Mathematical Model of Micropolar Elastic Thin Beams with Constrained Rotations and The Finite Element Method

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Abstract

In this paper hypotheses are accepted, which adequately replace properties of asymptotic solution of boundary-value problem of plane stress state of micropolar theory of elasticity with constrained rotation in thin two-dimensional region. On the basis of them applied model of bending deformation of thin beams is constructed. The appropriate algorithm of the finite element method is developed for solving boundary problems of statics and free oscillations of bending deformation of micropolar elastic thin beams with constrained rotation. On the basis of the analysis of the numerical results effective properties of the micropolarity of the material are established compared to the classical case.

1 Introduction

The question of mathematical modeling of an object [1] generates a clear action plan. It can be conditionally divided into three stages: model-algorithm-program.
This paper relates to the mathematical modeling of bending deformation of elastic thin beam in the formulation of the moment theory of elasticity with constrained rotation and with consideration of transverse shear strains, also to the development of the finite element method for solving applied problems in this field.
In the first part of the paper previously accepted approach approach [2-4] is developed and based on the equations of the plane problem of the moment theory of elasticity with constrained rotation [5] an applied model of bending deformation of thin beam is constructed with the derivation of the formula for the density of the potential energy of deformation.
The finite element method (FEM), which is closely connected to a personal computer and completely corresponds to it, is very popular now among the numerical methods. The main functional For FEM in the form of displacements is the total potential energy of the system [6,7].
In the second part of the paper an FEM is developed for solving concrete problems of determination the stress-strain state of the bending deformation of micropolar elastic thin beams with constrained rotation, which is realized on a personal computer.

2 Problem statement

An isotropic micropolar elastic parallelepiped of constant height $2h$, length a and thickness $2h_1 = 1$ is considered. The coordinate plane x_1x_3 is placed in the middle plane of the parallelepiped. The axis x_3 is directed along the height and x_1-along the length of the parallelepiped, which divides the height $2h$ in half. It is assumed that plane stress state is realized in direction of the axis x_2. Basic equations of the generalized plane stress state of micropolar theory of elasticity with constrained rotation are given in paper [5].

Our aim is to construct an applied (one-dimensional) model of bending deformation of thin beam with transverse shear deformations, taking the equations of the two-dimensional theory of the generalized plane stress state of micropolar elasticity with constrained rotation as a basis, with application of the already developed [2-4] approach.

The description of the law of the change of displacements and rotation along the beam's thickness is taken to be linear[2]:

$$V_3 = w(x_1), \;\; V_1 = x_3\psi_1(x_1), \;\; \omega_2 = \Omega_2(x_1). \tag{1.1}$$

In papers [2-4] the kinematic hypothesis (1.1) is called Timoshenko's generalized hypothesis for the micropolar case (since the formulas for displacements in (1.1) coincide with the formulas of Timoshenko's kinematic hypothesis [8] in the classical theory of elasticity).

In the micropolar theory of elasticity with constrained rotation, the rotations of body particles are expressed through displacements as in the classical theory of elasticity ($\overrightarrow{\omega} = \frac{1}{2} rot \overrightarrow{V}$), so that in this case we will have:

$$\omega_2 = \Omega_2(x_1) = \frac{1}{2}(\psi_1 - \frac{dw}{dx_1}). \tag{1.2}$$

Besides the kinematic hypothesis (1.1), static hypotheses were developed in paper [2] to reduce two-dimensional problem to one-dimensional one.

On the basis of these hypotheses one-dimensional model (applied model) of the bending deformation of micropolar elastic thin beams with constrained rotation will be obtained from the above mentioned two-dimensional theory:

Equilibrium equations(motion)

$$\frac{\partial N_{13}}{\partial x_1} = -2q(+2\rho h \frac{\partial^2 w}{\partial t^2}),$$

$$\frac{\partial M_{11}}{\partial x_1} - N_{31} = -h \cdot 2q_1(+\frac{2\rho h^3}{3}\frac{\partial^2 \psi_1}{\partial t^2}),$$

$$\frac{\partial L_{12}}{\partial x_1} + N_{31} - N_{13} = -2m_3(+2Jh\frac{\partial^2 \Omega_2}{\partial t^2}). \tag{1.3}$$

Elasticity relations

$$N_{13}+N_{31}=4h\mu\Gamma_{13},$$

$$M_{11}=\frac{2Eh^3}{3}K_{11},\ \ L_{12}=2Bhk_{12}. \tag{1.4}$$

Geometrical relations

$$\Gamma_{13}=\frac{\partial w}{\partial x_1}+\psi_1,\ \ K_{11}=\frac{\partial \psi_1}{\partial x_1},$$

$$k_{12}=\frac{\partial \Omega_2}{\partial x_1},\ \ \Omega_2=\frac{1}{2}(\psi_1-\frac{\partial w}{\partial x_1}). \tag{1.5}$$

