Cases of integrability corresponding to the motion of a pendulum in the four-dimensional space

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Abstract

In this activity, we systematize some results on the study of the equations of a motion of dynamically symmetric four-dimensional fixed rigid bodies-pendulums located in a nonconservative force fields. The form of these equations is taken from the dynamics of real fixed rigid bodies placed in a homogeneous flow of a medium. In parallel, we study the problem of a motion of a free four-dimensional rigid body also located in a similar force fields. Herewith, this free rigid body is influenced by a nonconservative tracing force; under action of this force, either the magnitude of the velocity of some characteristic point of the body remains constant, which means that the system possesses a nonintegrable servo constraint, or the center of mass of the body moves rectilinearly and uniformly; this means that there exists a nonconservative couple of forces in the system.

1 Introduction

Earlier (see [1, 2]), the author already proved the complete integrability of the equations of a plane-parallel motion of a fixed rigid body–pendulum in a homogeneous flow of a medium under the jet flow conditions when the system of dynamical equations possesses a first integral, which is a transcendental (in the sense of the theory of functions of a complex variable, i.e., it has essential singularities) function of quasi-velocities. It was assumed that the interaction of the medium with the body is concentrated on a part of the surface of the body that has the form of a (one-dimensional) plate. In [2, 3], the planar problem was generalized to the spatial (three-dimensional) case, where the system of dynamical equations has a complete set of transcendental first integrals. It was assumed that the interaction of the homogeneous medium flow with the fixed body (the spherical pendulum) is concentrated on a part of the body surface that has the form of a planar (two-dimensional) disk. Later on (see [4, 5]), the equations of motion of the fixed dynamically symmetric four-dimensional rigid bodies, where the force field is concentrated on a part of the body surface that has the form of a (three-dimensional) disk.
In this activity, the results relate to the case where all interaction of the homogeneous flow of a medium with the fixed body is concentrated on that part of the surface

of the body, which has the form of a three-dimensional disk, and the action of the force is concentrated in a direction perpendicular to this disk. These results are systematized and are presented in invariant form.

2 Model assumptions

Let consider the homogeneous three-dimensional disk $\mathcal{D}^3$ (with the center in the point D), the hyperplane of which perpendicular to the holder OD in the four-dimensional Euclidean space $\mathbf{E}^4$. The disk is rigidly fixed perpendicular to the tool holder OD located on the (generalized) spherical hinge O, and it flows about homogeneous fluid flow. In this case, the body is a physical (generalized spherical) pendulum. The medium flow moves from infinity with constant velocity $\mathbf{v} = \mathbf{v}_\infty \neq \mathbf{0}$. Assume that the holder does not create a resistance.

We suppose that the total force $\mathbf{S}$ of medium flow interaction perpendicular to the disk $\mathcal{D}^3$, and point N of application of this force is determined by at least the angle of attack α, which is made by the velocity vector $\mathbf{v}_D$ of the point D with respect to the flow and the holder OD; the total force is also determined by the angles β_1, β_2, which are made in the hyperplane of the disk $\mathcal{D}^3$ (thus, $(v, \alpha, \beta_1, \beta_2)$ are the (generalized) spherical coordinates of the tip of the vector $\mathbf{v}_D$), and also the reduced angular velocity tensor $\tilde{\omega} \cong l\tilde{\Omega}/v_D$, $v_D = |\mathbf{v}_D|$ (l is the length of the holder, $\tilde{\Omega}$ is the angular velocity tensor of the pendulum). Such conditions generalize the model of streamline flow around spatial bodies [3, 5, 6].

The vector $\mathbf{e} = \mathbf{OD}/l$ determines the orientation of the holder. Then $\mathbf{S} = s(\alpha)v_D^2\mathbf{e}$, where $s(\alpha) = s_1(\alpha)\mathrm{sign}\cos\alpha$, and the resistance coefficient $s_1 \geq 0$ depends only on the angle of attack α. By the axe-symmetry properties of the body–pendulum with respect to the point D, the function $s(\alpha)$ is even.

Let $Dx_1x_2x_3x_4$ be the coordinate system rigidly attached to the body, herewith, the axis Dx_1 has a direction vector $\mathbf{e}$, and the axes Dx_2, Dx_3 and Dx_4 lie in the hyperplane of the disk $\mathcal{D}^3$.

By the angles (ξ, η_1, η_2), we define the position of the holder OD in the four-dimensional space $\mathbf{E}^4$. In this case, the angle ξ is made by the holder and the direction of the over-running medium flow. In other words, the angles introduced are the (generalized) spherical coordinates of the point D of the center of a disk $\mathcal{D}^3$ on the three-dimensional sphere of the constant radius OD.

