

# The solutions of nonlinear equations of flat deformation of the crystal media allowing martensitic transformations

E. L. Aero, A. N. Bulygin, Yu. V. Pavlov

bulygin\_an@mail.ru

## Abstract

Mathematical methods of the solution of the equations of statics of flat nonlinear deformation of the crystal media with a complex lattice allowing martensitic transformations are developed. The equations of a statics represent system of four connected nonlinear equations. The vector of macroshifts is looked in the Papkovish-Neuber form. The system of the connected nonlinear equations is reduced to system of the separate equations. The vector of microshifts can be found from the sine Gordon equation with variable coefficient (amplitude) before the sine and Poisson equation. The class of doubly periodic solutions expressing in the Jacobi elliptic functions is found for a case of constant amplitude. It is shown that the nonlinear theory possesses a set of solutions which describe fragmentation of the crystal medium, emergence of defects of structure of different types, phase transformations and other topological features of the deformation which are implemented under the influence of intensive power loadings and which can't be described by classical mechanics of the continuous medium. Features of the found solutions are discussed.

## 1 Introduction

In recent years nanotechnologies are intensively developed. Practically all modern technologies for metals and alloys with ultrafine-grained structure are based on use of superhigh external impacts on material. The structure of material is significantly changed under the influence of intensive plastic deformations. Medium breaks up on separate nanoscale grains which are variously oriented. Grains divide big-angle borders in which defects like micropores and local consolidations are formed. The superlattice is formed and there are phase transformations.

Modern problems of technologies of obtaining and studying of the new materials set new problems for mechanics of continuous media. New analytical models are necessary for modern technologies of obtaining the new materials with the designated operational properties.

The classical continual model isn't adequate any more to those new problems which have arisen in connection with deep penetration into area the nano-scales. In works [1, 2] nonlinear model of deformation of crystal media with a complex lattice is offered. The offered model allows to describe specific processes of deformation, which are implemented in modern technologies of obtaining new materials.

## 2 Nonlinear model of deformation of crystal media with a complex lattice

In nonlinear model [1]–[3] deformation of crystal medium is described by vector of acoustic mode  $\mathbf{U}(x, y, z, t)$  and vector of optical mode  $\mathbf{u}(x, y, z, t)$ . Equations of motion defining  $\mathbf{U}(x, y, z, t)$  and  $\mathbf{u}(x, y, z, t)$ , are derived from Lagrange's variation principle. They have the form

$$\rho \ddot{U}_i = \sigma_{il,l}, \quad (1)$$

$$\sigma_{il} = \lambda_{ilmn} e_{mn} + C_{ilmn} \varepsilon_{mn} - s_{il} \Phi(u_s), \quad (2)$$

$$\mu \ddot{u}_i = \chi_{il,l} - \mathcal{P} \frac{\partial \Phi(u_s)}{\partial u_i}, \quad (3)$$

$$\chi_{il} = k_{ilmn} \varepsilon_{mn} + C_{ilmn} e_{mn}. \quad (4)$$

In Eqs. (1)–(4) and further the over point denotes time derivative while a comma in indexes defines spatial derivative. Besides, the following designations are entered:  $\rho$ ,  $\mu$  are density and the specified density of mass of couple of atoms,  $\sigma_{il}$ ,  $\chi_{il}$  are tensors of macro- and microstresses,  $\lambda_{ilmn}$ ,  $k_{ilmn}$ ,  $C_{ilmn}$  are the coefficients of elasticity, microelasticity and modules of interaction of acoustic and optical modes,  $e_{il}$ ,  $\varepsilon_{il}$  are tensors of deformation and microdeformation

$$e_{il} = \frac{1}{2} (U_{i,l} + U_{l,i}), \quad \varepsilon_{il} = \frac{1}{2} (u_{i,l} + u_{l,i}), \quad (5)$$