Here N_{13}, N_{31} are averaged forces along the beam thickness; M_{11}, L_{12} are averaged moments of power stress σ_{11} and moment stress μ_{12} along the beam thickness; Γ_{13} is shear deformation; K_{11} is beam axis bending (connected with transfer moment M_{11}), and k_{12} is beam axis bending (connected with transfer moment L_{12}); $2q$ is intensity of the load distributed normally to the beam axis; $2q_1$ is intensity of the load distributed parallel to the beam axis; $2m_3$ is intensity of external moment; E and μ are classical modules of elasticity and shear of beam material; B is new elastic constant of beam micropolar material.
Boundary conditions on the edge (on $x_1 = 0$ or $x_1 = a$) of the beam are the followings:

$$M_{11}=M_{11}^*,\ \text{ or }\ \psi_1=\psi_1^*,$$

$$N_{13}=N_{31}^*,\ \text{ or }\ w=w^*,$$

$$L_{12}=L_{12}^*,\ \text{ or }\ \Omega_2=\Omega_2^*. \tag{1.6}$$

General form of the total potential energy functional of the system is expressed as follows:

$$U=\int_0^a (W-2q_1h\psi_1-2qw-2m_3\Omega_2)\,dx_1-$$

$$-\left((M_{11}\psi_1+N_{13}w+L_{12}\Omega_2)_{x_1=a}-(M_{11}\psi_1+N_{13}w+L_{12}\Omega_2)_{x_1=0}\right), \tag{1.7}$$

where

$$W=E\frac{h^3}{3}K_{11}^2+h\mu\Gamma_{13}^2+Bhk_{12}^2\,. \tag{1.8}$$

W is linear density of deformation potential energy of micropolar beam during the bending (Ω_2 expressed by formula (1.2)).
Minimizing the functional (1.7) basic differential equations (1.3)-(1.5) and natural boundary conditions (1.6) will be obtained for bending deformation of micropolar beam.

3 Stiffness matrix of finite element of micropolar beam

Let's consider determination of stiffness matrix of micropolar beam finite element.

Following expansions in the form of cubic polynomials are chosen for deflection w, complete rotation ψ_1 of normal element:

$$w(x_1) = a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3,$$

$$\psi_1(x_1) = b_0 + b_1x_1 + b_2x_1^2 + b_3x_1^3. \tag{2.1}$$

Here a_i, b_i are coefficients, which are expressed with the help of nodal displacements and rotations. Nodal displacements are denoted as follows:

$$w(0) = \delta_1,\ w'(0) = \delta_2,\ \psi_1(0) = \delta_3,\ \psi_1'(0) = \delta_4,$$

$$w(a) = \delta_5,\ w'(a) = \delta_6,\ \psi_1(a) = \delta_7,\ \psi_1'(a) = \delta_8,\ . \tag{2.2}$$

As we can see above mentioned finite element has eight degrees of independence. Substituting (2.1) into (2.2), coefficients a_i, b_i will be expressed with the help of nodal displacements and rotations δ_k. Substituting a_i, b_i into (2.1), we obtain following approximations for displacements and rotations.

$$w(x_1) = \sum_{i=1,2,5,6} \delta_i N_i(x_1),$$

$$\psi_1(x_1) = \sum_{i=3,4,7,8} \delta_i N_i(x_1), \tag{2.3}$$

here $N_i(x)$ are form functions of the element:

$$N_1 = N_3 = 1 - \frac{3}{a^2}x_1^2 + \frac{2}{a^3}x_1^3,\ N_2 = N_4 = x_1 - \frac{2}{a}x_1^2 + \frac{1}{a^2}x_1^3,$$

$$N_5 = N_7 = \frac{3}{a^2}x_1^2 - \frac{2}{a^3}x_1^3,\ N_6 = N_8 = -\frac{1}{a}x_1^2 + \frac{1}{a^2}x_1^3. \tag{2.4}$$

Substituting (2.3) into functional (1.7), after integration we obtain function of eight independent variables $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8$. The minimization of functional (1.7) reduces to the determination of the minimum of function of eight independent variables:

$$\frac{\partial U}{\partial \delta_k} = 0 \quad (k = 1, 2, 3, \dots, 8).$$

Calculating corresponding partial derivatives, we obtain system of linear algebraic equations:

$$[K] \cdot \{\delta\} = \{P\}. \tag{2.5}$$

Here K is stiffness matrix of element with size 8×8, which is the most important concept of the finite element method; $\{\delta\}^T = \{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8\}$ is vector of nodal displacements and rotations; $\{P\}^T$ is vector concentrated nodal forces and moments.

