Dynamics of Gravitating System of Gas and Dust Cloud

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Abstract

Due to the gravitational force, cloud of dust and gas can contract and form planet. Here we present a simple model for the dynamics of one dimensional of self-gravitating spherical symmetrical gas and dust cloud.
We present analytic, similarity solution for the one dimensional of self-gravitating spherical symmetrically gas and dust cloud. In this paper we used a Cole-Hopf transformation to simplify the equations of dynamics and thereafter we applied method of characteristics to reduce partial differential equation to a system of completely solvable ordinary differential equations. The similarity solution method is applied to reduce the partial differential equations to a system of completely solvable ordinary differential equations. The Runga-Kutta method has been used for numerical calculation of the problem.
KEYWORDS: Cole-Hopf transformation, Self-Gravitating; Gas-Dust System; Similarity Reductions; Planet Formation

1 Introduction

Mechanical theories of gas-dust systems of cloud can be developed from two quite different starting points: We can introduce either the model of N gravitating mass points, or the model of a compressible fluid streaming in the phase space. The motion of this fluid will be determined by the gravitational fluid produced by itself. In this work we will study a model of a compressible fluid of dust and gas. Magnetic field, radiation force, rotation probably play important roles but to simplify the problem in this work we will ignoring these factors. The gravitational collapse of spheres has received considerable theoretical attention in literature,particularly in connection with the problem of star formation ([1] , [2]).
We shall consider spherically symmetric, self-gravitating dust-gas cloud. All the physical quantities will depend on two independent variables; radius and time $(r;t)$. Let $P(r,t)$, $\rho(r,t)$, $v(r,t)$ and $\Phi(r,t)$ be the pressure, mass density, radial velocity and gravitational potential respectively.

2 Fundamental Equations

The motion of the spherically symmetrical compressible fluid flow of self-gravitating dust-gas is governed by the following equations [3], [11]:

$$\rho_t + v\rho_r + \rho v_r + \frac{2}{r}v\rho = 0, \tag{1}$$

$$v_t + vv_r + \frac{1}{\rho}P_r + \Phi_r = 0, \tag{2}$$

$$\Phi_{rr} + \frac{2}{r}\Phi_r = 4\pi G\rho, \tag{3}$$

where G is Newton's gravitational constant.
These nonlinear partial differential equations are quite complicated and the general solution cannot be obtained. Here we will use two methods for finding particular solution, Hopf-Cole transformation method and finite-different numerical method.

3 Hopf-Cole Solution

Let us first consider one dimensional flow of fluid in self-gravitating field without pressure, the equation of dust state ($P = 0$). let $\sigma = r^2\rho$, now we can rewrite the equation (1)$-$(3) as:

$$\frac{\partial\sigma}{\partial t} + \frac{\partial(\sigma v)}{\partial r} = 0, \tag{4}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{\partial\Phi}{\partial r}, \tag{5}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = 4\pi G\sigma. \tag{6}$$

It is an amazing fact that the equations like (4)$-$(6) may be solved exactly using a trick discovered independently by Cole J. D. [5] and E. Hopf [6] about 1950. After hopf and Cole introduced the transformation, several attempts have been made to generalised Cole-Hopf transformation, we shall use here modified generalized Cole-Hopf method [7]. The trick is to change the dust velocity in the following form:

$$v(r,t) = -\frac{\theta_t}{\theta_r}, \tag{7}$$

where $\theta = \theta(r,t)$ is the auxiliary function, $\theta_t = \frac{\partial\theta}{\partial t}$, $\theta_r = \frac{\partial\theta}{\partial r}$.
Let consider θ_r to be

$$\theta_r = \sigma = r^2\rho, \tag{8}$$

By using generalised Cole-Hopf transformation we can find [Appendix]:

$$v(r,t) = -\frac{\theta_t}{\theta_r} = \pm\sqrt{b + r^{-1}}\sqrt{8\pi G}\sqrt{\theta}, \tag{9}$$

or

$$\theta_t \pm \sqrt{b + r^{-1}}\sqrt{8\pi G}\sqrt{\theta}\theta_r = 0, \tag{10}$$

where b is constant of integration.
By the substitution of new variable

$$\xi(r) = \int (b + r^{-1})^{-1/2} dr, \tag{11}$$

to equation (10), it takes the form

$$\theta_t \pm \sqrt{8\pi G}\sqrt{\theta}\theta_\xi = 0. \tag{12}$$

We use the method of characteristics to solve (12). The method will reduce PDE to ODE.
The characteristics of the PDE (12) are

$$\frac{dt}{1} = \pm\frac{d\xi}{\sqrt{8\pi G}\sqrt{\theta}} = \frac{d\theta}{0}, \tag{13}$$