The space of positions of this (generalized) spherical (physical) pendulum is the three-dimensional sphere

$$\mathbf{S}^3\{(\xi, \eta_1, \eta_2) \in \mathbf{R}^3 :\ 0 \leq \xi, \eta_1 \leq \pi,\ \eta_2 \bmod 2\pi\}, \tag{1}$$

and its phase space is the tangent bundle of the three-dimensional sphere

$$T_*\mathbf{S}^3\{(\dot{\xi}, \dot{\eta}_1, \dot{\eta}_2; \xi, \eta_1, \eta_2) \in \mathbf{R}^6 :\ 0 \leq \xi, \eta_1 \leq \pi,\ \eta_2 \bmod 2\pi\}. \tag{2}$$

The tensor (of the second-rank) $\tilde{\Omega}$ of the angular velocity in the coordinate system

$Dx_1x_2x_3x_4$, we define through the skew-symmetric matrix

$$\tilde{\Omega} = \begin{pmatrix} 0 & -\omega_6 & \omega_5 & -\omega_3 \\ \omega_6 & 0 & -\omega_4 & \omega_2 \\ -\omega_5 & \omega_4 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}, \ \tilde{\Omega} \in \mathrm{so}(4). \tag{3}$$

The distance from the center D of the disk $\mathcal{D}^3$ to the center of pressure (the point N) has the form $|\mathbf{r}_N| = r_N = DN\left(\alpha, \beta_1, \beta_2, l\Omega/v_D\right)$, where $\mathbf{r}_N = \{0, x_{2N}, x_{3N}, x_{4N}\}$ in system $Dx_1x_2x_3x_4$ (we omit the wave over Ω).

3 Set of dynamical equations in Lie algebra so(4)

Let a four-dimensional rigid body Θ of mass m with smooth three-dimensional boundary $\partial\Theta$ be under the influence of a nonconservative force field; this can be interpreted as a motion of the body in a resisting medium that fills up the four-dimensional domain of Euclidean space $\mathbf{E}^4$. We assume that the body is dynamically symmetric. In this case, there are two logical possibilities of the representation of its inertia tensor in the case of existence of *two* independent equations on the principal moments of inertia; i.e., either in some coordinate system $Dx_1x_2x_3x_4$ attached to the body, the operator of inertia has the form

$$\mathrm{diag}\{I_1, I_2, I_2, I_2\}, \tag{4}$$

or the form $\mathrm{diag}\{I_1, I_1, I_3, I_3\}$. In the first case, the body is dynamically symmetric in the hyperplane $Dx_2x_3x_4$ and in the second case, the two-dimensional planes Dx_1x_2 and Dx_3x_4 are planes of dynamical symmetry of the body.

The configuration space of a free, n-dimensional rigid body is the direct product $\mathbf{R}^n \times \mathrm{SO}(n)$ of the space $\mathbf{R}^n$, which defines the coordinates of the center of mass of the body, and the rotation group $\mathrm{SO}(n)$, which defines the rotations of the body about its center of mass and has dimension $n + n(n-1)/2 = n(n+1)/2$.

Respectively, the dimension of the phase space is equal to $n(n+1)$.

In particular, if Ω is the tensor of angular velocity of a four-dimensional rigid body (it is a second-rank tensor, see [3, 6, 7, 8]), $\Omega \in \mathrm{so}(4)$, then *the part of the dynamical equations of motion corresponding to the Lie algebra* so(4) has the following form (see [9, 10, 11, 12]):

$$\dot{\Omega}\Lambda + \Lambda\dot{\Omega} + [\Omega,\ \Omega\Lambda + \Lambda\Omega] = M, \tag{5}$$

$$\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\},\ \lambda_1 = \frac{-I_1 + I_2 + I_3 + I_4}{2},$$

$$\lambda_2 = \frac{I_1 - I_2 + I_3 + I_4}{2},\ \lambda_3 = \frac{I_1 + I_2 - I_3 + I_4}{2},\ \lambda_4 = \frac{I_1 + I_2 + I_3 - I_4}{2},$$

