Algebraic Poincaré equations

R.P. Moshkin[1]

[1]*M. V. Lomonosov Moscow State University, Moscow, Russia*

rmoshkin@yandex.ru

Abstract

As examples, we specify the Poincaré-Chetaev equations. The review article of V.V. Rumyantsev can be supplement with data on works devoted to other forms of equations of motion of nonholonomic systems.

The original source of this idea, the work of Poincaré published in 1901, actually introduced the expansion of the vector of generalized velocities with respect to a moving frame (although the term appeared later at E. Cartan) composed from vector fields forming the Lie algebra (in modern terminology), that is, their Lie brackets have constant ("structural") coefficients in the same fields. Thus, Poincaré has already used the idea of pseudo velocities, although this idea was clearly introduced by Boltzmann in 1902. It is interesting that the constancy of structural coefficients is not necessary for Poincaré equations in reality. The work of Hamel in 1904 has no reference to Poincaré, but clearly operates with vector fields. The Hamel equations are different from the Poincaré equations only terminologically. Hamel clearly point out that the constancy of structural coefficients is not required; he generalizes his "Euler-Lagrange equations" to nonholonomic systems.

It is interesting that, often, parallel works are also historically simultaneous. The next burst of activity in the field of equations of motion of nonholonomic systems was in 1926 and during next five years. On the one hand, N.G. Chetaev published the Hamiltonian version of the Poincaré equations and anticipates the Dirac approach to exposition of Hamiltonian mechanics. On the other hand, in 1926-31 Vranceanu, Singh and Shouten introduced and developed the notion of nonholonomic connection, due to which they tie a nonholonomic mechanics to Riemannian geometry. At the same time, they relied on the work of Ricci and Levi-Civita published in 1900!

There are a number of different forms of equations of motion of nonholonomic systems. We will consider the main of them and show that they can be obtained from the Maggi's equations. We will derive the most commonly used forms for noting the equations of motion of nonholonomic systems from the Maggi's equations.

1 Introduction

Earlier [1-3], the direct calculations showed the equivalence of the Poincaré equations of motion of nonholonomic systems to the Chaplygin, Appell, Hamel, Volterra, and Ferres equations and some other equations. The equivalence of the equations of motion in quasicoordinates to the Appell equations, as well as Chaplygin's equations, was proved [6] by derivation of these sets of equations from the d'Alembert-Lagrange principle. The Voronets equations were derived from the Poincaré equations (5.6) in [4].

We will show that the Poincaré equations are equivalent to some other forms of equations of motion of nonholonomic systems.

2 The equations of Volterra, Appell, Kane

Maggi [5] showed that the Appell and Volterra equations follow from the equations established by him. Maggi considered a mechanical system with coordinates $x_i\,(i=1,...,n)$, subject to linear constraints, which can be both holonomic and nonholonomic, explicitly dependent or independent on time. When solving the constraint equations in relation to $\dot{x}_i$, he presented the latter in the following form

$$\dot{x}_i = b_{is}(x)\eta_s, \qquad i = 0,1,...,n, \qquad s = 0,1,...,l, \qquad b_{i0} = \delta_{i0},$$

the quantities η_s (he denoted by e_s), are called motion characteristics of the system under consideration, where

$$b_{is} = \frac{\partial \dot{x}_i}{\partial \eta_s} = \frac{\partial \ddot{x}_i}{\partial \dot{\eta}_s}, \qquad s = 1,...,l = n - m.$$

Proceeding to the derivation of Volterra equations, Maggi converted his equations of the form

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} - Q_i\right)b_{is} = 0, \qquad s = 1,...,k$$

(in which the kinetic energy T occurs instead of L, while Q_i denotes all the active forces applied to the system) to the form

$$\frac{d}{dt}\frac{\partial T}{\partial \eta_r} = \frac{db_{ir}}{dt}\frac{\partial T}{\partial \dot{x}_i} + b_{ir}\frac{\partial T}{\partial x_i} + P_r, \qquad r = 1,...,l, \qquad P_r = Q_i b_{ir}. \tag{1}$$

Volterra [7] considered a system with N point masses, the velocities of which in a Cartesian system of coordinates are related to the motion characteristics of the form

$$\dot{x}_i = b_{is}\eta_s, \qquad i = 1,...,3N, \qquad s = 1,...,l,$$

where $b_{is} = b_{is}(x_1,...,x_{3N})$. In this case, the Maggi's equations (1) take the form of Volterra equations

$$\frac{d}{dt}\frac{\partial T}{\partial \eta_r} = c_{rs}^{(k)}\eta_k\eta_s + P_r, \qquad k,s = 1,...,l,$$

$$c_{rs}^{(k)} = m_i b_{ik}\frac{\partial b_{ir}}{\partial x_j}b_{js}, \qquad m_i = m_{i+1} = m_{i+2}, \qquad i,j = 1,...,3N,$$

where $T(x_1,...,x_{3N},\eta_1,...,\eta_l)-$ is the kinetic energy.

