A posteriori Error Bounds for numerical Solutions of Plate in Bending Problems

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Abstract

For the efficient error control of numerical solutions of the solid mechanics problems, the two requirements are important: an a posteriori error bound has sufficient accuracy and computation of the bound is cheap in respect to the arithmetic work. The first requirement can be formulated in a more specific form of consistency of an a posteriori bound, assuming that it is not improvable in the order and, at least, coincides in the order with the a priori error estimate. Several new a posteriori error bounds are presented, which improve accuracy and reduce the computational cost. Also for the first time a new consistent guaranteed a posteriori error bound is suggested. The presented a posteriori bounds bear on the counter variational Lagrange and Castigliano principles which are valid for a wide class of problems.

Introduction

The use of adaptive algorithms can considerably reduce the cost of the stress and deformation states analysis of structures. The key module of such algorithms implements some a posteriori error bound or error indicator which allows adequate local thickening, *e.g.*, of the FEM (finite element method) mesh in consecutive steps. In the literature, illustrations of the efficiency of adaptive algorithms, arranged in this way, are numerous. Here only the references to [1, 2] are given, where several popular error indicators are compared when used for the adaptive FEM stress state analysis, and where many additional references can be found. There are also other strong incentives for the development of efficient APEB's (a posteriori error bounds), and nowdays many commercial computer codes like ANSYS, ABAQUS, FLUENT etc. contain modules allowing not only to solve the problem, but also to evaluate the majorant for the error.

For derivation of APEB's many techniques have been developed, which are well illuminated in the books of Verfurt [3] and Ainsworth & Oden [4], Neittaanmaki & Repin [5] and, *e.g.*, in recent papers of Ainsworth, Demkowicz & Kim [6] and Braess & Schoeberl [7]. Primarily they are related to the problems described by the 2^{nd} order partial differential equations with much less attention paid to the thin plate and shell bending problems, which are widely used in structures and are discussed in this paper. We generalize upon thin plate bending problem the technique of Anufriev, Korneev & Kostylev [8, 9] based on the use of the *exactly equilibrated stress fields*. It allowed to suggest efficient a posteriori error bounds for the finite element solutions of the theory elasticity problems and other 2^{nd} order elliptic equations. In [8, 9] it was shown by many numerical experiments that these bounds are computationally cheap and provide very good effectiveness indices.

Classical formulations of thin plate and shell bending problems are described by the 4^{th} order elliptic partial differential equations and systems of equations. At present, numerical solutions of such equations are primarily obtained with the use of the mixed methods. However, at least sometimas, solution of the 4^{th} order elliptic equations is, for some reasons, preferable. This inspired development of the a posteriori estimators for the thin plate bending problem in classical formulations, including as conform [1, 10, 11] so different types of not conform and DG (discontinuous Galerkin) methods, see [12, 13, 14, 15]. A part of this paper concentrates on the conform finite element approximations and expand the technique of [8, 9], which in this case can be termed the *technique of the exactly equilibrated resultants*. Our main goals are to reduce the computational cost of the evaluation of the bounds and to make them sharper. For instance, in general the residuals, entering a posteriori bounds, contain second order derivatives of the approximate values of the moments. In the contrast, some bounds in this paper contain only first derivatives. More over, the norms in the right parts of the bounds contain not first derivatives of the moments, but one-dimensional integrals of them with variable upper limits. Clearly, both these features improve the accuracy of our a posteriori bounds. This is supported by the estimate of the order of convergence of the right part of the a posteriori bound, see Proposition 1. Finally, in Section 2, one of the most important a posteriori bounds of the paper is presented for the problem in an arbitrary sufficiently smooth domain. We term this bound *consistent* implying that the right part of it has the same order of smallness as predicted by the corresponding unimprovable a priori error bound.

To concentrate on the basic fatures of the exactly equilibrated resultants technique, the model problem of a thin plate in rectangle is considered Section 1. However, there is no principal difficulties visible for expansion of the bounds to the cases of more general domains and more general equations, e.g., of thin linearly elastic shells. For supporting the latter, we note that the equilibrated resultants and the spaces of the self-equilibrated resultants were defined for such problems in [16] and used there for construction of numerical algorithms.