$M = M_F$ is the natural projection of the moment of external forces $\mathbf{F}$ acting on the body in $\mathbf{R}^4$ on the natural coordinates of the Lie algebra so(4) and $[.,.]$ is the commutator in so(4). The skew-symmetric matrix corresponding to this second-rank tensor $\Omega \in \mathrm{so}(4)$ we represent in the form (3), where $\omega_1,\ \omega_2,\ \omega_3,\ \omega_4,\ \omega_5,\ \omega_6$ are

the components of the tensor of angular velocity corresponding to the projections on the coordinates of the Lie algebra so(4).
In this case, obviously, the following relations hold: $\lambda_i - \lambda_j = I_j - I_i$ for any $i, j = 1, \dots, 4$.
For the calculation of the moment of an external force acting on the body, we need to construct the mapping $\mathbf{R}^4 \times \mathbf{R}^4 \longrightarrow \mathrm{so}(4)$, than maps a pair of vectors $(\mathbf{DN}, \mathbf{F}) \in \mathbf{R}^4 \times \mathbf{R}^4$ from $\mathbf{R}^4 \times \mathbf{R}^4$ to an element of the Lie algebra so(4), where $\mathbf{DN} = \{0, x_{2N}, x_{3N}, x_{4N}\}$, $\mathbf{F} = \{F_1, F_2, F_3, F_4\}$, and $\mathbf{F}$ is an external force acting on the body. For this end, we construct the following auxiliary matrix

$$\begin{pmatrix} 0 & x_{2N} & x_{3N} & x_{4N} \\ F_1 & F_2 & F_3 & F_4 \end{pmatrix}.$$

Then the right-hand side of system (5) takes the form

$$M = \{M_1, M_2, M_3, M_4, M_5, M_6\} =$$

$$= \{x_{3N}F_4 - x_{4N}F_3, x_{4N}F_2 - x_{2N}F_4, -x_{4N}F_1, x_{2N}F_3 - x_{3N}F_2, x_{3N}F_1, -x_{2N}F_1\},$$

where M_1, M_2, M_3, M_4, M_5, M_6 are the components of tensor of the moment of external forces in the projections on the coordinates in the Lie algebra so(4),

$$M = \begin{pmatrix} 0 & -M_6 & M_5 & -M_3 \\ M_6 & 0 & -M_4 & M_2 \\ -M_5 & M_4 & 0 & -M_1 \\ M_3 & -M_2 & M_1 & 0 \end{pmatrix}.$$

In our case of a fixed pendulum, the case (4) is realized. Then the dynamical part of the equations of its motion has the following form:

$$\begin{aligned}
&(I_1 + I_2)\dot{\omega}_1 = 0,\ (I_1 + I_2)\dot{\omega}_2 = 0,\\
&2I_2\dot{\omega}_3 + (I_1 - I_2)(\omega_2\omega_6 + \omega_1\omega_5) = x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)v^2,\\
&(I_1 + I_2)\dot{\omega}_4 = 0,\\
&2I_2\dot{\omega}_5 + (I_1 - I_2)(\omega_4\omega_6 - \omega_1\omega_3) = -x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)v^2,\\
&2I_2\dot{\omega}_6 + (I_2 - I_1)(\omega_4\omega_5 + \omega_2\omega_3) = x_{2N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)v^2,
\end{aligned} \tag{6}$$

since the moment of the medium interaction force is determined by the following auxiliary matrix:

$$\begin{pmatrix} 0 & x_{2N} & x_{3N} & x_{4N} \\ -s(\alpha)v_D^2 & 0 & 0 & 0 \end{pmatrix},$$

where $\{-s(\alpha)v_D^2, 0, 0, 0\}$ is the decomposition of the force $\mathbf{S}$ of medium interaction in the coordinate system $Dx_1x_2x_3x_4$.
Since the dimension of the Lie algebra so(4) is equal to 6, the system of equations (6) is a group of dynamical equations on so(4), and, simply speaking, the motion equations.

We see, that in the right-hand side of Eq. (6), first of all, it includes the angles α, β_1, β_2, therefore, this system of equations is not closed. In order to obtain a complete system of equations of motion of the pendulum, it is necessary to attach several sets of kinematic equations to the dynamic equations on the Lie algebra so(4).

3.1 Cyclic first integrals

We immediately note that the system (6), by the existing dynamic symmetry

$$I_2 = I_3 = I_4, \tag{7}$$

possesses three cyclic first integrals

$$\omega_1 \equiv \omega_1^0 = \text{const},\ \omega_2 \equiv \omega_2^0 = \text{const},\ \omega_4 \equiv \omega_4^0 = \text{const}. \tag{8}$$

In this case, further, we consider the dynamics of our system at zero levels:

$$\omega_1^0 = \omega_2^0 = \omega_4^0 = 0. \tag{9}$$

Under conditions (7)–(9) the system (6) has the form of unclosed system of three equations:

$$\begin{gathered} 2I_2\dot{\omega}_3 = x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)v^2,\ 2I_2\dot{\omega}_5 = -x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)v^2, \\ 2I_2\dot{\omega}_6 = x_{2N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)v^2. \end{gathered} \tag{10}$$