Without giving Maggi's derivation of Appell equations from equations

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} - Q_i\right)b_{is} = 0, \qquad s = 1,...,k,$$

here, we note that they are simpler to derive directly from the following equations

$$\left(m_\nu \ddot{\bar{r}}_\nu - \bar{F}_\nu\right)\cdot \delta\bar{r}_\nu = 0, \qquad \nu = 1,...,N.$$

Differentiating equations

$$\dot{x}_i = b_{is}(x)\eta_s, \qquad i = 0,1,...,n, \qquad s = 0,1,...,l, \qquad b_{i0} = \delta_{i0}$$

with respect to time, we have $\ddot{x}_i = b_{is}\dot{\eta}_s + ...$, where three dots denote the members not containing $\dot{\eta}_s$.

Hence, we find that $\dfrac{\partial \ddot{x}_i}{\partial \eta_s} = b_{is}$, as a result of which, we obtain from $\left(m_\nu \ddot{\bar{r}}_\nu - \bar{F}_\nu\right)\cdot \delta\bar{r}_\nu = 0$, $\nu = 1,...,N$ the Appell equations

$$\frac{\partial S}{\partial \dot{\eta}_s} = \Pi_s, \qquad s = 1,...,l, \tag{2}$$

where $S = \dfrac{1}{2} m_\nu \ddot{\bar{r}}_\nu^2 -$ is the energy of accelerations, $\Pi_s = \bar{F}_\nu \cdot \bar{b}_{s\nu} -$ is the generalized force referred to the quasicoordinate π_s [6].

Finally, we will show that Kane's equations [8] are equivalent to the Poincaré equations. According to the relations

$$\delta f = \omega_r X_r f, \quad r = 1,...,k, \quad df = \eta_s X_s f dt, \quad s = 0,1,...,k,$$
$$\delta \bar{r}_\nu = \omega_s X_s \bar{r}_\nu \left(\nu = 1,...,N, s = 1,...,l\right).$$

Substitution of these expressions into $\left(m_\nu \ddot{\bar{r}}_\nu - \bar{F}_\nu\right)\cdot \delta\bar{r}_\nu = 0$, $\nu = 1,...,N$ leads to equations of motion of the form

$$m_\nu \ddot{\bar{r}}_\nu \cdot X_s \bar{r}_\nu = \bar{F}_\nu \cdot X_s \bar{r}_\nu, \qquad s = 1,...,l. \tag{3}$$

For a system with Lagrangian coordinates q_i, subject to nonintegrable constraints

$$\dot{q}_j = b_{js}(t,q)\dot{q}_s + b_j(t,q), \qquad j = l+1,...,n, \qquad s = 1,...,l,$$

and operators $X_s f = b_{is}\dfrac{\partial f}{\partial x_s}$, $s = 0,1,...,k$, $f(x) \in C^2$ of the form

$$X_s f = \frac{\partial f}{\partial q_s} + b_{js}\frac{\partial f}{\partial q_j},$$

equations (3) coincide with Kane's equations (19) [8]

$$K_{q_s} + K'_{q_s} = 0, \quad s = 1,...,l,$$

which, consequently, are equivalent to the Poincaré equations.

3 The equations of Chaplygin and Voronets

Let us suppose that stationary linear nonholonomic constraints are imposed on the system under consideration, which equations can be represented in the form

$$\dot{q}^{l+k} = \beta_\lambda^{l+k}(q)\dot{q}^\lambda, \qquad \chi = 1,2,...,k; \qquad \lambda = 1,2,...,l. \tag{4}$$

Then, assuming

$$\upsilon_*^\lambda = \dot{q}^\lambda, \quad \lambda = 1,2,...,l;$$
$$\upsilon_*^{l+\chi} = \dot{q}^{l+\chi} - \beta_\lambda^{l+\chi}(q)\dot{q}^\lambda, \qquad \chi = 1,2,...,k,$$

we have

$$\frac{\partial \dot{q}^\mu}{\partial \upsilon_*^\lambda} = \begin{cases} 1, & \mu = \lambda, \\ 0, & \mu \neq \lambda, \end{cases} \qquad \mu = 1,2,...,l,$$

$$\frac{\partial \dot{q}^{l+\chi}}{\partial \upsilon_*^\lambda} = \beta_\lambda^{l+\chi}, \qquad \chi = 1,2...,k; \qquad \lambda = 1,2,...,l.$$

From these expressions it follows that for nonholonomic constraints given in the form (4), the Maggi's equations

$$(M\omega_\sigma - Q_\sigma)\frac{\partial \dot{q}^\sigma}{\partial \upsilon_*^\lambda} = 0, \qquad \lambda = 1,2,...,l.$$

can be written in the form:

$$M\omega_\lambda + M\omega_{l+\chi}\beta_\lambda^{l+\chi} = Q_\lambda + Q_{l+\chi}\beta_\lambda^{l+\chi}, \tag{5}$$
$$\lambda = 1,2,...,l; \qquad \chi = 1,2,...,k.$$

Let us suppose that kinetic energy T does not depend on the generalized coordinates and $q^{l+\chi}$ и $Q_{l+\chi} = 0 \quad (\chi = 1,2,...,k).$ Then equations (5) can be represented in the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\lambda} - \frac{\partial T}{\partial q^\lambda} + \beta_\lambda^{l+\chi}\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{l+\chi}} = Q_\lambda, \qquad \lambda = 1,2,...,l. \tag{6}$$

Let us transform equations (6). Let us eliminate all the velocities $\dot{q}^{l+\chi}$ from the expression for the kinetic energy T using the constraint equations (4), and denote the resulting expression for the kinetic energy by T_*.