The paper is arranged as follows. In Section 1 we consider approach based on direct of the equilibrated testing moments. In Subsection 1.1 the problem of thin linearly elastic plate bending is formulated. Also an example of a posteriori error bounds by means of exactly equilibrated moments of smooth approximate solutions are presented. Subsection 1.2 is allocated for a posteriory estimator of the type termed often functional error majorants. The use of smoothed moments, recovered from FEM, for the error estimation is considered in Subsection 1.3. The consistent a posteriori bound is given in Section 2.

The notation $\|\phi\|_{H^k(\mathcal{Q})}$ will stand for the norms in the Sobolev's spaces $H^k(\mathcal{Q})$ on a domain $\mathcal{Q}$

$$\|\phi\|^2_{H^k(\mathcal{Q})} = \|\phi\|^2_{L_2(\mathcal{Q})} + \sum_{l=1}^{k} |\phi|^2_{H^l(\mathcal{Q})}, \ |\phi|^2_{H^l(\mathcal{Q})} = \sum_{q_1+q_2=l} \int_{\mathcal{Q}} (\partial^l \phi / \partial x_1^{q_1} \partial x_2^{q_2})^2 dx,$$

where $\|\phi\|^2_{L_2(\mathcal{Q})} = \int_{\mathcal{Q}} \phi^2 dx_1 dx_2$. If $\mathcal{Q} = \Omega$, the simpler notations $\|\cdot\|_0$, $\|\cdot\|_k$ and $|\cdot|_k$ will be used for $\|\cdot\|_{L_2(\Omega)}$, $\|\cdot\|_{H^k(\Omega)}$ and $|\cdot|_{H^k(\Omega)}$, respectively. The finite element space is denoted as $V(\Omega)$ with $\mathring{V}(\Omega) = \{v \in V(\Omega) : v|_{\partial\Omega} = 0\}$ and it is always assumed that the finite element assemblage satisfies the generalized conditions of quasiuniformity with some mesh parameter $h > 0$, see [17, Section 3.2].

1 A posteriori error bounds by means of exactly equilibrated moments

1.1 Problem and equilibrated bounds

For a model problem, we use the problem of a thin homogeneous linearly elastic plate in the square $\pi_1 = (0,1) \times (0,1)$ of the constant thickness h and clamped at the boundary $\partial\pi_1$. Deflection of the middle surface of the plate under the transverse load is described by the equation

$$D\Delta\Delta u = f(x), \ x = (x_1, x_2) \in \pi_1, \ u(y) = \frac{\partial u}{\partial n}(y) = 0, \ y = (y_1, y_2) \in \partial\pi_1 \quad (1)$$

where $D = Eh^3/(12(1-\nu^2))$ and n is the external normal to the boundary, E and ν are the elasticity module and Poisson coefficient. The vector $M = (M_{1,1}, M_{2,2}, M_{1,2})^{\mathrm{T}}$ of moments satisfy the equilibrium equation

$$L_M M \equiv \frac{\partial^2 M_{1,1}}{\partial x_1^2} + 2\frac{\partial^2 M_{1,2}}{\partial x_1 \partial x_2} + \frac{\partial^2 M_{2,2}}{\partial x_2^2} = f. \quad (2)$$

The *energy norms* are important characteristics of the error of approximate solutions of the problem, the squares of two such norms can be written as

$$\begin{aligned}
\lfloor v \rceil^2_U &= \frac{D}{2} \int_{\pi_1} \left[(\frac{\partial^2 v}{\partial x_1^2})^2 + (\frac{\partial^2 v}{\partial x_2^2})^2 + 2\nu \frac{\partial^2 v}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} + 2(1-\nu)(\frac{\partial^2 v}{\partial x_1 \partial x_2})^2 \right] dx, \\
\lfloor M \rceil^2_M &= \int_{\pi_1} M^{\mathrm{T}} \Xi M dx \\
&= \frac{1}{2D(1-\nu^2)} \int_{\pi_1} \left[M_{1,1}^2 + M_{2,2}^2 - 2\nu M_{1,1} M_{2,2} + 2(1+\nu) M_{1,2}^2 \right] dx \quad (3)
\end{aligned}$$

where for the matrix Ξ stands

$$\Xi = \frac{1}{2D(1-\nu^2)} \begin{pmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{pmatrix}.$$

