# On the selection of snapshot computation for Proper Orthogonal Decomposition in structural dynamics

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**Abstract**

The analysis of structural dynamic systems usually involves a large number of finite elements and time steps. In order to save computational resources, model order reduction (MOR) approaches have been developed. The Proper Orthogonal Decomposition (POD) is one MOR technique, which defines from a training stage, so called snapshot computation, a reduced basis in which the dynamic equations may be solved easily and quickly. In this contribution, the efficiency of POD in terms of computational cost and accuracy is investigated depending on the load considered during the training stage for dynamic applications.

## 1 Introduction

In civil or mechanical engineering, dynamic systems are often studied using the finite element method (FEM). This leads to the discrete system of equations $M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = f(t)$, where $x$ is the set of degrees of freedom (dofs) defining the system, i.e. $x$ contains the displacement in each direction for all the nodes and for any time $t$. $f(t)$ is the time depending loading and $M$, $D$, $K$ are the mass matrix, viscous damping matrix and stiffness matrix respectively. They are here considered as constant and symmetric. Despite powerful computational capabilities, some analyses and design problems still cannot be solved within a reasonable computing time using standard methods when the number of dofs $N$ becomes very large. It is then advisable to construct reduced models which approximate the behaviour of the original model by much less dofs while maintaining an acceptable accuracy [12, 10].
Different model order reduction (MOR) approaches have been proposed in the literature [3]. They are based on Galerkin projection onto a subspace of the Sobolev FEM space. This space may depend on the time step $t$ for non-linear problems [12]. The reduced system for the displacement approximation $\tilde{x}$ reads

$$\Phi^T M \Phi \ddot{\tilde{x}}(t) + \Phi^T D \Phi \dot{\tilde{x}}(t) + \Phi^T K \Phi \tilde{x}(t) = \Phi^T f(t), \tag{1}$$

where $\Phi$ is a transformation matrix defining the reduced space and the number of dofs is significantly smaller.
Modal basis contains the natural eigenforms of the structure [9, 4]. As this basis is orthogonal with respect to the scalar product of $M$ and $K$, the dynamic system turns to be diagonal, which reduces drastically the computational cost [12]. The load-dependent Ritz method avoids to compute the eigenvalue problem, which may be costly [7]. Condensation methods, such as Guyan method or dynamic condensation, are explored in [11]. From the comparison between alternative MOR techniques for quasi-static cases [5], or for the frequence response analysis of proportional and non-proportional damped systems [13], Proper Orhogonal Decomposition (POD) appears as an interesting alternative.
POD defines a basis from the result $\{x_T(t)\}$ of a first simulation referred to as the training stage, as summarized in figure 1.

Training stage:

Full computation of a reference problem: $\mathsf{N}$ dofs

↓

SVD (or KLE or PCA) of the reference solution

↓

Choice of the number of $\mathfrak{l}$ modes involved in ROB
depending on the distribution of the singular values

↓

Definition of the reduced basis: $\mathfrak{l}$ POD modes

Reduced Computation:

Computation using Galerkin approach
of the probem of interest projected into ROB

Figure 1: Schematic representation of the POD steps (ROB: Reduced Order Basis)

Using for example singular value decomposition (SVD), a space-time decomposition of the snapshot matrix $X_T(p,t)$ representing the results $\{x_T(t)\}$ provides a set of left singular vectors $L$ and right singular vectors $R$ [5] as $X = (x_1, x_2, ..x_n) = L\lambda R^T$. $\lambda$ is pseudo diagonal, $L$ and $R$ are orthogonal. By normalizing the singular vectors, this decomposition is unique. $L$ and $R$ describe the space and time dependences of the solution respectively. As the space part of the problem is generally the most computationally costly, POD is based on the space matrix considering $l$ vectors of the matrix $L$. The decomposition could also be provided by principal component analysis (PCA) or by Karhunen-Loève expansion (KLE) [8].
Once the number $l$ of space modes has been chosen depending on the required accuracy [6], the reduced problem is computed on the basis of POD modes. POD coefficients which represent the time dependence are computed, while the space dependence of the deformation of the structure is described by the POD modes. Updating POD basis during the computation has been proposed by several authors e.g. [1]. Some drawbacks of POD is that the full model still needs to be computed during the training stage, and that the accuracy of the computation largely depends on the training stage characteristics such as its loading or boundary conditions.
An open question is which training problem has to be considered to establish the POD basis for the problem of interest. This question includes time interval to be

considered for the training stage, the loading with respect to time as well as the position of the applied load. Some authors propose for dynamics to consider the first time steps of the problem of interest to establish the ROB, without a detailed investigation, a time length corresponding to the fundamental period of vibration is heuristically suggested in [5]. But it has been outlined that this time has not been optimised. In this contribution, the influence of the time interval and the loading case of the training problem on the POD approximations is investigated. Computational savings offered by POD strategies are also explored to evaluate the potential interest of this approach for dynamic applications.