In this case, the following equalities are correct

$$\frac{\partial T_*}{\partial \dot{q}^\lambda} = \frac{\partial T}{\partial \dot{q}^\lambda} + \frac{\partial T}{\partial \dot{q}^{l+\chi}}\frac{\partial \dot{q}^{l+\chi}}{\partial \dot{q}^\lambda} = \frac{\partial T}{\partial \dot{q}^\lambda} + \frac{\partial T}{\partial \dot{q}^{l+\chi}}\beta_\lambda^{l+\chi}, \tag{7}$$

$$\frac{\partial T_*}{\partial q^\lambda} = \frac{\partial T}{\partial q^\lambda} + \frac{\partial T}{\partial \dot{q}^{l+\chi}}\frac{\partial \dot{q}^{l+\chi}}{\partial q^\lambda} = \frac{\partial T}{\partial q^\lambda} + \frac{\partial T}{\partial \dot{q}^{l+\chi}}\frac{\partial \beta_\mu^{l+\chi}}{\partial q^\lambda}\dot{q}^\mu, \tag{8}$$
$$\lambda, \mu = 1,2,...,l.$$

Let us suppose that the coefficients $\beta_\lambda^{l+\chi}$ do not depend on $q^{l+\chi}$, $\chi = 1,2,...,k.$ Then, differentiating the expression (7) with respect to time, we obtain

$$\frac{d}{dt}\frac{\partial T_*}{\partial \dot{q}^\lambda} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\lambda} + \beta_\lambda^{l+\chi}\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{l+\chi}} + \frac{\partial T}{\partial \dot{q}^{l+\chi}}\frac{d}{dt}\beta_\lambda^{l+\chi} =$$
$$= \frac{d}{dt}\frac{\partial T_*}{\partial \dot{q}^\lambda} + \beta_\lambda^{l+\chi}\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{l+\chi}} + \frac{\partial T}{\partial \dot{q}^{l+\chi}}\frac{\partial \beta_\lambda^{l+\chi}}{\partial q^\mu}\dot{q}^\mu, \qquad \lambda, \mu = 1,2,...,l. \tag{9}$$

Computing the quantities $d/dt(\partial T/\partial \dot{q}^\lambda)$ and $\partial T/\partial q^\lambda$ by formulas (9) and (8) and substituting them into equations (6), we get

$$\frac{d}{dt}\frac{\partial T_*}{\partial \dot{q}^\lambda} - \frac{\partial T_*}{\partial q^\lambda} - \frac{\partial T}{\partial \dot{q}^{l+\chi}}\left(\frac{\partial \beta_\lambda^{l+\chi}}{\partial q^\mu} - \frac{\partial \beta_\mu^{l+\chi}}{\partial q^\lambda}\right)\dot{q}^\mu = Q_\lambda, \tag{10}$$
$$\lambda, \mu = 1,2,...,l; \quad \chi = 1,2,...,k.$$

These equations were obtained by S.A. Chaplygin [10].

If in the equations (10) we eliminate the dependent velocities $\dot{q}^{l+1}, \dot{q}^{l+2}, ..., \dot{q}^{l+\chi}$, in the expressions $\partial T/\partial \dot{q}^{l+\chi}$ using the constraint equations (4), then we obtain a system of l equations with respect to unknown functions $q^1, q^2, ..., q^l$. Thus, Chaplygin's equations allow us independently of the constraints (4) to find $q^1(t), q^2(t), ..., q^l(t)$, then to define the remainder $q^{l+1}(t), q^{l+2}(t), ..., q^{l+k}(t)$ from equations (4).

The Chaplygin's equations are transformed into ordinary Lagrange's equations of the second kind, if the constraints (4) are integrable, that is, the coefficients $\beta_\lambda^{l+\chi}$ satisfy the following conditions:

$$\frac{\partial \beta_\mu^{l+\chi}}{\partial q^\lambda} - \frac{\partial \beta_\lambda^{l+\chi}}{\partial q^\mu} = 0, \qquad \lambda, \mu = 1,2,...,l; \; \chi = 1,2,...,k. \tag{11}$$

Let us suppose, for example, at $\lambda = 1$ we have

$$\frac{\partial \beta_\mu^{l+\chi}}{\partial q^1} - \frac{\partial \beta_1^{l+\chi}}{\partial q^\mu} = 0, \qquad \mu = 1,2,...,l; \; \chi = 1,2,...,k. \tag{12}$$