4 First set of kinematic equations

In order to obtain a complete system of equations of motion, it needs the set of kinematic equations which relate the velocities of the point D (i.e., the center of the disk $\mathcal{D}^3$) and the over-running medium flow:

$$\mathbf{v}_D = v_D \cdot \mathbf{i}_v(\alpha, \beta_1, \beta_2) = \tilde{\Omega}\mathbf{l} + (-v_\infty)\mathbf{i}_v(-\xi, \eta_1, \eta_2),\ \mathbf{l} = \{l, 0, 0, 0\}, \tag{11}$$

$$\mathbf{i}_v(\alpha, \beta_1, \beta_2) = \begin{pmatrix} \cos\alpha \\ \sin\alpha\cos\beta_1 \\ \sin\alpha\sin\beta_1\cos\beta_2 \\ \sin\alpha\sin\beta_1\sin\beta_2 \end{pmatrix}. \tag{12}$$

The equation (11) expresses the theorem of addition of velocities in projections on the related coordinate system $Dx_1x_2x_3x_4$.

Indeed, the left-hand side of Eq. (11) is the velocity of the point D of the pendulum with respect to the flow in the projections on the related with the pendulum coordinate system $Dx_1x_2x_3x_4$. Herewith, the vector $\mathbf{i}_v(\alpha, \beta_1, \beta_2)$ is the unit vector along the axis of the vector $\mathbf{v}_D$. The vector $\mathbf{i}_v(\alpha, \beta_1, \beta_2)$ has the spherical coordinates $(1, \alpha, \beta_1, \beta_2)$ which determines the decomposition (12).

The right-hand side of the Eq. (11) is the sum of the velocities of the point D when you rotate the pendulum (the first term), and the motion of the flow (the second term). In this case, in the first term, we have the coordinates of the vector вектора $\mathbf{OD} = \{l, 0, 0, 0\}$ in the coordinate system $Dx_1x_2x_3x_4$.
We explain the second term of the right-hand side of Eq. (11) in more detail. We have in it the coordinates of the vector $(-\mathbf{v}_\infty) = \{-v_\infty, 0, 0, 0\}$ in the immovable space. In order to describe it in the projections on the related coordinate system $Dx_1x_2x_3x_4$, we need to make a (reverse) rotation of the pendulum at the angle $(-\xi)$ that is algebraically equivalent to multiplying the value $(-v_\infty)$ on the vector $\mathbf{i}_v(-\xi, \eta_1, \eta_2)$.
Thus, the first set of kinematic equations (11) has the following form in our case:

$$\begin{aligned} &v_D \cos\alpha = -v_\infty \cos\xi,\ v_D \sin\alpha \cos\beta_1 = l\omega_6 + v_\infty \sin\xi \cos\eta_1, \\ &v_D \sin\alpha \sin\beta_1 \cos\beta_2 = -l\omega_5 + v_\infty \sin\xi \sin\eta_1 \cos\eta_2, \\ &v_D \sin\alpha \sin\beta_1 \sin\beta_2 = l\omega_3 + v_\infty \sin\xi \sin\eta_1 \sin\eta_2. \end{aligned} \tag{13}$$

5 Second set of kinematic equations

We also need a set of kinematic equations which relate the angular velocity tensor $\tilde{\Omega}$ and coordinates $\dot{\xi}, \dot{\eta_1}, \dot{\eta_2}, \xi, \eta_1, \eta_2$ of the phase space (2) of pendulum studied, i.e., the tangent bundle $T_*\mathbf{S}^3\{\dot{\xi}, \dot{\eta_1}, \dot{\eta_2}; \xi, \eta_1, \eta_2\}$.
We draw the reasoning style allowing arbitrary dimension. The desired equations are obtained from the following two sets of relations. Since the motion of the body takes place in a Euclidean space $\mathbf{E}^n, n = 4$ formally, at the beginning, we express the tuple consisting of a phase variables $\omega_3, \omega_5, \omega_6$, through new variable z_1, z_2, z_3 (from the tuple z). For this, we draw the following turn by the angle η_1, η_2:

$$\begin{pmatrix} \omega_3 \\ \omega_5 \\ \omega_6 \end{pmatrix} = T_{1,2}(\eta_2) \circ T_{2,3}(\eta_1) \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}, \tag{14}$$

$$T_{2,3}(\eta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\eta_1 & -\sin\eta_1 \\ 0 & \sin\eta_1 & \cos\eta_1 \end{pmatrix},\ T_{1,2}(\eta_2) = \begin{pmatrix} \cos\eta_2 & -\sin\eta_2 & 0 \\ \sin\eta_2 & \cos\eta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In other words, the relations