# 2 Investigation of POD for dynamic computations

To explore POD capabilities depending on different training stage strategies, a cantilever model with length 25 m, height 1.45 m, and width 3 m is used as structural example. A linear elastic material behaviour with Young's modulus of 210 GPa, Poisson's ratio of 0.3, and mass density of 7850 $\frac{\mathrm{kg}}{\mathrm{m}^3}$ is considered.

## 2.1 Load cases and numerical discretization

The model is studied under different load cases causing bending ($B$) and/or normal tension and compression ($C$), as illustrated in figure 2. For the bending load case, the force is applied at the free end once and in subsequently computations at different positions (load cases $B_1$ and $B_2$). Different loads with respect to time are taken into account, e.g. a Heaviside step, a Dirac impulse, or a harmonic loading, see figure 3. For the analysis, the dynamic response of the centre point at the free end section is considered as quantity of interest.

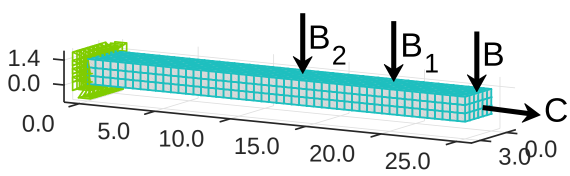



Figure 2: FE model with positions of applied loads

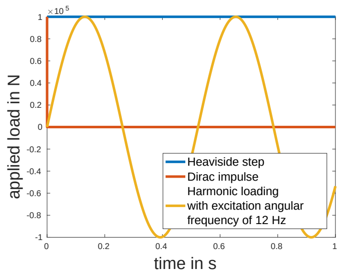
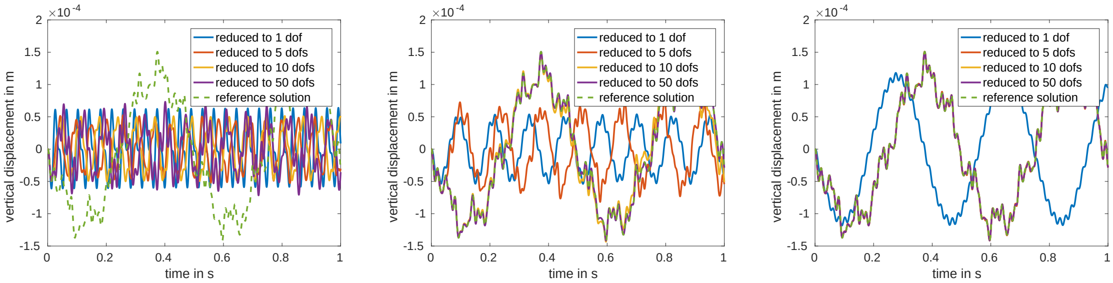



Figure 3: Applied loads with respect to time

The model is discretized using three-dimensional eight-node finite elements with linear shape functions resulting in 4284 dofs. Time integration is performed using an explicit central difference scheme to avoid numerical damping due to implicit solvers and to obtain an accurate estimation of the dynamical response. The time interval of 10 s is divided into 100 000 time steps, leading to a time step interval of $10^{-4}$ s to guarantee a stable solution of the explicit solver.

## 2.2 Training stage: Snapshot time window and number of reduced dofs

At first, the influence of the POD training interval on the accuracy of the POD computation considering the same kind of loading is investigated. The results obtained are depicted in figure 4 for a harmonic loading with excitation angular frequency of 325 Hz. A training interval of 0.01 s is too small to gather enough information on the system behaviour for POD construction, see figure 4 (a). The amplitude of the POD approximated response is too small and the response frequency is too high independently of the number of dofs considered in the reduced model. Therefore, the training interval is successively increased. For a smooth, continuous loading, like the harmonic loading, already a training interval of 0.025 s provides good results, if the system is reduced to 10 or 50 dofs, compare figure 4 (b). Using less dofs results into inaccurate approximations. Augmenting the time interval, less dofs are required to obtain a good approximation with the reduced model as illustrated in figure 4 (c). For a non-smooth loading, like the Dirac impulse loading, the time interval of the training stage needs to be increased to at least 0.05 s. It is possible to use only one dof in the reduced system, if the training interval has been large enough.
The required training interval is relatively small, here 0.05 s represents less than 10 % of the period of the first eigenfrequency.




(a) $\Delta t_{Training} = 0.01\,\mathrm{s}$ (b) $\Delta t_{Training} = 0.025\,\mathrm{s}$ (c) $\Delta t_{Training} = 0.05\,\mathrm{s}$

Figure 4: POD dynamic responses under harmonic loading with excitation angular frequency of 325 Hz using different training intervals $\Delta t_{Training}$. The reference solution is computed using a modal subspace reduced model of 100 dofs.