Let us derive the functions $u^{l+\chi} = u^{l+\chi}(q^1, q^2, ..., q^s)$ as follows:

$$u^{l+\chi} = \int_{q^{1,0}}^{q^1} \beta_1^{l+\chi} dq^1, \qquad \chi = 1,2,...,k,$$

where $q^{1,0}$ - is an arbitrary constant value. Using conditions (12), we obtain

$$\frac{\partial u^{l+\chi}}{\partial q^\mu} = \int_{q^{1,0}}^{q^1} \frac{\partial \beta_1^{l+\chi}}{\partial q^\mu} dq^1 = \int_{q^{1,0}}^{q^1} \frac{\partial \beta_\mu^{l+\chi}}{\partial q^1} dq^1 =$$
$$= \beta_\mu^{l+\chi}(q^1, q^2, ..., q^s) - \beta_\mu^{l+\chi}(q^{1,0}, q^2, ..., q^s),$$

or

$$\beta_\mu^{l+\chi} = \frac{\partial u^{l+\chi}}{\partial q^\mu} + \beta_\mu^{l+\chi}(q^{1,0}, q^2, ..., q^s), \qquad \mu = 2,3,...,l; \; \chi = 1,2...,k.$$

Hence and from expressions

$$\beta_1^{l+\chi} = \frac{\partial u^{l+\chi}}{\partial q^1}; \qquad \chi = 1,2,...,k,$$

it follows that equation (10) by the coordinate q^1 will have the form of Lagrange's equation of the second kind in the case when the constraint equations (4) can be reduced to the form

$$\dot{q}^{l+\chi} = \dot{u}^{l+\chi} + \beta_\mu^{l+\chi}(q^{1,0}, q^2, ..., q^s)\dot{q}^\mu, \qquad \mu = 2,3...,l; \; \chi = 1,2,...,k.$$

Now we derive the equations of motion in the form obtained by P.V. Voronets [11]. We consider a mechanical system with constraints given in the form (4) without making those additional assumptions that lead to the Chaplygin's equations. In the case, when the kinetic energy T depends on all the coordinates, the Maggi's equations (5) will be written in the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\lambda} - \frac{\partial T}{\partial q^\lambda} + \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{l+\chi}} - \frac{\partial T}{\partial q^{l+\chi}}\right)\beta_\lambda^{l+\chi} = Q_\lambda + Q_{l+\chi}\beta_\lambda^{l+\chi}, \tag{13}$$
$$\lambda = 1,2,...,l; \; \chi = 1,2,...,k.$$

In order to bring these equations to the Voronets equations, we proceed similarly to the previous case. Relations (8) save their form, and expressions (9), taking into account that now the coefficients $\beta_\lambda^{l+\chi}$ depend on all q^σ, take the form

$$\frac{d}{dt}\frac{\partial T_*}{\partial \dot{q}^\lambda} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\lambda} - \beta_\lambda^{l+\chi}\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{l+\chi}} - \frac{\partial T}{\partial \dot{q}^{l+\chi}}\frac{\partial \beta_\lambda^{l+\chi}}{\partial q^\mu}\dot{q}^\mu +$$
$$+ \frac{\partial T}{\partial \dot{q}^{l+\chi}}\frac{\partial \beta_\lambda^{l+\chi}}{\partial q^{l+\nu}}\beta_\mu^{l+\nu}\dot{q}^\mu, \tag{14}$$
$$\lambda, \mu = 1,2,...,l; \; \chi, \nu = 1,2,...,k.$$

In this case, along with relations (8) and (14), we should take into account the following equalities

$$\beta_\lambda^{l+\chi}\frac{\partial T_*}{\partial q^{l+\chi}} = \beta_\lambda^{l+\chi}\left(\frac{\partial T}{\partial q^{l+\chi}} + \frac{\partial T}{\partial \dot{q}^{l+\nu}}\frac{\partial \beta_\mu^{l+\chi}}{\partial q^{l+\chi}}\dot{q}^\mu\right).$$

This expression, as well as relations (8) and (14) allows us to represent the equations (13) in the form

$$\frac{d}{dt}\frac{\partial T_*}{\partial \dot q^\lambda}-\frac{\partial T_*}{\partial q^\lambda}-\beta_\lambda^{l+\chi}\frac{\partial T_*}{\partial q^{l+\chi}}-\frac{\partial T}{\partial \dot q^{l+\chi}}\beta_{\lambda\mu}^{l+\chi}\dot q^\mu =$$
$$= Q_\lambda + Q_{l+\chi}\beta_\lambda^{l+\chi}; \qquad \lambda,\mu = 1,2,...,l; \ \chi = 1,2,...,k, \qquad (15)$$

where

$$\beta_{\lambda\mu}^{l+\chi} = \frac{\partial \beta_\lambda^{l+\chi}}{\partial q^\mu}-\frac{\partial \beta_\mu^{l+\chi}}{\partial q^\lambda}+\frac{\partial \beta_\lambda^{l+\chi}}{\partial q^{l+\nu}}\beta_\mu^{l+\nu}-\frac{\partial \beta_\lambda^{l+\chi}}{\partial q^{l+\nu}}\beta_\lambda^{l+\nu}.$$

The equations (15) are called the *Voronets equations*. Joining the constraint equations (15) with the equations of motion (4), we will obtain a system of differential equations for obtaining the functions $q^\sigma(t), \ \sigma = 1,2,...,s$.