From here we find the general solution

$$\xi \pm \sqrt{8\pi G}\sqrt{\theta}t = F(\theta). \tag{14}$$

4 Initial and Boundary Conditions

One of the main problems with model calculations the formation of planets is fact that initial conditions of the cloud are not known. It is obvious for the density become zero at the surface of the cloud, then let we look for the solution of the problem with initial conditions

$$\rho(r, 0) = \frac{1}{r^2}\frac{\partial\theta_0(r)}{\partial r} \tag{15}$$

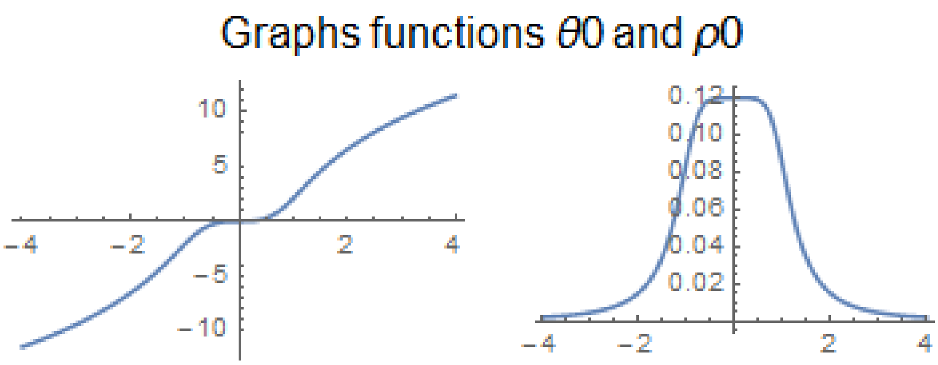


Figure 1

where $\theta_0(r) = \theta(r, 0)$.
Let

$$\theta_0(r) = \frac{M}{4\pi} \operatorname{arsinh}(r^3), \tag{16}$$

where M is mass of cloud.
Differentiating equation (16) and using (8), we find $\rho(r, 0)$, (see Figure 1).

$$\rho_0 = \rho(r, 0) = \frac{3M}{4\pi(r^6 + 1)^{1/2}} > 0, \tag{17}$$

$$v(r, 0) = 0; \tag{18}$$

and assume that we have boundary condition

$$\Phi(R, t) = -\frac{GM}{R}, \tag{19}$$

where R is radius of the cloud.

5 Analytical Solution of Fundamental Equations

Now we want to get the solution of the equation (12) with initial condition (16) and (17).
By plugging initial condition (16) into equation (12), we get

$$\theta - \frac{M}{4\pi} \operatorname{arsinh}(\frac{3}{2}(\xi \pm \sqrt{8\pi G}\sqrt{\theta}t))^2 = 0, \tag{20}$$

When $b = 0$ in equation (11) we obtain

$$\xi(r) = \frac{2}{3} r^{\frac{3}{2}}. \tag{21}$$

Now we can rewrite (20) as

$$\theta - \frac{M}{4\pi} \operatorname{arsinh}(r^{3/2} \pm \frac{3}{2}\sqrt{8\pi G}\sqrt{\theta}t)^2 = 0, \tag{22}$$

this is hyperbolic transcendental equation, we can solve it numerically to find θ.
Differentiating the equation (22) with respect to r, we find