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = T_{2,3}(-\eta_1) \circ T_{1,2}(-\eta_2) \begin{pmatrix} \omega_3 \\ \omega_5 \\ \omega_6 \end{pmatrix}$$

hold, i.e.,

$$\begin{aligned} z_1 &= \omega_3 \cos\eta_1 + \omega_5 \sin\eta_2, \\ z_2 &= -\omega_3 \cos\eta_1 \sin\eta_2 + \omega_5 \cos\eta_1 \cos\eta_2 + \omega_6 \sin\eta_1, \\ z_3 &= \omega_3 \sin\eta_1 \sin\eta_2 - \omega_5 \sin\eta_1 \cos\eta_2 + \omega_6 \cos\eta_1. \end{aligned}$$

Then we substitute the following relationship instead of the variable z:

$$z_3 = \dot{\xi},\ z_2 = -\dot{\eta_1}\frac{\sin\xi}{\cos\xi},\ z_1 = \dot{\eta_2}\frac{\sin\xi}{\cos\xi}\sin\eta_1. \tag{15}$$

Thus, two sets of Eqs. (14) and (15) give the second set of kinematic equations:

$$\begin{aligned} \omega_3 &= \dot{\xi}\sin\eta_1\sin\eta_2 + \dot{\eta}_1\frac{\sin\xi}{\cos\xi}\cos\eta_1\sin\eta_2 + \dot{\eta}_2\frac{\sin\xi}{\cos\xi}\sin\eta_1\cos\eta_2, \\ \omega_5 &= -\dot{\xi}\sin\eta_1\cos\eta_2 - \dot{\eta}_1\frac{\sin\xi}{\cos\xi}\cos\eta_1\cos\eta_2 + \dot{\eta}_2\frac{\sin\xi}{\cos\xi}\sin\eta_1\sin\eta_2, \\ \omega_6 &= \dot{\xi}\cos\eta_1 - \dot{\eta}_1\frac{\sin\xi}{\cos\xi}\sin\eta_1. \end{aligned} \tag{16}$$

We see that three sets of the relations (10), (13), and (16) form the closed system of equations.
These three sets of equations include the following functions:

$$x_{2N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v_D}\right),\ x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v_D}\right),\ x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v_D}\right),\ s(\alpha).$$

In this case, the function s is considered to be dependent only on α, and the functions x_{2N}, x_{3N}, x_{4N} may depend on, along with the angles α, β_1, β_2, generally speaking, the reduced angular velocity tensor $l\tilde{\Omega}/v_D$.

6 Case where the moment of nonconservative forces depends on the angular velocity

6.1 Dependence on the angular velocity

This section is devoted to dynamics of the four-dimensional rigid body in the four-dimensional space. Since this subsection is devoted to the study of the case of the motion where the moment of forces depends on the angular velocity tensor, we introduce this dependence in the general case; this will allow us to generalize this dependence to multi-dimensional bodies.
Let $x = (x_{1N}, x_{2N}, x_{3N}, x_{4N})$ be the coordinates of the point N of application of a nonconservative force (interaction with a medium) on the three-dimensional disk $\mathcal{D}^3$, and $Q = (Q_1, Q_2, Q_3, Q_4)$ be the components independent of the angular velocity. We introduce only the linear dependence of the functions $(x_{1N}, x_{2N}, x_{3N}, x_{4N})$ on the angular velocity tensor Ω since the introduction of this dependence itself is not a priori obvious (see [1, 3, 5]).
Thus, we accept the following dependence: $x = Q + R$, where $R = (R_1, R_2, R_3, R_4)$ is a vector-valued function containing the angular velocity tensor Ω. Here, the dependence of the function R on the angular velocity is gyroscopic:

$$R = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} = -\frac{1}{v_D}\begin{pmatrix} 0 & -\omega_6 & \omega_5 & -\omega_3 \\ \omega_6 & 0 & -\omega_4 & \omega_2 \\ -\omega_5 & \omega_4 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix},$$

where (h_1, h_2, h_3, h_4) are certain positive parameters (comp. with [2, 4]).