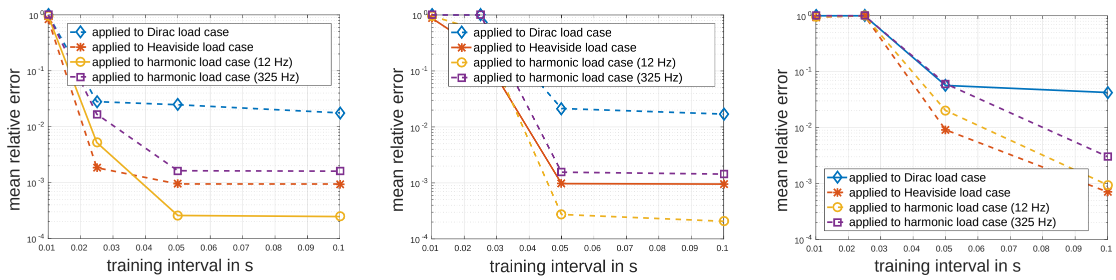
To evaluate the approximations obtained when the applied load of the snapshot compuation and the target computation differ with respect to time, a mean relative error is defined as

$$e_{rel} = \frac{1}{nsteps} \sum_{t_i=1}^{nsteps} \frac{|u_{ref}(t_i) - u_{red}(t_i)|}{|u_{ref}(t_i)|},$$

comparing $u_{ref}$ the reference solution computed with modal subspace reduced model of 100 dofs to $u_{red}$ the solution of the POD reduced model for the centre node at the free end over the whole number of time steps ($nsteps$).
If the aim is to compute the system under a Dirac impulse load case, POD modes from snapshots of a Heaviside step or harmonic loading lead to non-sufficient approximations, see figures 5 (a) and (b). On the other hand, POD modes computed

from snapshots of a Dirac impulse loading can be applied successfully for target computations under Heaviside step and harmonic load cases if the training interval is chosen large enough, see figure 5 (c).

The accuracy of POD approximation using snapshot computations from Heaviside step or harmonic loading, compare figures 5 (a) and (b), converge to a limit with respect to the training time interval, while using the Dirac impulse as load case, the approximation can still be improved by using a larger training time interval as depicted in figure 5 (c). This is due to the range of frequencies excited by the different load cases. The Heaviside step and harmonic loading only excite a specific range of frequencies whereas the Dirac impulse loading excites theoretically all frequencies. Therefore, more information about the dynamic behaviour of the model is captured in the Dirac snapshots which can then be extracted by SVD.




(a) Harmonic loading (12 Hz) snapshots

(b) Heaviside step snapshots

(c) Dirac impulse snapshots

Figure 5: Comparison of the applicability of snapshot computations from different load cases for reduced models with 25 dofs

## 2.3 Comparison of POD bases

To have a better understanding of the method's behaviour, the POD subspace appearing from SVD of the snapshot computations are compared. In figure 6 the first 50 POD modes obtained from a computation with a Dirac loading within a time interval of 0.1 s are compared to POD modes obtained for the same loading but a smaller time interval by computing the scalar product of each vector pair. A light white point symbolises a scalar product of zero, i.e. these vectors are orthogonal, whereas a dark black point marks a scalar product close to one, i.e. these vectors are collinear. The POD modes from a training stage of 0.01 s and 0.1 s differ, only about 5 similar modes are observed. While enlarging the training time interval, the POD modes converge to a final set of modes. Even for a non-harmonic loading, enlargement of the time interval will not change the determined set of POD modes.

POD modes represent the space dependence of the training stage. Therefore, considering a pure bending load case as training stage, first POD modes are similar to the first pure bending eigenmodes, see figure 7. Similarly, for a longitudinal compression deformation as training problem, the first POD modes correspond to the first eigenmodes describing this kind of deformation. For POD computation considering

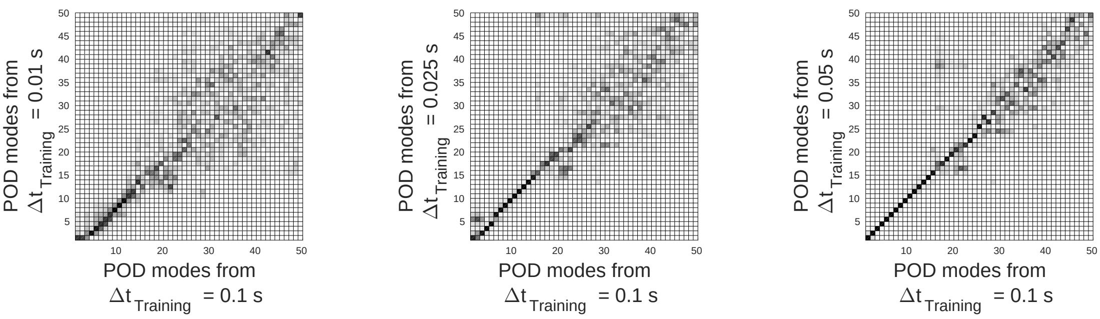



Figure 6: Comparison of POD modes from different training intervals under Dirac loading

$B$ and $C$ load coupled, the eigenmodes corresponding to a bending and compression deformation are extracted by the POD computation.