In the case of motion of a constrained system under action of forces, which have a potential, the equations (15) take the form

$$\frac{d}{dt}\frac{\partial T_*}{\partial \dot q^\lambda}-\frac{\partial (T_*+U)}{\partial q^\lambda}-\beta_\lambda^{l+\chi}\frac{\partial (T_*+U)}{\partial q^{l+\chi}}-\frac{\partial T}{\partial \dot q^{l+\chi}}\beta_{\lambda\mu}^{l+\chi}\dot q^\mu = 0, \qquad \lambda,\mu = 1,2,...,l.$$

In the particular case when the coordinates $q^{l+1}, q^{l+2},...,q^{l+k}$, corresponding to the eliminated velocities is not explicitly included into the relations for kinetic and potential energy, as well as into the constraint equations, the Voronets equations (15) coincide with the Chaplygin's equations (10).

4 The equations of motion in quasicoordinates (Hamel-Novoselov, Voronets-Hamel, Poincaré – Chetaev equations)

In the case of rotation of a rigid body around a fixed point, it was shown that the projections of the vector of instantaneous angular velocity $\overline{\omega}$ on the fixed axes cannot be considered as derivatives with respect to the new angles that uniquely determine the position of rigid body. Similarly, it may turn out that quantities υ_*^ρ, which are one-to-one connected with generalized velocities by the relations

$$\upsilon_*^\rho = \upsilon_*^\rho(t,q,\dot q), \qquad \rho = 1,2,...,s,$$

and

$$\hat\upsilon^\sigma \equiv \dot q^\sigma = \dot q^\sigma(t,q,\upsilon_*), \qquad \upsilon_* = (\upsilon_*^1,\upsilon_*^2,...,\upsilon_*^s),$$

cannot be considered as derivatives with respect to the certain new coordinates q_*^ρ, that is, cannot be supposed that $\upsilon_*^\rho = \dot q_*^\rho$. In this case, the quantities υ_*^ρ are called the *quasivelocities*, and the variables $\tilde\upsilon_*^\rho$, given by formulas

$$\tilde\upsilon_*^\rho = \int_{t_0}^{t}\upsilon_*^\rho dt$$

are called quasicoordinates.

In the expression for the kinetic energy T, the generalized velocities $\dot q^\sigma$ are replaced by quasivelocities υ_*^ρ. We denote the resulting function by T^*. We find out, which form can have the Maggi's equations $(M\omega_\sigma - Q_\sigma)\dfrac{\partial \dot q^\sigma}{\partial \upsilon_*^\lambda}=0, \qquad \lambda = 1,2,...,l$, written in the form

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\sigma}}-\frac{\partial T}{\partial q^{\sigma}}-Q_{\sigma}\right)\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_{*}^{\lambda}}=0, \qquad \lambda=1,2,...,l;\ \ \sigma=1,2,...,s, \tag{16}$$

when using the function T^{*}.

Taking into account the relations

$$\frac{\partial T^{*}}{\partial \upsilon_{*}^{\lambda}}=\frac{\partial T}{\partial \dot{q}^{\sigma}}\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_{*}^{\lambda}}, \qquad \frac{\partial T^{*}}{\partial q^{\sigma}}=\frac{\partial T}{\partial q^{\sigma}}+\frac{\partial T}{\partial \dot{q}^{\rho}}\frac{\partial \dot{q}^{\rho}}{\partial q^{\sigma}},$$

$$\lambda=1,2...,l;\ \ \rho,\sigma=1,2,...,s,$$

we have

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\sigma}}\right)\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_{*}^{\lambda}}=\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^{\sigma}}\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_{*}^{\lambda}}\right)-$$

$$-\frac{\partial T}{\partial \dot{q}^{\sigma}}\frac{d}{dt}\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_{*}^{\lambda}}=\frac{d}{dt}\frac{\partial T^{*}}{\partial \upsilon_{*}^{\lambda}}-\frac{\partial T}{\partial \dot{q}^{\sigma}}\frac{d}{dt}\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_{*}^{\lambda}}, \tag{17}$$

$$\frac{\partial T}{\partial q^{\sigma}}\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_{*}^{\lambda}}=\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_{*}^{\lambda}}\left(\frac{\partial T^{*}}{\partial q^{\sigma}}-\frac{\partial T}{\partial \dot{q}^{\rho}}\frac{\partial \dot{q}^{\rho}}{\partial q^{\sigma}}\right)=$$

$$=\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_{*}^{\lambda}}\frac{\partial T^{*}}{\partial q^{\sigma}}-\frac{\partial T}{\partial \dot{q}^{\rho}}\frac{\partial \dot{q}^{\rho}}{\partial q^{\sigma}}\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_{*}^{\lambda}}. \tag{18}$$