Suppose $\tilde{u}(x)$ is an approximation of the exact solution u, satisfying the boundary condition in (1), and $\Phi = (\Phi_{1,1}, \Phi_{2,2}, \Phi_{1,2})^{\mathrm{T}}$ is the corresponding vector of moments.

For the first step, the case of a smooth $\tilde{u}(x)$, *e.g.*, having 4^{th} bounded derivatives is considered. If $Z = (Z_{1,1}, Z_{2,2}, Z_{1,2})^{\mathrm{T}}$ is any vector of sufficiently smooth moments, which satisfy (2), then the bound

$$\lfloor u - \tilde{u} \rfloor_U^2 \leq \lfloor Z - \Phi \rfloor_M^2 \tag{4}$$

is the direct consequence of the Lagrange and Castigliano principles, see, *e.g.*, [18]. In order to get satisfactory bounds (4), the equilibrated vector of moments Z should be as close as possible to the exact one. Since as a rule the most reliable information about the exact solution is contained in the approximate solution $\tilde{u}$, it is natural to use it for definition of Z. The simplest way to arrange this is to adopt $Z_{k,l} = \Phi_{k,l}$ for two components of the vector Z and define the remaining one from the equilibrium equation (2). For instance, one can set $Z_{k,k} = \Phi_{k,k},\ k = 1, 2$, substitute them into (2) and obtain the equality

$$\frac{\partial^2 Z_{1,2}}{\partial x_1 \partial x_2} = 0.5 \left[f - \frac{\partial^2 \Phi_{1,1}}{\partial x_1^2} - \frac{\partial^2 \Phi_{2.2}}{\partial x_2^2} \right] \text{ in } \pi_1. \tag{5}$$

Taking these facts, the boundary conditions and the notation $\mathbb{D} = 2D(1 - \nu)$ into account yields the bound (4) of the form

$$\lfloor u - \tilde{u} \rfloor_U^2 \leq \mathbb{D}^{-1} \int_{\pi_1} \left\{ 2\Phi_{1,2} - \int_0^{x_2} \int_0^{x_1} f(\eta) d\eta + \right.$$
$$\left. + \sum_{k=1,2} \int_0^{x_{3-k}} \left[\frac{\partial \Phi_{k,k}}{\partial x_k}(x_k, \eta_{3-k}) - \frac{\partial \Phi_{k,k}}{\partial x_k}(0, \eta_{3-k}) \right] d\eta_{3-k} \right\}^2 dx\,. \tag{6}$$

Boundary conditions in (1) are *essential* and, at the numerical solution, e.g., by Galerkin and FE methods, can be satisfied exactly, implying that $\Phi_{1,2} = 0$ on Γ_0. With the use of this, the a posteriori error bound can be also transformed into

$$\lfloor u - \tilde{u} \rfloor_U^2 \leq \mathbb{D}^{-1} \int_{\pi_1} \left[\int_0^{x_2} \int_0^{x_1} \left[f(\eta) - L_M \Phi(\eta) \right] d\eta \right]^2 dx. \tag{7}$$

The bounds (6), (7) are more "symmetric" than the corresponding bounds at the choice, e.g., $Z_{1,1} = \Phi_{1,1}$, $Z_{1,2} = \Phi_{1,2}$ and computation of $Z_{2,2}$ from (2). An important distinction of (6) from the residual tipe bounds is that the former contains only first derivatives of the moments in the right part whereas the latter contains second derivatives. Therefore the accuracy of the former bounds can be much higher. It should be also underlined that, in the contrast with a posteriori error bounds of many other types, the evaluation of (6), (7) do not require any other computations (e.g., solution of global or local systems of algebraic equations for the rezidual minimization or local discrete equilibration etc.) except the evaluation of integrals.