Now, for our problem, since $x_{1N} = x_N \equiv 0$, we have

$$x_{2N} = Q_2 - h_1\frac{\omega_6}{v_D},\ x_{3N} = Q_3 + h_1\frac{\omega_5}{v_D},\ x_{4N} = Q_4 - h_1\frac{\omega_3}{v}.$$

Thus, the function $\mathbf{r}_N$ is selected in the following form (the disk $\mathcal{D}^3$ is defined by the equation $x_{1N} \equiv 0$):

$$\mathbf{r}_N = \begin{pmatrix} 0 \\ x_{2N} \\ x_{3N} \\ x_{4N} \end{pmatrix} = R(\alpha)\mathbf{i}_N - \frac{1}{v_D}\tilde{\Omega}h, \tag{17}$$

$$\mathbf{i}_N = \mathbf{i}_v\left(\frac{\pi}{2}, \beta_1, \beta_2\right),\ h = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix},\ \tilde{\Omega} = \begin{pmatrix} 0 & -\omega_6 & \omega_5 & -\omega_3 \\ \omega_6 & 0 & -\omega_4 & \omega_2 \\ -\omega_5 & \omega_4 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

(see (3), (12)).
Thus, the following relations

$$x_{2N} = R(\alpha)\cos\beta_1 - h_1\omega_6/v_D,\ x_{3N} = R(\alpha)\sin\beta_1\cos\beta_2 + h_1\omega_5/v_D,$$

$$x_{4N} = R(\alpha)\sin\beta_1\sin\beta_2 - h_1\omega_3/v_D,$$

hold, which show that an additional dependence of the damping (or accelerating in some domains of the phase space) moment of the nonconservative forces is also present in the system considered (i.e., the moment depends on the angular velocity tensor).
And so, for the construction of the force field, we use the pair of dynamical functions $R(\alpha), s(\alpha)$; the information about them is of a qualitative nature. Similarly to the choice of the Chaplygin analytical functions (see [1, 2]), we take the dynamical functions s and R as follows:

$$R(\alpha) = A\sin\alpha,\ s(\alpha) = B\cos\alpha,\ A, B > 0. \tag{18}$$

6.2 Reduced systems

Theorem 6.1. *The simultaneous equations (6), (13), (16) under conditions (7)–(9), (17), (18) can be reduced to the dynamical system on the tangent bundle (2) of the three-dimensional sphere (1).*

Indeed, if we introduce the dimensionless parameters and the differentiation by the formulas

$$b_* = ln_0,\ n_0^2 = \frac{AB}{2I_2},\ H_{1*} = \frac{h_1 B}{2I_2 n_0},\ <\cdot>= n_0 v_\infty <'>, \tag{19}$$

then the obtained equations have the following form ($b_* > 0,\ H_{1*} > 0$):

$$\begin{gathered}
\xi'' + (b_* - H_{1*})\xi'\cos\xi + \sin\xi\cos\xi - [\eta_1'^2 + \eta_2'^2\sin^2\eta_1]\frac{\sin\xi}{\cos\xi} = 0, \\
\eta_1'' + (b_* - H_{1*})\eta_1'\cos\xi + \xi'\eta_1'\frac{1+\cos^2\xi}{\cos\xi\sin\xi} - \eta_2'^2\sin\eta_1\cos\eta_1 = 0, \\
\eta_2'' + (b_* - H_{1*})\eta_2'\cos\xi + \xi'\eta_2'\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + 2\eta_1'\eta_2'\frac{\cos\eta_1}{\cos\eta_1} = 0.
\end{gathered} \tag{20}$$

After the transition from the variables z (about the variables z see (15)) to the intermediate dimensionless variables w

$$z_k = n_0 v_\infty(1 + b_* H_{1*})Z_k,\ k = 1,2,\ z_3 = n_0 v_\infty(1 + b_* H_{1*})Z_3 - n_0 v_\infty b_* \sin\xi,$$

system (20) is equivalent to the system

$$\xi' = (1 + b_* H_{1*})Z_3 - b_*\sin\xi, \tag{21}$$

$$Z_3' = -\sin\xi\cos\xi + (1 + b_* H_{1*})(Z_1^2 + Z_2^2)\frac{\cos\xi}{\sin\xi} + H_{1*}Z_3\cos\xi, \tag{22}$$

$$Z_2' = -(1 + b_* H_{1*})Z_2 Z_3\frac{\cos\xi}{\sin\xi} - (1 + b_* H_{1*})Z_1^2\frac{\cos\xi}{\sin\xi}\frac{\cos\eta_1}{\sin\eta_1} + H_{1*}Z_2\cos\xi, \tag{23}$$

$$Z_1' = -(1 + b_* H_{1*})Z_1 Z_3\frac{\cos\xi}{\sin\xi} + (1 + b_* H_{1*})Z_1 Z_2\frac{\cos\xi}{\sin\xi}\frac{\cos\eta_1}{\sin\eta_1} + H_{1*}Z_1\cos\xi, \tag{24}$$