$\Phi(u_s)$  is the energy of interaction of sublattices. In pioneer work [4] and in majority of modern works [5, 6] one is accepted, that

$$\Phi(u_s) = 1 - \cos u_s. \quad (6)$$

The argument is

$$u_s = \mathbf{B} \cdot \mathbf{u}, \quad (7)$$

where  $\mathbf{B}$  is the vector of inverse lattice [7]. For crystal of cubic system with length of elementary cell  $b$  one has

$$\mathbf{B} = \frac{1}{b} (\mathbf{i} + \mathbf{j} + \mathbf{k}), \quad (8)$$

$$u_s = \frac{1}{b} (u_x + u_y + u_z). \quad (9)$$

Multiplier  $\mathcal{P} = p - s_{il}e_{il}$  is an effective interatomic barrier, where  $p$  is half of energy of activation of rigid shift of lattices, and  $s_{il}$  is a tensor of nonlinear mechanostriction. For the cubic crystal one has  $s_{il} = s\delta_{il}$ , and

$$\mathcal{P} = p - s \operatorname{div} \mathbf{U}, \quad (10)$$

The material tensors  $\lambda_{ilmn}$ ,  $k_{ilmn}$ ,  $C_{ilmn}$  have only three independent components for this crystal. Let they will be in Voigt's designations [8]

$$\lambda_{11}, \lambda_{12}, \lambda_{44}, \quad k_{11}, k_{12}, k_{44}, \quad C_{11}, C_{12}, C_{44}.$$

Take for a measure of anisotropy of a cubic crystal [9]

$$a_1 = \frac{2\lambda_{44}}{\lambda_{11} - \lambda_{12}}, \quad a_2 = \frac{2k_{44}}{k_{11} - k_{12}}, \quad a_3 = \frac{2C_{44}}{C_{11} - C_{12}}. \quad (11)$$

For isotropic medium we have

$$a_1 = a_2 = a_3 = 1. \quad (12)$$

Material ratios for the media of cubic system have the form

$$\sigma_{il} = \begin{cases} (\lambda_{11} - \lambda_{12})e_{il} + (C_{11} - C_{12})\varepsilon_{il} + (\lambda_{12}e + C_{12}\varepsilon - s\Phi(u_s))\delta_{il} & (i = l), \\ 2\lambda_{44}e_{il} + 2C_{44}\varepsilon_{il} & (i \neq l), \end{cases} \quad (13)$$

$$\chi_{il} = \begin{cases} (k_{11} - k_{12})\varepsilon_{il} + (C_{11} - C_{12})e_{il} + (k_{12}\varepsilon + C_{12}e)\delta_{il} & (i = l), \\ 2k_{44}\varepsilon_{il} + 2C_{44}e_{il} & (i \neq l), \end{cases} \quad (14)$$

$$e = e_{xx} + e_{yy} + e_{zz}, \quad \varepsilon = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}. \quad (15)$$

### 3 Flat deformation. Statics equations

We will call the deformed state as statically flat and parallel to an axis  $x_3$  if

$$U_x = U_x(x, y), \quad U_y = U_y(x, y), \quad U_z = 0, \quad (16)$$

$$u_x = u_x(x, y), \quad u_y = u_y(x, y), \quad u_z = 0. \quad (17)$$

Taking into account (16), (17) in material ratios (13) and (14), and substituting the expressions for  $\sigma_{il}$  and  $\chi_{il}$  in Eqs. (1) and (3), we find the equations of static in movements for flat deformation of nonlinear model:

$$\begin{aligned} & \lambda_{44}\Delta\mathbf{U} + (\lambda_{12} + \lambda_{44})\operatorname{grad} \operatorname{div} \mathbf{U} + \\ & C_{44}\Delta\mathbf{u} + (C_{12} + C_{44})\operatorname{grad} \operatorname{div} \mathbf{u} - s\operatorname{grad} \Phi(u_s) = 0, \end{aligned} \quad (18)$$

$$\begin{aligned} & k_{44}\Delta\mathbf{u} + (k_{12} + k_{44})\operatorname{grad} \operatorname{div} \mathbf{u} + \\ & C_{44}\Delta\mathbf{U} + (C_{12} + C_{44})\operatorname{grad} \operatorname{div} \mathbf{U} - \mathbf{B}(p - s\operatorname{div} \mathbf{U})\sin u_s = 0. \end{aligned} \quad (19)$$

In Eqs. (18) and (19) one has

$$\mathbf{U} = U_x\mathbf{i} + U_y\mathbf{j}, \quad \mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j}, \quad \mathbf{B} = \frac{\mathbf{i} + \mathbf{j}}{b}, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and, besides, restrictions (12) are accepted.