1.2 Functional error majorants

For some purposes it is convenient to have an a posteriori bound with an arbitrary testing resultant vector, *i.e.*, not satisfying the equilibrium equations. For instance, this is the case, when the right part of an a posteriori bound is supposed to be minimized with the help of such vectors. Such bounds directly follow from the bounds of the previous section. Let $Y = (Y_{1,1}, Y_{2,2}, Y_{1,2})^{\mathrm{T}}$ be an arbitrary vector, sufficiently smooth on $\overline{\pi}_1$, and $Y^{Eq} = Y + \delta Y$ be the equilibrated vector. The latter can be defined by Y exactly, as it was suggested to define Z by Φ and, in particular, by setting $Y^{Eq}_{k,k} = Y_{k,k}$, $k = 1, 2$, and defining $Y^{Eq}_{1,2}$ from (5), in which $\Phi_{k,k}$ are replaced by $Y_{k,k}$. At that, there is no difficulties to satisfy the boundary condition $Y_{1,2} = 0$ on $\partial\pi_1$. In this way, we come to the bound

$$\lfloor u - \tilde{u} \rfloor_U \leq \left\lfloor Y^{Eq} - \Phi \right\rfloor_M \leq \lfloor Y - \Phi \rfloor_M + \lfloor \delta Y \rfloor_M \leq \lfloor Y - \Phi \rfloor_M + \Xi(Y_{1,2}, f, Y^{[1,2]}), \quad (8)$$

in which $Y^{[1,2]} = (Y_{1,1}, Y_{2,2}, 0)$ and

$$\Xi^2(Y_{1,2}, f, Y^{[1,2]}) = \mathbb{D}^{-1} \int_{\pi_1} \left\{ 2Y_{1,2} - \int_0^{x_2} \int_0^{x_1} f(\eta) d\eta + \right.$$

$$\left. + \sum_{k=1,2} \int_0^{x_{3-k}} \left[\frac{\partial Y_{k,k}}{\partial x_k}(x_k, \eta_{3-k}) - \frac{\partial Y_{k,k}}{\partial x_k}(0, \eta_{3-k}) \right] d\eta_{3-k} \right\}^2 dx.$$

Another bound follows from the first one by the triangular inequality and can be written in simpler form

$$\lfloor u - \tilde{u} \rfloor_U \leq \left\lfloor Y^{Eq} - \Phi \right\rfloor_M \leq \left\lfloor Y^{[1,2]} - \Phi^{[1,2]} \right\rfloor_M + \Xi(\Phi_{1,2}, f, Y^{[1,2]}). \quad (9)$$

Clearly, the bound (7) turns into

$$\lfloor u - \tilde{u} \rfloor_U \leq \lfloor Y - \Phi \rfloor_M + \left\{ \mathbb{D}^{-1} \int_{\pi_1} \left[\int_0^{x_2} \int_0^{x_1} [f(\eta) - L_M\, Y(\eta)]\, d\eta \right]^2 dx \right\}^{1/2}. \quad (10)$$

The drawback of this bound in comparison with (8), (9) is that it contains 2^{nd} derivatives of the components of Y instead first derivatives in (8), (9). For any smooth function $F(x)$ on rectangle $\Pi = (0, a) \times (0, b)$, vanishing on the intersections of $\partial\Pi$ with the axes, there is valid the inequality $\int_\Pi F^2 dx \leq c \int_\Pi \left(\partial^2 F / \partial x_1 \partial x_2\right)^2 dx$ with $c \leq 16a^2b^2/\pi^4$, from where and (10) it also follows that

$$\lfloor u - \tilde{u} \rfloor_U \leq \lfloor Y - \Phi \rfloor_M + \left\{ \frac{c}{\mathbb{D}} \int_{\pi_1} [f(x) - L_M Y(x)]^2\, dx \right\}^{1/2}. \quad (11)$$

The bound is only by the constant different from the popular bound, found in Neittaanmaki & Repin [5], in which $c = c_\Omega$ is the depending only on the domain constant from the Friedrichs type inequality. It is assumed, as is often realized in practice, that the appropriate Y is found by minimization procedure applied to the right part of (11).