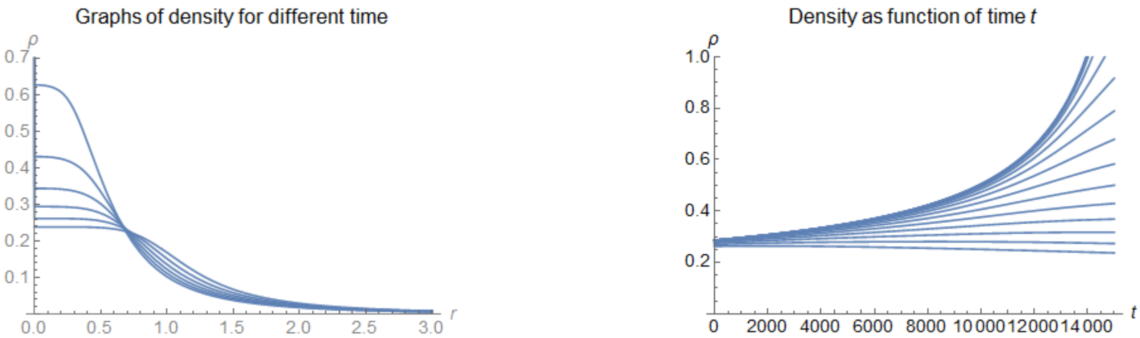
$$\theta_r = \frac{6M\sqrt{\theta}w(r, t)\sqrt{r}}{8\pi\sqrt{\theta}\sqrt{1 + w^4} - 3M\lambda t w(r, t)}, \tag{23}$$

where

$$w(r, t) = (r^{\frac{3}{2}} \pm \frac{3}{2}\lambda\sqrt{\theta}t), \qquad \lambda = \sqrt{8\pi G}. \tag{24}$$

Now we can calculate density ρ through function $\theta(r, t)$

$$\rho = \frac{1}{r^2}\theta_r = \frac{6M\sqrt{\theta}(r^{\frac{3}{2}} \pm \frac{3}{2}\lambda t\sqrt{\theta})r^{-\frac{3}{2}}}{8\pi\sqrt{\theta}\sqrt{1 + (r^{\frac{3}{2}} \pm \frac{3}{2}\lambda\sqrt{\theta}t)^4} - 3M\lambda t(r^{\frac{3}{2}} \pm \frac{3}{2}\lambda\sqrt{\theta}t)}. \tag{25}$$



(a) (b)

Figure 2

Differentiating equation (20) with respect to t, we obtain

$$\theta_t = \frac{6M\theta w(r,t)}{8\pi\sqrt{\theta}\sqrt{1+w^4} - 3M\lambda t w(r,t)}. \tag{26}$$

Now we can calculate velocity v through function θ

$$v(r,t) = -\frac{\theta_t}{\theta_r} = -\frac{\lambda\theta^{\frac{1}{2}}}{r^{\frac{1}{2}}}. \tag{27}$$

From equations (5) and (19)

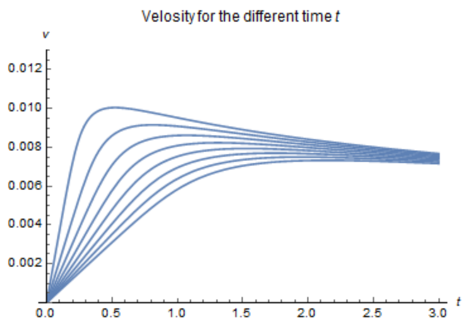


Figure 3

$$\Phi(r,t) = \lambda^2\theta. \tag{28}$$

6 Similarity Solution

In this section we are seeking solution of the equations of spherically symmetrical compressible fluid flow of self-gravitating dust-gas (1)−(3).
Let for example we look for the solution of the problem with boundary conditions

$$v(0,t) = 0, \tag{29}$$

$$\rho(0,t) = \rho_0, \tag{30}$$

$$\Phi(0,t) = 0 \tag{31}$$

By introducing the similarity variable [2] [1]

$$\xi = \frac{r}{\sqrt{4\lambda\pi G t}} \tag{32}$$

and seek a solution in the following form

$$v(r,t) = \sqrt{4\lambda\pi G} w(\xi), \tag{33}$$

$$\rho(r,t) = \frac{q(\xi)}{4\pi G t^2}, \tag{34}$$

$$\Phi(r,t) = 4\lambda\pi G \Psi(\xi) \tag{35}$$

Here λ - some dimensional factor, functions w, q, Ψ are dimensionless functions
Let we denote

$$\Psi^{'} = \Omega \tag{36}$$

Let us first we seeking solution for the problem without pressure, which corresponds