### 3.1 General solution of the equations of statics

The equations of static (18), (19) are a system of four coupled nonlinear equations. We will seek a vector of macroshifts  $\mathbf{U}$  in Papkovish-Neuber form

$$\mathbf{U} = a\mathbf{A} + \text{grad } \chi. \quad (20)$$

Here  $(a, \chi, \mathbf{A})$  are an arbitrary constant, a scalar function  $\chi(x, y)$  and a vector function  $\mathbf{A}(x, y)$ . If one substitutes the expression (20) in Eq. (18), then it will be solved if vectors  $\mathbf{A}$  and  $\mathbf{u}$  are satisfy to equation

$$a\lambda_{44}\Delta\mathbf{A} + C_{44}\Delta\mathbf{u} = 0, \quad (21)$$

and the scalar function  $\chi$  is a solution of the Poisson equation

$$\Delta\chi = \frac{1}{\lambda_{12} + 2\lambda_{44}} [s\Phi(u_s) - a(\lambda_{12} + \lambda_{44}) \text{div } \mathbf{A} - (C_{12} + C_{44}) \text{div } \mathbf{u}]. \quad (22)$$

After substitution Eq. (20) in Eq. (19), we can see that this equation will be solved if

$$\text{div } \mathbf{A} = \text{div } \mathbf{u}, \quad a = \frac{k_{12} + k_{44} - C(C_{12} + C_{44})}{(\lambda_{12} + \lambda_{44})C - (C_{12} + C_{44})}, \quad C = \frac{C_{12} + 2C_{44}}{\lambda_{12} + 2\lambda_{44}}, \quad (23)$$

$$k_{44}\Delta\mathbf{u} + aC_{44}\Delta\mathbf{A} + sC \text{grad } \Phi(u_s) - \mathbf{B}(p - s \text{div } \mathbf{U}) \sin u_s = 0. \quad (24)$$

One can exclude  $\Delta\mathbf{A}$  from Eq. (24) with help (21). Then the components of microshift vector  $(u_x, u_y)$  are the solutions of the equations

$$\begin{aligned} K\Delta u_x + sC \frac{b^2}{2} \sin u_s \frac{\partial u_s}{\partial x} - \mathbf{B}(p - s \text{div } \mathbf{U}) \sin u_s &= 0, \\ K\Delta u_y + sC \frac{b^2}{2} \sin u_s \frac{\partial u_s}{\partial y} - \mathbf{B}(p - s \text{div } \mathbf{U}) \sin u_s &= 0, \\ K &= \frac{b^2}{2} \left( k_{44} - \frac{C_{44}^2}{\lambda_{44}} \right). \end{aligned} \quad (25)$$

Eqs. (25) can be transformed to more simple form. After summation of (25) and elementary algebraic transformations we find the equation for  $u_s$

$$K\Delta u_s = P \sin u_s, \quad (26)$$

$$P = p - \frac{s}{\lambda_{12} + 2\lambda_{44}} [\text{div } \mathbf{u}_+ + s\Phi(u_s)],$$

$$\mathbf{u}_+ = (a\lambda_{44} - C_{12} - C_{44}) \mathbf{u} + \frac{b^2}{2} (C_{12} + 2C_{44}) \mathbf{B} u_s.$$

To find a vector of microshifts  $\mathbf{u}$  it is necessary to add the equation for  $u_x$  and  $u_y$  to the Eq. (26). They have the form

$$\Delta \left( u_x - \frac{b}{2} u_s \right) = f, \quad \Delta \left( u_y - \frac{b}{2} u_s \right) = -f, \quad (27)$$

where

$$f = \lambda \left( \frac{\partial u_s}{\partial y} - \frac{\partial u_s}{\partial x} \right) \sin u_s, \quad \lambda = \frac{s}{4K} C b^2. \quad (28)$$

Thus, realization of nonlinear model is reduced to the solution of the equation (26). If function  $u_s$  is found, then the functions  $u_x$  and  $u_y$  can be found from the Poisson equation with the known function  $f$ , as we can see from (27) and (28). Problems of finding of scalar function  $\chi$  and vector  $\mathbf{A}$  from Eq. (21) also lead to Poisson equation

$$\Delta \chi = \frac{1}{\lambda_{12} + 2\lambda_{44}} \{ s\Phi(u_s) - [(C_{12} + C_{44}) + a(\lambda_{12} + \lambda_{44})] \operatorname{div} \mathbf{u} \}. \quad (29)$$

### 3.2 Solutions of the equations of optical mode and structures of microdeformation corresponding to them

In literature there are no analytical methods for solution of sine Gordon (SG) equation with a variable amplitude. Functionally invariant solutions of the (2+1)- and (3+1)-dimensional SG equations are constructed for a wide, but specific type of amplitudes in [10]–[12]. The Eq. (26) can be reduced to well studied cases if to make some restrictions for model or for the field of microdeformations. So, if not to consider dependence of potential of interaction of sublattices on deformation of the medium, i.e. to accept  $s = 0$ , then Eq. (28) becomes the SG equation with constant coefficients  $(K, p)$

$$K\Delta u_s = p \sin u_s. \quad (30)$$

Also it will be if

$$\operatorname{div} \mathbf{u}_+ + \frac{s}{\lambda_{12} + 2\lambda_{44}} \Phi(u_s) = 0. \quad (31)$$