The bound (8), (9) has the advantages summarized as follows.

i) They do not contain constants beside one naturally entering the energy norm.

ii) The minimization of the right parts is done with respect only of two components of Y, and, therefore, the dimension of the system of algebraic equations to be solved is reduced by $\sim 1/3$.

iii) (8), (9) contain on the right only first derivatives of $Y_{k,k}$, $k = 1, 2$, implying that the finite elements of the class C can be used for definition of $Y_{k,k}$.

iv) Under the integral over π_1 we have additional one-dimensional integrals of $\partial Y_{k,k}/\partial x_k$, which can additionally improve accuracy.

1.3 The use of smoothed finite element solutions

Suppose the approximate solution $\tilde{u}$ is the finite element solution, $u \in H^{p+1}(\Omega),\ p \geq 3$, and we have the a priori convergence estimates

$$\|(u - \tilde{u})\|_k \leq \hat{c}_{k,l} h^{l-k} \|u\|_l\,, \quad k = 0, 1, 2, \quad 2 \leq l \leq p+1, \quad \hat{c}_{k,l} = \text{const}\,. \quad (12)$$

This assumes existence in $\overset{\circ}{V}(\Omega)$ of such approximations $\tilde{u}_{\text{A}}$ of u that

$$\|(u - \tilde{u}_{\text{A}})\|_k \leq c_{k,l} h^{l-k} \|u\|_l\,, \quad c_{k,l} = \text{const}\,. \quad (13)$$

The minimization of the right parts of the a posteriori bounds can be circumvented if, before the equilibration, the FEM solution is subjected to some procedure of smoothing, as it is often done in many other approaches to the a posteriori error estimation, see [3, 4]. For instance, one can use the bound (6) with the equilibrated vector Z, in which $Z_{k,k} = \tilde{\Phi}_{k,k},\ k = 1, 2$, where $\tilde{\Phi}_{k,k}$ are smoothed fields $\Phi_{k,k}$, whereas $Z_{1,2}$ is defined from (5) with $\Phi_{k,k}$ replaced by $\tilde{\Phi}_{k,k}$. This transforms the estimate (9) into

$$\lfloor u - \tilde{u} \rfloor_U \leq \Psi(h, u), \quad \Psi(h, u) = \text{where}$$
$$= \left\{ \int_{\pi_1} \left[(\Lambda_{1,1})^2 + (\Lambda_{2,2})^2 - 2\nu \Lambda_{1,1} \Lambda_{2,2} \right] dx \right\}^{1/2} + \Xi(\Phi_{1,2}, f, \tilde{\Phi}_{k,k}), \quad (14)$$

where $\Lambda_{k,k} = \Phi_{k,k} - \tilde{\Phi}_{k,k}$. One of the options is conveniently to define the components $\tilde{\Phi}_{k,k}$ as functions of a finite element space $V(\pi_1)$ of the class C^1 or even of the much simpler class C. In particular, it can be the same finite element space, which is used for solving the problem in (1). For each node $x^{(i)} \in \overline{\pi}_1$, the nodal parameters, *i.g.*, values of the moments and, if necessary, their derivatives, are evaluated by the averaging with weights of the corresponding values of $\Phi_{k,k}$ and their derivatives at $x^{(i)}$ over all finite elements having $x^{(i)}$ for the node. We will call the procedure of obtaining of smothed moment $\tilde{\Phi}_{k,l}$ *consistent*, if for $\tilde{\Phi}_{k,l}$ and $\Phi_{k,l}$ the same in the order of smallness estimates of convergence hold.