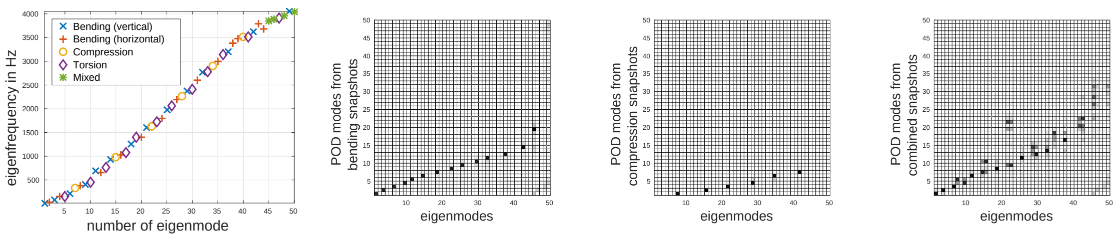



Figure 7: Comparison POD modes from a training interval of 0.1 s with eigenmodes

When modifying the position where the bending load is applied the extracted POD modes differ. 12 modes computed from 10 000 snapshots are similar for load case $B$ and $B_1$ and only 9 modes are similar for load case $B$ and $B_2$. Similar are the first POD modes which correspond to the pure bending eigenmodes.
Hence, the application point of the load has a large influence on the definition of the POD basis, and subsequently on the accuracy of the POD computation.

## 2.4 POD computational effort

Finally, the required computational effort is compared. For the reduction process a singular value problem needs to be computed and the original system is transformed by a Galerkin projection onto a subspace. The computation times for the reduction are presented in figure 8. The more snapshot computations are used and the more dofs the reduced system consists of, the larger the computational times. The computations of snapshots are not considered here.
In figure 9, the computational times for solving the full system and the reduced systems are compared for a time interval of 10 s, corresponding to a computation of 100 000 time steps. The computational effort of the reduced systems here includes the snapshot computations, the reduction process as well as the solution of the reduced system. The number of dofs of the reduced model contributes only slightly to the computational times compared to the snapshot computations.
The savings with respect to the number of snapshot computations are presented in figure 10. For a large number of time steps, POD is clearly more efficient than solving

the full system directly. The advantage of the reduced model depends directly on the required number of time steps for the training time interval and the number of time steps of the target computation. This factor equals to the computational savings. The computational effort of the singular value decomposition and the projection onto the subspace are insignificant. However, for this linear example modal decomposition performs better than POD.

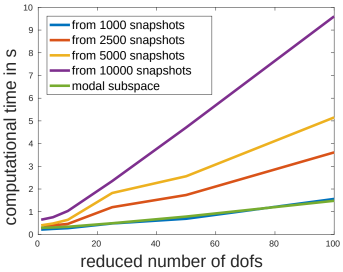



Figure 8: Computational times to reduce the system by SVD

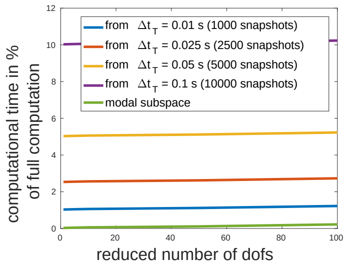



Figure 9: Comparison of computational times for a time interval of 10 s

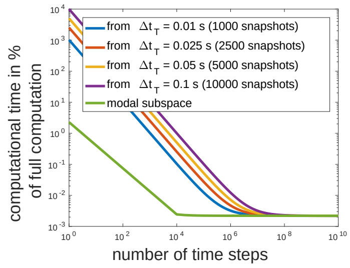



Figure 10: Computational times of reduced models with 100 dofs

# 3 Conclusion

Here, POD performance has been investigated for linear dynamics, in particular the required characteristics of the training stage have been explored. Concerning the training stage time, it has been seen that a relatively short time, corresponding to less that 10 % of the period of the first eigenmode, is enough to guarantee a good accuracy of the POD approximations. POD computational savings is drastically significant, and becomes larger when the POD computations tackle a long time interval. POD approach seems limited to some load positions which are close to the one of the training stage. Modal decomposition is more flexible regarding different load positions and more efficient in the matter of computational effort in this linear case, but will be reaching its performance limit when including non-linearities. New strategies to overcome that shall be explored in the future.

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