In literature the method of the solution of the SG equation (30) based on substitution

$$u_s = 4 \operatorname{arctg}(G(x, y)), \quad G(x, y) = \Phi_1(x)\Phi_2(y), \quad (32)$$

is widely known. The solution (32) assumes that  $K > 0$ . If  $K < 0$ , then solution of Eq. (30) is

$$u_s = \pi + 4 \operatorname{arctg}(G(x, y)). \quad (33)$$

The solution (32) is connected with G. L. Lamb Jr. [13], though the first time it was used by Steuerwald [14].

The functions  $\Phi_1(x)$  and  $\Phi_2(y)$  can be found by inversion of the corresponding elliptic integrals by Legendre's method. The Legendre method is rather difficult. In [15] the method of finding of functions  $\Phi_1(x)$  and  $\Phi_2(y)$  is offered based on the differential equations which satisfy elliptic functions of Jacobi (the modified Lamb method). The offered approach allows to receive a wide class of doubly periodic solutions of

the SG equation. Two solutions from this class are given below

$$1) \quad \begin{aligned} A &= \frac{\text{cn}(\xi, \nu_1)}{\text{cn}(\eta, \nu_2)}, & B^2 &= \frac{(1+A^2)[(1+A^2)\nu_2^2 - 1]}{A^2} K^2(\nu_2), \\ & & A^4 &= \frac{\nu_1^2(1-\nu_2^2)}{\nu_2^2(1-\nu_1^2)}, \end{aligned} \quad (34)$$

$$2) \quad \begin{aligned} A &= \frac{A}{\text{cn}(\xi, \nu_1)} \frac{\text{sn}(\eta, \nu_2)}{\text{dn}(\eta, \nu_2)}, & B^2 &= \frac{(\nu_2^2 - A^2)(A^2 + 1 - \nu_2^2)}{A^2} K^2(\nu_2), \\ & & A^4 &= \frac{\nu_2^2(1-\nu_1^2)(1-\nu_2^2)}{\nu_1^2}, \quad \nu_1^2 + \nu_2^2 > 1, \end{aligned} \quad (35)$$

In examples (34), (35) values  $K(\nu_1)$ ,  $K(\nu_2)$  are full elliptic integrals of the first sort and variables are

$$\xi = \frac{x}{lH} K(\nu_1), \quad \eta = \frac{y}{lB} K(\nu_2), \quad l = \sqrt{K/p}. \quad (36)$$

On Fig. 1 and Fig. 2 microdeformations corresponding to the solutions (34) and (35) are shown. As we can see, the solution (34) describes creating in crystal lattice

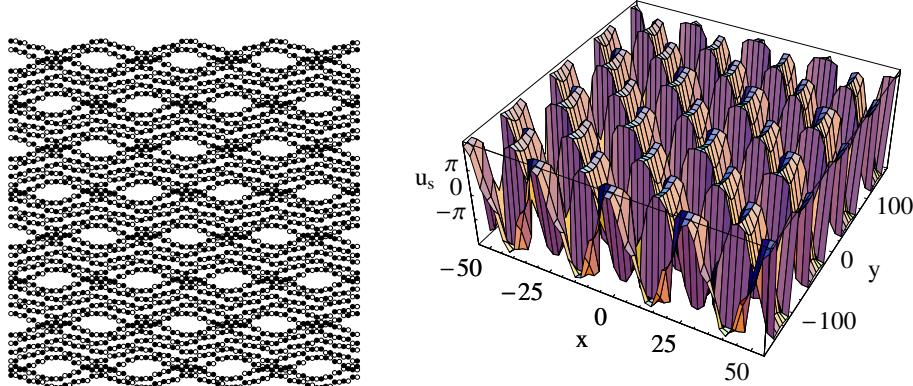


Figure 1: Microdeformations of crystal and solution (34) for  $\nu_1 = 0.99999$ ,  $\nu_2 = 0.9$ .

of system of regularly located micropores, and (35) describes of system of regularly located microconsolidations.