Proposition 1. *Suppose the FE mesh is a square mesh of size h, $u \in H^{p+1}(\Omega)$ and the convergence estimates (12) hold. Suppose also that the procedure of obtaining of smoothed moments $\tilde{\Phi}_{k,k}$ is consistent. Then $\Psi(h, u)$ satisfies the inequality $\Psi(h, u) \leq ch^{p-5/2} |u|_{H^{p+1}(\Omega)}$ and can be calculated for $\mathcal{O}(N)$ arithmetic operations, where N is the dimension of the FE space.*

The proof containing application of standard well known bounds is omitted. The multiplier $h^{1/2}$ in the boung of $\Psi(h, u)$ is lost due to the presence in it boundary integrals. Also it is worth mentioning that it is completed without use of the super-convergence property of finite element solutions, which under some additional conditions can take place and allows to improve the multiplier $h^{p-5/2}$ in the above estimate up to $h^{p-3/2}$. If the mesh is not rectangular, the computational cost of $Z_{1,2}$ can became super-linear with respect to N, even in the case of the quasiuniform mesh. This can be caused by the $1d$ numerical integration procedures. However, there exist some techniques, not discussed in this paper, which can be implemented to reduce the computational cost.

2 Consistent functional error majorant

The most desirable a posteriori error bound provides the exact order of accuracy, *i.e.*, not improvable in the order of smallness for the particular numerical method, used for solution of the boundary value problem. In this section, we will present such an a posteriori error bound.
We consider the problem

$$\Delta\Delta u = f(x), \quad x \in \Omega, \qquad u(y) = \frac{\partial u}{\partial n}(y) = 0, \quad y \in \partial\Omega, \tag{15}$$

under the assumptions that the domain Ω with the boundary $\partial\Omega$ are sufficiently smooth and $f \in L_2(\Omega)$. From these assumptions it follows that for any $f \in L_2(\Omega)$, the solution u of the problem (15) satisfies the inequality

$$\|u\|_4 \le c_\Omega \|f\|_0 , \quad c_\Omega = const. \tag{16}$$

We assume that $\tilde{u} \in H^2(\Omega)$ is the approximate solution of the problem by a compatible finite element method, *i.e.*, of the class C^1, which satisfies boundary conditions of (15) and that the finite element assemblage satisfies generalized conditions of quasiuniformity (see [17]) with the mesh parameter h.
To formulate our result we need to introduce the spaces $H^2(\Omega, \Delta\Delta) = \{v \in H^2(\Omega) : \Delta\Delta v \in L_2(\Omega)\}$ and $\mathbf{L}_2(\Omega, L_M) = \{Y \in (L_2(\Omega))^2 : L_M Y \in L_2(\Omega)\}$.

Theorem 1. *Let u be the solutin of the problem and above assumptions be fulfilled. Then the error of the finite element solution of the problem (15) satisfies the a posteriori bounds*

$$\begin{aligned} |u - \tilde{u}|_2^2 &\le ch^4 \|f - \Delta\Delta w\|_0^2 + 2\, |w - \tilde{u}|_2^2 , \\ |u - \tilde{u}|_2^2 &\le ch^4 \|f - L_M Y\|_0^2 + 2 \left[\int_\Omega \sum_{k+l=2} \left(\frac{\partial^2 \tilde{u}}{\partial x_1^k \partial x_2^l} - Y_{k,l}\right)^2\right] , \end{aligned} \tag{17}$$

where w and Y are arbitrary function and vector field from the spaces $H^2(\Omega, \Delta\Delta)$, and $\mathbf{L}_2(\Omega, L_M)$, respectively, and $c = 4c_{2,4}^2 c_\Omega^2$ is a constant with $c_{2,4}$ from (13).