$$\eta_1' = -(1 + b_* H_{1*})Z_2\frac{\cos\xi}{\sin\xi}, \tag{25}$$

$$\eta_2' = (1 + b_* H_{1*})Z_1\frac{\cos\xi}{\sin\xi\sin\eta_1}, \tag{26}$$

on the tangent bundle $T_*\mathbf{S}^3\{(Z_3, Z_2, Z_1; \xi, \eta_1, \eta_2) \in \mathbf{R}^6 :\ 0 \le \xi, \eta_1 \le \pi,\ \eta_2 \bmod 2\pi\}$ of the three-dimensional sphere $\mathbf{S}^3\{(\xi, \eta_1, \eta_2) \in \mathbf{R}^3 :\ 0 \le \xi, \eta_1 \le \pi,\ \eta_2 \bmod 2\pi\}$.
We see that the independent fifth-order subsystem (21)–(25) (due to cyclicity of the variable η_2) can be substituted into the sixth-order system (21)–(26) and can be considered separately on its own five-dimensional manifold.

6.3 Complete list of the first integrals

We turn now to the integration of the desired sixth-order system (21)–(26) (without any simplifications, i.e., in the presence of all coefficients).
Similarly, for the complete integration of sixth-order system (21)–(26), in general, we need five independent first integrals. However, after the change of variables

$$w_3 = -Z_3,\ w_2 = \sqrt{Z_2^2 + Z_1^2},\ w_1 = \frac{Z_2}{Z_1}, \tag{27}$$

the system (21)–(26) splits as follows:

$$\left.\begin{aligned}
&\xi' = -(1+b_*H_{1*})w_3 - b_*\sin\xi,\\
&w_3' = \sin\xi\cos\xi - (1+b_*H_{1*})w_2^2\frac{\cos\xi}{\sin\xi} + H_{1*}w_3\cos\xi,\\
&w_2' = (1+b_*H_{1*})w_2w_3\frac{\cos\xi}{\sin\xi} + H_{1*}w_2\cos\xi,
\end{aligned}\right\} \tag{28}$$

$$\left.\begin{aligned}
&w_1' = d_1(w_3,w_2,w_1;\xi,\eta_1,\eta_2)\frac{1+w_1^2}{w_1}\frac{\cos\eta_1}{\sin\eta_1},\\
&\eta_1' = d_1(w_3,w_2,w_1;\xi,\eta_1,\eta_2),
\end{aligned}\right\} \tag{29}$$

$$\eta_2' = d_2(w_3,w_2,w_1;\xi,\eta_1,\eta_2), \tag{30}$$

$$\begin{aligned}
&d_1(w_3,w_2,w_1;\xi,\eta_1,\eta_2) =\\
&= -(1+b_*H_{1*})Z_2(w_3,w_2,w_1)\frac{\cos\xi}{\sin\xi} = \mp\frac{w_1w_2}{\sqrt{1+w_1^2}}\frac{\cos\xi}{\sin\xi},\\
&d_2(w_3,w_2,w_1;\xi,\eta_1,\eta_2) =\\
&= (1+b_*H_{1*})Z_1(w_3,w_2,w_1)\frac{\cos\xi}{\sin\xi\sin\eta_1} = \pm\frac{w_2}{\sqrt{1+w_1^2}}\frac{\cos\xi}{\sin\xi\sin\eta_1},
\end{aligned}$$

in this case $Z_k = Z_k(w_3,w_2,w_1)$, $k=1,2,3$, are the functions by virtue of change (27).

We see that the independent third-order subsystem (28) (which can be considered separately on its own three-dimensional manifold), the independent second-order subsystem (29) (after the change of independent variable) can be substituted into the sixth-order system (28)–(30), and also Eq. (30) on η_2 is separated (due to cyclicity of the variable η_2).

Thus, for the complete integration of the system (28)–(30), it suffices to specify two independent first integrals of system (28), one first integral of system (29), and an additional first integral that "attaches" Eq. (30) (*i.e., only four*).