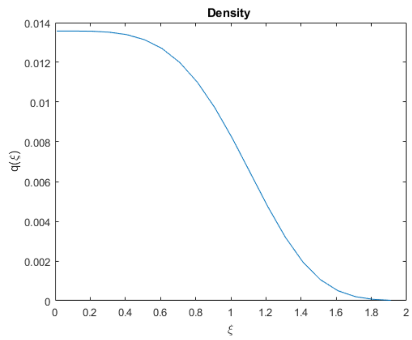
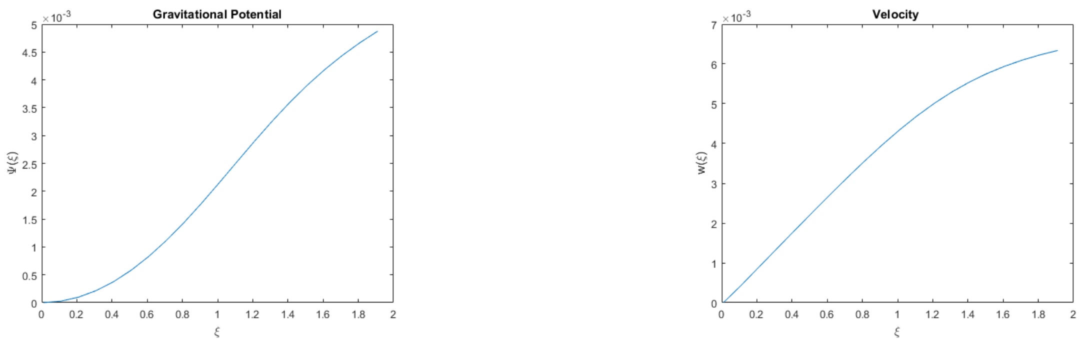


Figure 4

to the equation of state (P = 0).
Using the introduced dimensionless variables, after some manipulation we can reduce



(a) (b)

Figure 5

the system of fundamental equations (1), (2) and (3) to the following system of ordinary differential equations:

$$\Psi^{'} = \Omega, \tag{37}$$

$$w^{'} = \frac{\Omega}{\xi - w}, \tag{38}$$

$$\Omega^{'} = \frac{\xi q - 2\Omega}{\xi}, \tag{39}$$

$$q^{'} = q\frac{\Omega\xi - 2(\xi - w)^2}{\xi(\xi - w)^2} \tag{40}$$

where the prime denotes differentiation with respect to the ξ.
In term of the similarity variables the boundary conditions take the form

$$w(\xi = 0) = 0 \tag{41}$$

$$q(\xi = 0) = 4\pi G\rho_0 t^2, \tag{42}$$

$$\Psi^{'}(\xi = 0) = 0 \tag{43}$$

We obtain the similarity solution by integrating equations (37)−(40) by numerical

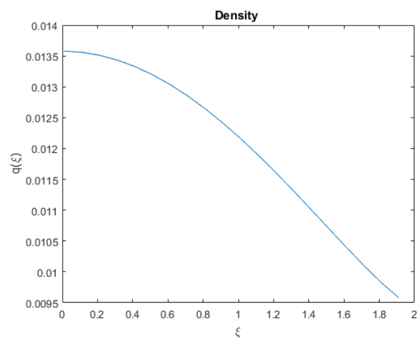


Figure 6

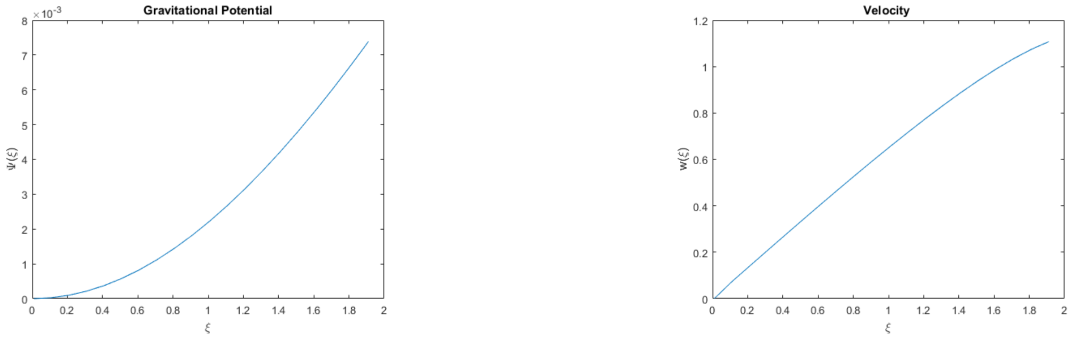
methods.
Now we try to find solution when the equation of state of flow which is a mixture of gas and small solid particle is taken to be