In the right-hand side of expression (18) in the double sum, we exchange the indices of summing ρ and σ. As a result, we have

$$\frac{\partial T}{\partial q^{\sigma}}\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_{*}^{\lambda}}=\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_{*}^{\lambda}}\frac{\partial T^{*}}{\partial q^{\sigma}}-\frac{\partial T}{\partial \dot{q}^{\rho}}\frac{\partial \dot{q}^{\rho}}{\partial \upsilon_{*}^{\lambda}}\frac{\partial \dot{q}^{\sigma}}{\partial q^{\rho}}. \tag{19}$$

Let us consider the operator

$$\frac{\partial}{\partial \tilde{\upsilon}_{*}^{\rho}}=\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_{*}^{\rho}}\frac{\partial}{\partial q^{\sigma}}, \qquad \rho,\sigma=1,2,...,s, \tag{20}$$

which, under assumption $\upsilon_{*}^{\rho}=\dot{\tilde{\upsilon}}_{*}^{\rho}=\dot{q}_{*}^{\rho}$, passes into the operator of partial derivative with respect to the new coordinate q_{*}^{ρ}, since we have

$$\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_{*}^{\rho}}\frac{\partial}{\partial q^{\sigma}}=\frac{\partial \dot{q}^{\sigma}}{\partial \dot{q}_{*}^{\rho}}\frac{\partial}{\partial q^{\sigma}}=\frac{\partial q^{\sigma}}{\partial q_{*}^{\rho}}\frac{\partial}{\partial q^{\sigma}}=\frac{\partial}{\partial q_{*}^{\rho}}.$$

The relation (19), taking into account expression (20), can be written in the form

$$\frac{\partial T}{\partial q^{\sigma}}\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_{*}^{\rho}}=\frac{\partial T^{*}}{\partial \tilde{\upsilon}_{*}^{\lambda}}-\frac{\partial T}{\partial \dot{q}^{\sigma}}\frac{\partial \dot{q}^{\sigma}}{\partial \tilde{\upsilon}_{*}^{\lambda}}.$$

Hence and from expression (17) it follows that the Maggi's equations (16) can be represented in the form

$$\frac{d}{dt}\frac{\partial T^{*}}{\partial \upsilon_{*}^{\lambda}}-\frac{\partial T^{*}}{\partial \tilde{\upsilon}_{*}^{\lambda}}-\frac{\partial T}{\partial \dot{q}^{\sigma}}\left(\frac{d}{dt}\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_{*}^{\lambda}}-\frac{\partial \dot{q}^{\sigma}}{\partial \tilde{\upsilon}_{*}^{\lambda}}\right)=Q_{\lambda}^{*}, \tag{21}$$

$$\lambda=1,2,...,l;\ \ \sigma=1,2,...,s.$$

Here

$$Q_{*}^{\lambda}=Q_{\sigma}\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_{*}^{\lambda}}. \tag{22}$$

Equations (21) are sometimes called the equations of Chaplygin's type [12].

Let us consider the particular case when the generalized velocities $\dot{q}^{\sigma}$ are related to quasivelocities υ_{*}^{ρ} by linear, homogeneous, stationary relations:

$$\upsilon_*^{\rho} = \alpha_{\sigma}^{\rho}(q)\dot{q}^{\sigma},$$
$$\rho,\sigma = 1,2,...,s, \tag{23}$$
$$\dot{q}^{\sigma} = \beta_{\rho}^{\sigma}(q)\upsilon_*^{\rho},$$

and the constraint equations are the following:

$$\upsilon_*^{l+\chi} \equiv \alpha_{\sigma}^{l+\chi}(q)\dot{q}^{\sigma} = 0. \tag{24}$$

In this case, using expressions (23) and operator (20), and also taking into account that after performing the operations of differentiation it can be assumed that $\upsilon_*^{l+\chi} = 0 \quad (\chi = 1,2,...,k)$, we have

$$\frac{d}{dt}\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_*^{\lambda}} = \frac{d}{dt}\beta_{\lambda}^{\sigma}(q) = \frac{\partial \beta_{\lambda}^{\sigma}}{\partial q_{\sigma}^{\rho}}\dot{q}^{\rho} = \frac{\partial \beta_{\lambda}^{\sigma}}{\partial q^{\rho}}\beta_{\mu}^{\rho}\upsilon_*^{\mu} =$$
$$= \upsilon_*^{\mu}\frac{\partial \dot{q}^{\rho}}{\partial \upsilon_*^{\mu}}\frac{\partial \beta_{\lambda}^{\sigma}}{\partial q^{\rho}} = \upsilon_*^{\mu}\frac{\partial \beta_{\lambda}^{\sigma}}{\partial \tilde{\upsilon}_*^{\mu}}, \qquad \lambda,\mu = 1,2,...,l;\ \ \rho,\sigma = 1,2,...,s;$$
$$\frac{\partial \dot{q}^{\sigma}}{\partial \tilde{\upsilon}_*^{\lambda}} = \frac{\partial \dot{q}^{\rho}}{\partial \upsilon_*^{\lambda}}\frac{\partial \dot{q}^{\sigma}}{\partial q^{\rho}} = \frac{\partial \dot{q}^{\rho}}{\partial \upsilon_*^{\lambda}}\frac{\partial \beta_{\mu}^{\sigma}}{\partial q^{\rho}}\upsilon_*^{\mu} =$$
$$= \upsilon_*^{\mu}\frac{\partial \beta_{\mu}^{\sigma}}{\partial \tilde{\upsilon}_*^{\lambda}}, \qquad \lambda,\mu = 1,2,...,l;\ \ \rho,\sigma = 1,2,...,s.$$