First, we compare the third-order system (28) with the nonautonomous second-order system

$$\begin{aligned}
\frac{dw_3}{d\xi} &= \frac{\sin\xi\cos\xi - (1+b_*H_{1*})w_2^2\cos\xi/\sin\xi + H_{1*}w_3\cos\xi}{-(1+b_*H_{1*})w_3 - b_*\sin\xi},\\
\frac{dw_2}{d\xi} &= \frac{(1+b_*H_{1*})w_2w_3\cos\xi/\sin\xi + H_{1*}w_2\cos\xi}{-(1+b_*H_{1*})w_3 - b_*\sin\xi}.
\end{aligned} \tag{31}$$

Using the substitution $\tau = \sin\xi$, we rewrite system (31) in the algebraic form:

$$\begin{aligned}
\frac{dw_3}{d\tau} &= \frac{\tau - (1+b_*H_{1*})w_2^2/\tau + H_{1*}w_3}{-(1+b_*H_{1*})w_3 - b_*\tau},\\
\frac{dw_2}{d\tau} &= \frac{(1+b_*H_{1*})w_2w_3/\tau + H_{1*}w_2}{-(1+b_*H_{1*})w_3 - b_*\tau}.
\end{aligned} \tag{32}$$

Further, if we introduce the uniform variables by the formulas $w_3 = u_2\tau,\ w_2 = u_1\tau$, we reduce system (32) to the following form:

$$\begin{aligned} \tau\frac{du_2}{d\tau} &= \frac{(1+b_*H_{1*})(u_2^2-u_1^2)+(b_*+H_{1*})u_2+1}{-(1+b_*H_{1*})u_2-b_*}, \\ \tau\frac{du_1}{d\tau} &= \frac{2(1+b_*H_{1*})u_1u_2+(b_*+H_{1*})u_1}{-(1+b_*H_{1*})u_2-b_*}. \end{aligned} \tag{33}$$

We compare the second-order system (33) with the nonautonomous first-order equation

$$\frac{du_2}{du_1} = \frac{1-(1+b_*H_{1*})(u_1^2-u_2^2)+(b_*+H_{1*})u_2}{2(1+b_*H_{1*})u_1u_2+(b_*+H_{1*})u_1}, \tag{34}$$

which can be easily reduced to the exact differential equation

$$d\left(\frac{(1+b_*H_{1*})(u_2^2+u_1^2)+(b_*+H_{1*})u_2+1}{u_1}\right) = 0.$$

Therefore, Eq. (34) has the following first integral:

$$\frac{(1+b_*H_{1*})(u_2^2+u_1^2)+(b_*+H_{1*})u_2+1}{u_1} = C_1 = \text{const}, \tag{35}$$

which in the old variables has the form

$$\Theta_1(w_3, w_2; \xi) = \\ = \frac{(1+b_*H_{1*})(w_3^2+w_2^2)+(b_*+H_{1*})w_3\sin\xi+\sin^2\xi}{w_2\sin\xi} = C_1 = \text{const}. \tag{36}$$

Then the additional first integral has the following structure:

$$\Theta_2(w_3, w_2; \xi) = G\left(\sin\xi, \frac{w_3}{\sin\xi}, \frac{w_2}{\sin\xi}\right) = C_2 = \text{const}. \tag{37}$$

Thus, we have found two first integrals (36), (37) of the independent third-order system (28). For its complete integrability, it suffices to find one first integral for the system (29), and an additional first integral that "attaches" Eq. (30).
Indeed, the desired first integrals have the following forms:

$$\Theta_3(w_1; \eta_1) = \frac{\sqrt{1+w_1^2}}{\sin\eta_1} = C_3 = \text{const}, \tag{38}$$

$$\Theta_4(w_1; \eta_1, \eta_2) = \eta_2 \pm \operatorname{arctg}\frac{\cos\eta_1}{\sqrt{C_3^2\sin^2\eta_1-1}} = C_4 = \text{const}, \tag{39}$$

in this case, in the left-hand side of Eq. (39), we must substitute instead of C_3 the first integral (38).

Theorem 6.2. *The sixth-order system (28)–(30) possesses the sufficient number (four) of the independent first integrals (36), (37), (38), (39).*

Theorem 6.3. *Three sets of relations (6), (13), (16) under conditions (7)–(9), (17), (18) possess four the first integrals (the complete set), which are the transcendental function (in the sense of complex analysis) and are expressed as a finite combination of elementary functions.*

6.4 Topological analogies

We can present two groups of analogies, which describes the motion of a free body in the presence of a tracking force [1, 10, 11]. Thus, we have the following topological and mechanical analogies in the sense explained above.
(1) A motion of a fixed physical pendulum on a (generalized) spherical hinge in a flowing medium (nonconservative force fields under assumption of additional dependence of the moment of the forces on the angular velocity).
(2) A spatial free motion of a four-dimensional rigid body in a nonconservative force field under a tracing force (in the presence of a nonintegrable constraint under assumption of additional dependence of the moment of the forces on the angular velocity).
(3) A composite motion of a four-dimensional rigid body rotating about its center of mass, which moves rectilinearly and uniformly, in a nonconservative force field under assumption of additional dependence of the moment of the forces on the angular velocity.

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