$$P = K\rho \tag{44}$$

where K is a constant physically characterizes the central pressure and central density in our model.
Similar equations to (37)−(40) obtained when taking into account the equation of state (44)

$$\Psi^{'} = \Omega, \tag{45}$$

$$w^{'} = \frac{(\xi\Omega - 2)(w - \xi)}{\xi(1 - (w - \xi)^2)}, \tag{46}$$



(a) Gravitational Potential (b) Velocity

Figure 7

$$\Omega' = \frac{\xi q - 2\Omega}{\xi}, \tag{47}$$

$$q' = q\frac{\xi\Omega - 2(\xi - w)^2}{((\xi - w)^2 - 1)\xi} \tag{48}$$

Similarly we can find numerical solution of the equations (45)−(48) for boundary conditions (41)−(43).

Equations (37)−(40) and (45)−(48) have been integrated by Runge-Kutta method for boundary conditions (41)−(43). The graphs of flow variable are given in Figure 4 , Figure 5a , Figure 5b , Figure 6 , Figure 7a and Figure 7b . It is clear from the graphs that the density increases as we move towards the centre, while velocity and gravitational potential decreases.

7 Summary

We investigated the The motion of the spherically symmetrical compressible fluid flow of self-gravitating dust-gas cloud, In certain cases we have tried to find solution for the system of equations presented in (1)−(3). In the case when $P = 0$, we find particular analytical solution with help of modified Cole-Hopf transformation and special initial condition.

In this work, similarity solutions are obtained for one dimensional flow of fluid in self-gravitating field without pressure and with pressure.

Further possible study my be the investigation of the system (1)−(3) with other possible equation of state, which my be different than the ones considered here.

8 Appendix

Following [7], let us consider the representation for dust velocity in the following form:

$$v(r,t) = -\frac{\theta_t}{\theta_r} \tag{49}$$

where $\theta = \theta(r, t)$ is the auxiliary function (generalized Cole-Hopf transformation) The equivalent representation (49) has the form of the equation

$$\theta_t + v(r, t)\theta_r = 0 \tag{50}$$

There are several simple identities that follow from (50) and hold for any differentiable function $\theta(r, t)$ [7] , the first of them has the form

$$\left[\frac{\partial}{\partial t} + v(r, t)\frac{\partial}{\partial r}\right] F(\theta) = F'(\theta)(\theta_t + v\theta_r) \tag{51}$$

Differentiating (50) with respect to r, we obtain

$$\frac{\partial}{\partial t}\theta_r + \frac{\partial}{\partial r}\left[v(r, t)\theta_r\right] = 0 \tag{52}$$

From (50) and (51) there follows anther identity of the form

$$\frac{\partial}{\partial t}\left[f(r)F(\theta)\right] + v\frac{\partial}{\partial r}\left[f(r)F(\theta)\right] = f'(r)F(\theta)v \tag{53}$$

Relation (52) takes the form of continuity equation if the density $\rho(r, t)$ is considered to be

$$\rho(r, t) = \frac{1}{r^2}\theta_r \tag{54}$$

In this case (52) is equivalent to the continuity equation (4)

$$\frac{\partial \sigma}{\partial t} + \frac{\partial}{\partial r}(v\sigma) = 0, \tag{55}$$

We reduce the Poisson equation (6) to the form

$$\Phi_r = \frac{1}{r^2}4\pi G\theta \tag{56}$$

From equation (5) and (56), we obtain

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{4\pi G\theta}{r^2} \tag{57}$$

Let

$$v = S(r)T(\theta) \tag{58}$$

where $S(r)$ and $T(\theta)$ are so far undefined functions. Using the identities we obtain

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = S'(r)T(\theta)v = S'(r)S(r)T^2(\theta) \tag{59}$$

Comparing (57) and (59), we obtain

$$T(\theta) = \sqrt{4\pi G}\sqrt{\theta} \tag{60}$$

$$S(r) = \pm\sqrt{2}\sqrt{b + r^{-1}} \tag{61}$$

where b is constant of integration.
From (49), (58), (60), and (61)

$$v(r, t) = -\frac{\theta_t}{\theta_r} = \pm\sqrt{b + r^{-1}}\sqrt{8\pi G}\sqrt{\theta} \tag{62}$$

or

$$\theta_t \pm \sqrt{b + r^{-1}}\sqrt{8\pi G}\sqrt{\theta}\theta_r = 0 \tag{63}$$

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