The modified method allows to construct solutions which are expressed through circular or hyperbolic functions from solutions which are expressed through elliptic functions of Jacobi. It can be done, if to use the known limit ratios

1.  $\nu \rightarrow 0$ ,  $\text{sn}(u, \nu) \rightarrow \sin u$ ,  $\text{cn}(u, \nu) \rightarrow \cos u$ ,  $\text{dn}(u, \nu) \rightarrow 1$ ,
2.  $\nu \rightarrow 1$ ,  $\text{sn}(u, \nu) \rightarrow \text{th } u$ ,  $\text{cn}(u, \nu) \rightarrow \frac{1}{\text{ch } u}$ ,  $\text{dn}(u, \nu) \rightarrow \frac{1}{\text{ch } u}$ .

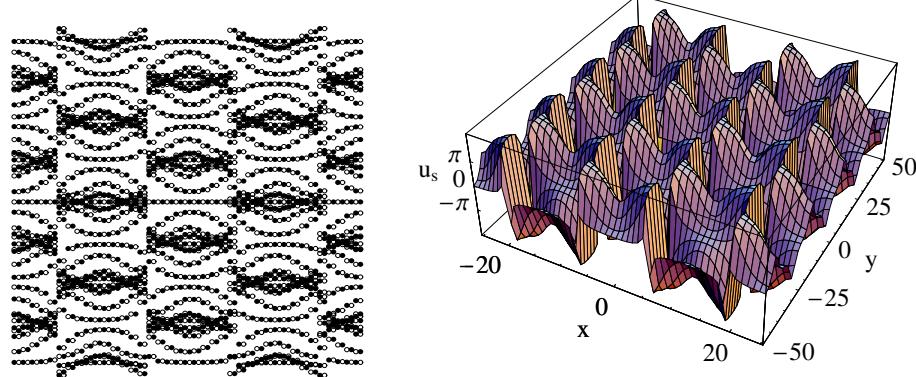


Figure 2: Microdeformations of crystal and solution (35) for  $\nu_1 = \nu_2 = 0.999$ .

Using (37), one finds

$$G = \begin{cases} \operatorname{tg} \psi \frac{\operatorname{sh}(x \cos \psi)}{\operatorname{sh}(y \sin \psi)} \\ \operatorname{tg} \psi \frac{\operatorname{ch}(x \cos \gamma)}{\operatorname{ch}(y \sin \gamma)}, \\ \frac{1}{\operatorname{th} \psi} \frac{\cos(x \operatorname{sh} \psi)}{\operatorname{sh}(y \cosh \psi)}, \end{cases} \quad (38)$$

where  $\psi$  is an arbitrary constant.

On Fig. 3–5 microdeformations corresponding to the solutions (38) are shown. For

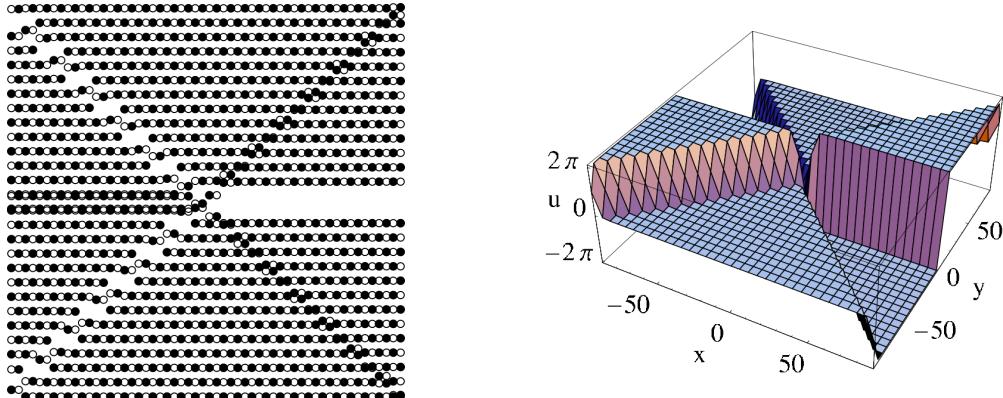


Figure 3: Microdeformations and solutions for the first case in (38) with  $\psi = \pi/4$ .

the first solution the plane of  $y = 0$  is the plane with defects. On half-plane  $x > 0$  defect like the main crack is formed, half-plane  $x < 0$  contains the defects caused by introduction of the excess crystal planes. It is visible that defects are also the inclined planes. For the second solution the plane of  $y = 0$  is not the singular plane. The third solution (38) describes system of micropores which are located in  $y = 0$  plane.