Consequently, equations (21) take the form

$$\frac{d}{dt}\frac{\partial T^*}{\partial \upsilon_*^{\lambda}} - \frac{\partial T^*}{\partial \tilde{\upsilon}_*^{\lambda}} - \frac{\partial T}{\partial \dot{q}^{\sigma}}\left(\frac{\partial \beta_{\lambda}^{\sigma}}{\partial \tilde{\upsilon}_*^{\mu}} - \frac{\partial \beta_{\mu}^{\sigma}}{\partial \tilde{\upsilon}_*^{\lambda}}\right)\upsilon_*^{\mu} = Q_{\lambda}^*, \tag{25}$$
$$\lambda,\mu = 1,2,...,l;\ \ \sigma = 1,2,...,s.$$

These equations are usually called the *Chaplygin's equations in quasicoordinates* [6]. Let us note that the equations (21) and (25) should be considered together with the equations of nonholonomic constraints given respectively in the form

$$\varphi^{\chi}(t,q,\dot{q}) = 0, \qquad \chi = 1,2,...,k,$$

and (24).

Equations (21) include both the function T^* and the function T. Now, we reduce the Maggi's equations (16) to the form that involves the function T^* only.

The following relation

$$\frac{\partial T}{\partial \dot{q}^{\sigma}} = \frac{\partial T^*}{\partial \upsilon_*^{\rho}}\frac{\partial \upsilon_*^{\rho}}{\partial \dot{q}^{\sigma}}, \qquad \rho,\sigma = 1,2,...,s,$$

yield the relation

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\sigma}}\right)\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_*^{\lambda}} = \frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_*^{\lambda}}\frac{d}{dt}\left(\frac{\partial T^*}{\partial \upsilon_*^{\rho}}\frac{\partial \upsilon_*^{\rho}}{\partial \dot{q}^{\sigma}}\right) =$$
$$= \left(\frac{d}{dt}\frac{\partial T^*}{\partial \upsilon_*^{\rho}}\right)\frac{\partial \upsilon_*^{\rho}}{\partial \dot{q}^{\sigma}}\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_*^{\lambda}} + \frac{\partial T^*}{\partial \upsilon_*^{\rho}}\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_*^{\lambda}}\frac{d}{dt}\frac{\partial \upsilon_*^{\rho}}{\partial \dot{q}^{\sigma}}.$$

Since

$$\frac{\partial \upsilon_*^{\rho}}{\partial \dot{q}^{\sigma}}\frac{\partial \dot{q}^{\sigma}}{\partial \upsilon_*^{\lambda}} = \begin{cases} 1, & \rho = \lambda, \\ 0, & \rho \neq \lambda, \end{cases}$$

we have

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\sigma}\right)\frac{\partial \dot{q}^\sigma}{\partial \upsilon_*^\lambda} = \frac{d}{dt}\frac{\partial T^*}{\partial \upsilon_*^\lambda} + \frac{\partial T^*}{\partial \upsilon_*^\rho}\frac{\partial \dot{q}^\sigma}{\partial \upsilon_*^\lambda}\frac{d}{dt}\frac{\partial \upsilon_*^\rho}{\partial \dot{q}^\sigma}. \quad (26)$$

Taking into account the expressions

$$\frac{\partial T}{\partial q^\sigma} = \frac{\partial T^*}{\partial q^\sigma} + \frac{\partial T^*}{\partial \upsilon_*^\rho}\frac{\partial \upsilon_*^\rho}{\partial q^\sigma}$$

and operator (20), we obtain

$$\frac{\partial T}{\partial q^\sigma}\frac{\partial \dot{q}^\sigma}{\partial \upsilon_*^\lambda} = \frac{\partial T^*}{\partial \tilde{\upsilon}_*^\lambda} + \frac{\partial T^*}{\partial \upsilon_*^\rho}\frac{\partial \dot{q}^\sigma}{\partial \upsilon_*^\lambda}\frac{\partial \upsilon_*^\rho}{\partial q^\sigma}.$$