For simplicity, below we give the proof which main steps are quite similar to the proof of theorem 1 and is aimed at quite similar to (17) a posteriori error bound for the finite element solution of the Poisson equation

$$Lu \; = \; -\Delta u = f(x), \; x \in \Omega, \qquad u\,|_{\partial\Omega} = 0. \tag{18}$$

With respect to the problem (18) and its finite element solution three assumptions will be used. i) The domain Ω is sufficiently smooth, $f \in L_2(\Omega)$ and, therefore, for each such f the inequality

$$\|u\|_2 \le c_\Omega \|f\|_0 , \; c_\Omega = const, \tag{19}$$

holds, see Ladyzhenskaya and UralŸtseva [19]. ii) The FEM mesh is a quasi-uniform mesh of size h and the conform FEM solutuion $\tilde{u}$ satisfies boundary condition $\tilde{u}|_{\partial\Omega} = 0$. iii) The solution u of the problem belongs to $H^l(\Omega)$ with some integer l, $1 \le l \le p+1$, and

$$\|u - \tilde{u}_{\mathrm{A}}\|_k \le \underset{\sim}{c}_{k,l} h^{l-k} \|u\|_l , \; k = 0, 1,$$

where p is the order of polynomials in the space on the respective reference element and $\tilde{u}_{\mathrm{A}}$, $\tilde{u}_{\mathrm{A}}|_{\partial\Omega} = 0$, is some finite element approximation of u, *i.e.*, L_2-projection of u upon $\overset{\circ}{V}(\Omega)$ or other.
According to the assumptions and the technique of Sea, see, *e.g.*, [20], the error in the $L_2(\Omega)$-norm satisfies also the inequality

$$\|(u-\tilde{u})\|_0 \le \underset{\sim}{c}_{1,2} c_\Omega h \,\|u-\tilde{u}\|_1 = c_{1,2,\Omega} h \,\|u-\tilde{u}\|_1\,,\; c_{1,2,\Omega} = \underset{\sim}{c}_{1,2} c_\Omega = \text{const.} \quad (20)$$

The subsidiary problem

$$Lu = -\Delta u + \sigma u = f_1(x),\; x \in \Omega,\; u\,|_{\partial\Omega} = 0, \quad (21)$$

with $f_1 = f + \sigma u$ and an arbitrary positive number σ, obviously, has the same solution with (18). According to (18), (19) and Theorem 22 of Aubin [20],

$$\|u-\tilde{u}\|_1^2 + \sigma\,\|u-\tilde{u}\|_0^2 \le \sigma^{-1}\,\|f_1 - \sigma\tilde{u} - \Delta w\|_0^2 + \|u-w\|_1^2 = \sigma^{-1}\,\|f - \sigma(u-\tilde{u}) - \Delta w\|_0^2 + \|\tilde{u}-w\|_1^2, \quad (22)$$

where,

$$\sigma^{-1}\,\|f - \sigma(u-\tilde{u}) - \Delta w\|_0^2 = \sigma^{-1}\,\|f-\Delta w\|_0^2 + \sigma\,\|(u-\tilde{u})\|_0^2 + 2\int_\Omega (u-\tilde{u})(f-\Delta w)dx, \quad (23)$$

and, therefore, for any $\varepsilon{>}0$

$$2\int_\Omega (u-\tilde{u})(f-\Delta w)dx \le \varepsilon\,\|(u-\tilde{u})\|_0^2 + \frac{1}{\varepsilon}\,\|f-\Delta w\|_0^2\,. \quad (24)$$

If to set $\varepsilon = 0.5\;\|(u-\tilde{u})\|_1^2/\|u-w\|_0^2$ and to take into account (11), one comes to the inequality $1/\varepsilon \le 2\;c_{1,2,\Omega}^2 h^2$, which, when combined with (22) – (24), yields

$$0.5\,\|u-\tilde{u}\|_1^2 \le [\sigma^{-1}{+}2c_{1,2,\Omega}^2 h^2]\,\|f-\Delta w\|_0^2{+}\|\tilde{u}-w\|_1^2 \le ch^2\,\|f-\Delta w\|_0^2{+}\|\tilde{u}-w\|_1^2$$

Since σ can be any positive number, this inequality above approves the bound

$$\|u-\tilde{u}\|_1^2 \le ch^2\,\|f-\Delta w\|_0^2 + 2\,\|\tilde{u}-w\|_1^2,\; c = 4c_{1,2,\Omega}^2\;. \quad (25)$$