The given examples show that the nonlinear model describes features of deformation which are implemented in the field of big external tension, are observed on experience and not described by classical mechanics of the continuous medium.

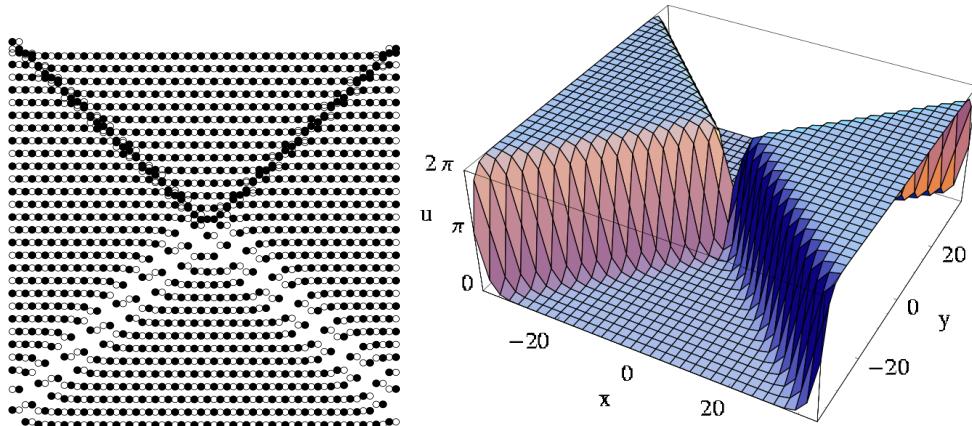


Figure 4: Microdeformations and solutions for the second case in (38) with  $\psi = \pi/4$ .

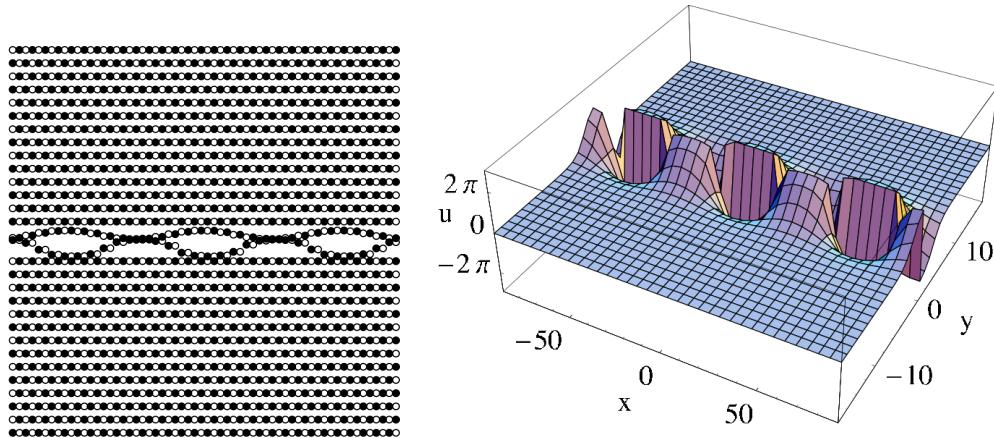


Figure 5: Microdeformations and solutions for third case in (38) with  $\psi = 0.1$ .

## 4 Conclusion

The flat deformation plays an extremely important role in classical mechanics of continuous medium. It is caused by the fact that effective analytical and numerical methods are developed for the solution of problems of flat deformation. The problems of flat deformation can be reduced to boundary problems of theory of the functions of complex variable. It allowed for their solution to apply both methods of classical mathematical physics, and the methods of the theory of functions of complex variable (conformal mappings, Riemann problem, theory of singular integrable equations, etc.) As a result exact analytical solutions of a large number of concrete cases of deformation of one- and multiply connected domains have been found [16]. The found solutions have formed a scientific basis of modern materials science. However the linear classical model of the continuous medium doesn't answer inquiries of modern technologies of materials with internal structure. It doesn't describe processes of deformation and specific nanoscale changes (defects, phase transformations, fragmentation, etc.) which are implemented in the field of intensive plastic deformations. More adequately these processes can be described by the nonlinear theory of deformation of crystal media with a complex lattice. However, the development

of mathematical methods for realization of the nonlinear model is necessary that it became a scientific basis for engineering calculations in modern technologies.

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*E. L. Aero, A. N. Bulygin, Yu. V. Pavlov,  
Institute for Problems in Mechanical Engineering of Russian Academy of Sciences,  
61 Bol’shoy, V.O., Saint Petersburg, 199178, Russia*