Then from the above and from formulas (22) and (26), it follows that the Maggi's equations (16) can be represented in the form

$$\frac{d}{dt}\frac{\partial T^*}{\partial \upsilon_*^\lambda} - \frac{\partial T^*}{\partial \tilde{\upsilon}_*^\lambda} + \frac{\partial T^*}{\partial \upsilon_*^\rho}\frac{\partial \dot{q}^\sigma}{\partial \upsilon_*^\lambda}\left(\frac{d}{dt}\frac{\partial \upsilon_*^\rho}{\partial \dot{q}^\sigma} - \frac{\partial \upsilon_*^\rho}{\partial q^\sigma}\right) = Q_\lambda^*, \quad (27)$$

$$\lambda = 1,2,...,l; \quad \rho,\sigma = 1,2,...,s.$$

The equations (21) and (27) can be applied to both holonomic and nonholonomic systems, with either the linear or nonlinear with respect to velocities ideal constraints. In the case when the time does not enter into the kinetic energy and the constraint equations in explicit form, the equations (21) and (27) were obtained by G. Hamel [14] and in the general case by V.S. Novoselov. Therefore, these equations should be called the *Hamel-Novoselov equations.*

In the case when the quasivelocities are defined by formulas (23) and the constraints are given by equations (24), we have

$$\frac{\partial \dot{q}^\sigma}{\partial \upsilon_*^\lambda}\frac{d}{dt}\frac{\partial \upsilon_*^\rho}{\partial \dot{q}^\sigma} = \beta_\lambda^\sigma \frac{d\alpha_\sigma^\rho}{dt} = \beta_\lambda^\sigma \frac{\partial \alpha_\sigma^\rho}{\partial q^\tau}\dot{q}^\tau = \beta_\lambda^\sigma \beta_\mu^\tau \frac{\partial \alpha_\sigma^\rho}{\partial q^\tau}\upsilon_*^\mu,$$

$$\frac{\partial \dot{q}^\sigma}{\partial \upsilon_*^\lambda}\frac{\partial \upsilon_*^\rho}{\partial \dot{q}^\sigma} = \beta_\lambda^\sigma \frac{\partial \alpha_\tau^\rho}{\partial q^\sigma}\dot{q}^\tau = \beta_\lambda^\sigma \beta_\mu^\tau \frac{\partial \alpha_\tau^\rho}{\partial q^\sigma}\upsilon_*^\mu,$$

$$\lambda,\mu = 1,2,...,l; \quad \rho,\sigma,\tau = 1,2,...,s.$$

Consequently, in this case, equations (27) take the form

$$\frac{d}{dt}\frac{\partial T^*}{\partial \upsilon_*^\lambda} - \frac{\partial T^*}{\partial \tilde{\upsilon}_*^\lambda} + c_{\lambda\mu}^\rho \upsilon_*^\mu \frac{\partial T^*}{\partial \upsilon_*^\rho} = Q_\lambda^*,$$

$$c_{\lambda\mu}^\rho = \left(\frac{\partial \alpha_\sigma^\rho}{\partial q^\tau} - \frac{\partial \alpha_\tau^\rho}{\partial q^\sigma}\right)\beta_\lambda^\sigma \beta_\mu^\tau, \quad (28)$$

$$\lambda,\mu = 1,2,...,l; \quad \rho,\sigma,\tau = 1,2,...,s.$$

In the case of $l = s$ these equations, as well as the expressions for the coefficients $c_{\sigma\tau}^\rho$ were obtained first by P.V. Voronets in 1901. In 1904 for $l < s$ these results were obtained once more by G. Hamel [15]. Therefore, these equations are usually called the *Voronets-Hamel equations*, but Hamel himself called them the Euler-Lagrange equations. We remark that in the literature they are also called the *Hamel-Boltzmann equations*.

A little before the work of P.V. Voronets, it was appeared an article by H. Poincaré [16], who obtained equations highly close to equations (28). Poincaré equations correspond to the case when in equations (28) for $l = s$ the coefficients $c_{\sigma\tau}^\rho$ are constant and the forces are expressed via the forcing function U :

$$Q_\tau^* = \beta_\tau^\sigma \frac{\partial U}{\partial q^\sigma}, \qquad \sigma,\tau = 1,2,...,s.$$

In this case, equations (28) can be written in the form proposed by H. Poincaré:

$$\frac{d}{dt}\frac{\partial L^*}{\partial \upsilon_*^\tau} = c_{\sigma\tau}^\rho \upsilon_*^\delta \frac{\partial L^*}{\partial \upsilon_*^\rho} + \beta_\tau^\sigma \frac{\partial L^*}{\partial q^\sigma}, \qquad L^*(q,\upsilon_*) = T^* + U,$$

$$\rho,\sigma,\tau = 1,2,...,s.$$

On derivation of equations of motion, H. Poincaré used the group theory. The Poincaré approach was subsequently developed in the works of N.G. Chetaev, L.M. Markhashov, V.V. Rumyantsev, Fam Guen. They generalized the Poincaré equations to the case when the coefficients of $c_{\sigma\tau}^\rho$ are not constant and the motion occurs under the action of both potential and non-potential forces. The equations derived by them describing the motion of nonholonomic systems are called *Poincaré-Chetaev equations* [4], [13].

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