The a posteriori bound (17) possesses several good properties. Suppose that the a priori error estimates (12) hold. If to take for w a properly smoothed finite element solution, which is denoted as $\tilde{u}_{sm}$, then the right part of (17) has the same order h^{p-k+1} as the right part of the a priori error estimate (12), assuming $u \in H^{p+1}(\Omega)$, $p \ge 3$. Therefore, the bound (17) is consistent as well as the bound (25). They greatly improve the accuracy of the similar type posteriori bounds obtained for the approximate solutions of 2^{nd} and 4^{th} order elliptic equations earlier [4, 8, 10]. The reason is in the appearance of the additional multipliers h^2 and h^4 before the L_2-norms in (25), (17), respectively, instead of constants in the earlier bounds.

3 Concluding remarks

L_2-norms of the residual type terms multiplied by constants are a common place in the a posteriori error majorants of approximate solutions of the elliptic equations. In the case of the thin plate bending problem this term depends on the 2^{th}-order derivatives of the testing vector of moments. In the paper we obtained the majorant in which this term is replaced by the other one depending only on the first derivatives. This considerably simplifies the numerical realization of the error majorant and at the same time essentially improves accuracy. Additionally we obtained the majorant with the same as in the papers of other authors L_2-norm of the residual type term, but in our case with the additional multiplier $\mathcal{O}(h^2)$ instead of a constant. This improves the accuracy of the majorant in the two orders of h making the majorant consistent with the a priori error bounds. At the derivation of these majorant, only general properties of solutions of the finite element method and elliptic boundary value problems were used. For this reason, there are no difficulties to generalize such error bounds to a wider range of finite element methods and elliptic equations [22, 21]. In particular, generalization of the consistent a posteriori error majorants on the elliptic equations of the order $2n$ is considered in the forthcoming paper [21]. I addition, we note that there exits another proof which results in the bounds (17), (25) with simpler constants. Namely, constants c in (17), (25) can be replaced by $c_{0,2}$ and $\underline{c}_{0,1}$, respectively.
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In general, any smooth w(x), $x \in \pi_1$, can be represented as

$$w(x) = \int_0^{x_2} \int_0^{x_1} \frac{\partial^2 w}{\partial x_1 \partial x_2}(\eta) d\eta + w(0, x_2) + w(x_1, 0) - w(0,0). \tag{26}$$

and, therefore,

$$Z_{1,2} = \frac{1}{2} \int_0^{x_2} \int_0^{z_1} \left[f(\eta) d\eta - \sum_{k=1,2} (\frac{\partial^2 \Phi_{1,1}}{\partial \eta_1^2}(\eta)) \right] d\eta + \varphi_1(x_2) + \varphi_2(x_1) + c_\circ, \tag{27}$$

where $\varphi_k(x_{3-k})$ and $c_\circ$ are arbitrary functions and a number, respectively. However, from the boundary condition in (1), it follows that $M_{1,2} = 0$ on $\partial \pi_1$, and it can be accepted $Z_{1,2} = 0$ on the set Γ_0, which is the union of the left and the lower edges of π_1 and the point (0,0). As the result of this one can take $\varphi_k(x_{3-k}) \equiv 0$, $c_\circ = 0$ and set

$$\begin{aligned} Z_{1,2} = \frac{1}{2} \Bigg\{ \int_0^{x_2} \int_0^{z_1} f(\eta) d\eta - \int_0^{x_2} \left[\frac{\partial \Phi_{1,1}}{\partial x_1}(x_1, \eta_2) - \frac{\partial \Phi_{1,1}}{\partial x_1}(0, \eta_2) \right] d\eta_2 \\ - \int_0^{x_1} \left[\frac{\partial \Phi_{2,2}}{\partial x_2}(\eta_1, x_2) - \frac{\partial \Phi_{2,2}}{\partial x_2}(\eta_1, 0) \right] d\eta_1 \Bigg\}. \end{aligned} \tag{28}$$