation θ_i. For example, with $\theta_1 = 1$, $\theta_2 = \theta_3 = \theta_4 = 0$ we get the first column of the matrix. And to find Ω_i we need to compute the jumps of moments at the points s_i with $\Theta = 0$.
One of the main purposes of this paper is to calculate transmission ratio ω_2/ω_1. Due to the effect of elastic microslip it is not equal to the pulleys radii ratio (one in the case of equal radii as in the present work). Its value reduces with the increase of load - the resistance moment on the driven pulley:

$$T = -k^{-1} \int_{s_2}^{s_3} \tilde{q}_t ds. \tag{22}$$

The function under the integral $\tilde{q}_t$ is the friction force. It is determined from the first equation of (5) after $\tilde{Q}_t$, $\tilde{Q}_n$. It must not exceed (by absolute value) the dry friction limit.
The taken continuity of moment follows from the obvious impossibility of lumped moment reactions. If there exist no lumped contact forces, then the vector **Q** must be continuous too - it gives eight matching conditions. However we have no remaining values to satisfy these conditions. Even if we admit the translation of points $s_1, ..., s_4$, it gives just four additional unknowns whereas we need eight.
Therefore we conclude: the chosen problem formulation is hardly suitable for smooth belt without lumped contact forces. We can recommend the constructed solution only for the toothed belts where the lumped forces are possible (the concentration at the boundary teeth).

6 Numerical example

We consider a benchmark example with parameters: $k_0 = 4\,\text{m}^{-1}$ is the initial curvature, $k_1 = 10\,\text{m}^{-1}$ is the pulley circle curvature, $E = 10^9\,\text{Pa}$ is the Young modulus, $\nu = 0.5$ is the Poisson coefficient, $2h \times b = 0.01 \times 0.01\,\text{m}^2$ is the cross section. The loading parameters: $P = 200\,\text{N}$ is the force taking the pulleys apart (first stage), $\omega_1 = 0.2$, $\omega_2 = -0.2$ are the pulley rotation (chosen for illustrative purpose).
The belt form is shown in Fig. 2. Points indicate the contact boundaries.
The variations of distributed contact reactions are presented in Fig. 3 for the first pulley, the picture for the second pulley is symmetrical. The resulting torques

Table 5: Lumped contact forces

	$\omega_1 = 0.2$, $\omega_2 = -0.2$		$\omega_1 = 0.2$, $\omega_2 = 0$	
	$[\![\tilde{Q}_t]\!]$, N	$[\![\tilde{Q}_n]\!]$, N	$[\![\tilde{Q}_t]\!]$, N	$[\![\tilde{Q}_n]\!]$, N
s_1	-2076	-11253	-802	-10562
s_2	-1256	-12001	-64	-1052
s_3	2076	11253	1274	692
s_4	1256	12001	1192	10949

equal $T_1 = -T_2 = 1912\,\mathrm{Nm}$ (in the case $\omega_1 = 0.2$, $\omega_2 = 0$ they equal $T_1 = 1666\,\mathrm{Nm}$, $T_2 = -246\,\mathrm{Nm}$). We calculate the variations of lumped contact forces as the jumps of the force components, their values are written in Table 5; $[\![...]\!] = ...|_{s=s_i+0} - ...|_{s=s_i-0}$.

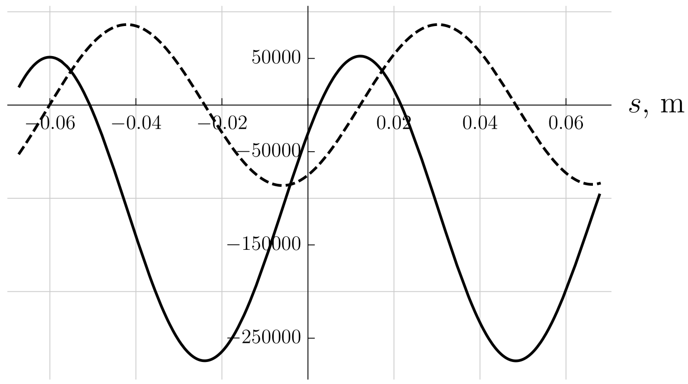

Figure 3: Variations of contact reactions. *Dashed line* denotes normal component, *full line* is tangent one

7 Conclusion

In the paper we presented the solution of the static problem for belt-pulley contact interaction taking into account stick and deformations caused by it. We derived the equations in variations superposed upon the stress state of fitting the belt on the pulleys. We assumed that the points of contact areas move along the pulley circles, hence the distributed contact reactions arise, both normal and tangential, and distributed moment as well. The proposed model is suitable only for the toothed belts, because the lumped contact reactions arise at the boundaries of contact zones.

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