Expressions for the elements of the stiffness matrix of a finite element are introduced below:

$$K_{11} = -K_{15} = K_{55} = \frac{6h(5B + 2a^2\mu)}{5a^3},$$

$$K_{12} = K_{16} = -K_{25} = -K_{76} = \frac{3Bh}{a^2} + \frac{h\mu}{5},$$

$$K_{13} = K_{17} = -K_{35} = -K_{57} = -h\mu,$$

$$K_{14} = -K_{18} = -K_{23} = K_{27} = K_{36} = -K_{45} = K_{58} = -K_{67} = \frac{Bh}{2a} - \frac{ha\mu}{5},$$

$$K_{22} = K_{66} = \frac{2Bh}{a} + \frac{4ah\mu}{15}, \quad K_{24} = K_{68} = \frac{Bh}{4}, \quad K_{26} = \frac{Bh}{a} - \frac{ha\mu}{15},$$

$$K_{28} = -K_{46} = -\frac{Bh}{4} + \frac{a^2h\mu}{30}, \quad K_{33} = K_{77} = \frac{h(21B + 26a^2\mu + 28h^2E)}{35a},$$

$$K_{34} = -K_{78} = \frac{h(21B + 44a^2\mu + 28h^2E)}{420},$$

$$K_{37} = \frac{h(-21B + 9a^2\mu - 28h^2E)}{35a},$$

$$K_{38} = -K_{47} = \frac{h(21B - 26a^2\mu + 28h^2E)}{420},$$

$$K_{44} = K_{88} = \frac{ah(21B + 6a^2\mu + 28h^2E)}{315},$$

$$K_{48} = -\frac{ah(21B + 18a^2\mu + 28h^2E)}{1260}.$$

4 Model calculation of micropolar elastic beams with constrained rotation for the static problem

As an example we'll consider problem of the bending of the beam when evenly distributed load with intensity q is acting along the axis x_1 (in this case $q_1 = 0, q \neq 0, m_3 = 0$) and the edges are hinged-supported. Boundary conditions for hinged supported beam are follows:

$$w = 0, \quad M_{11} = 0, \quad L_{12} = 0, \quad \text{on } x_1 = 0; a. \tag{3.1}$$

We obtain following expression for functional (1.7) with consideration of (3.1):

$$U = \int_0^a (W - 2qw)dx_1.$$

After the constructing of the stiffness matrix K, the vector of equivalent nodal forces and moments P, with consideration of the boundary conditions (3.1), we form a system of linear algebraic equations (2.5) corresponding to the considered problem for different numbers of dividing the beam into finite elements.

We consider the case when the beam is divided into two finite elements. Numerical results (maximum deflection) of the calculation are given for the case, when the physical constants have following values: $\mu = 0,75MPa$, $E = 191MPa$, $B = 1000N$, load is $q = 0,5 \cdot 10^3 Pa$, and geometrical dimensions of the beam are the followings: $a = 8mm$, $h = 0.2mm$ (we also introduce the result for classical theory of elastic thin beam, when it is bent).

As can be seen from the given values of Table 1, the micropolarity of the material of the beam increases the stiffness of the beam compared with the classical case of the material.

Table 11: The maximum deflection of micropolar and classical beam.

	Micropolar beam			Classical beam			
w_{max}	Exact value	2 finite elements	4 finite elements	Exact value	2 finite elements	4 finite elements	$\frac{w^{cl}_{max}-w^{mic}_{max}}{w^{cl}_{max}}$
(m)	$2,86\cdot 10^{-8}$	$2,59\cdot 10^{-8}$	$2,77\cdot 10^{-8}$	$4\cdot 10^{-8}$	$3,62\cdot 10^{-8}$	$3,86\cdot 10^{-8}$	$0,285$

5 Dynamic problem of a micropolar elastic beam with constrained rotation

The general form of the functional of the total mechanical energy (the sum of the potential energy of deformation and kinetic energy) of a micropolar-elastic beam for bending deformation is expressed as follows:

$$\widetilde{U} = \int_0^a (W + \rho h \frac{\partial^2 w}{\partial t^2} \cdot w + \frac{\rho h^3}{3} \frac{\partial^2 \psi_1}{\partial t^2} \cdot \psi_1 + Jh \frac{\partial^2 \Omega_2}{\partial t^2} \cdot \Omega_2) dx_1. \tag{4.1}$$

In case of free oscillations the main kinematic functions of the problem are introduced in this way:

$$w(x_1, t) = (a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3) \sin \omega t,$$
$$\psi_1(x_1, t) = (b_0 + b_1 x_1 + b_2 x_1^2 + b_3 x_1^3) \sin \omega t, \tag{4.2}$$

where ω is frequency of the natural oscillation.
Substituting (4.2) into (4.1), the problem of minimizing the functional (4.1) is reduced to the obtaining of the minimum of the function of eight independent variables ($\frac{\partial U}{\partial \delta_k} = 0, k = 1, 2, 3, ...8$).
Calculating the corresponding partial derivatives, we obtain the following matrix equation:

$$(K - \omega^2 M) \cdot \{\delta\} = 0, \tag{4.3}$$

where K is stiffness matrix of finite element, M is the matrix of masses of a finite element.
Expressions for the elements of the matrix of masses of the finite element are introduced below:

$$M_{11} = M_{55} = \frac{3hJ}{5a} + \frac{26ha\rho}{35}, \quad M_{12} = -M_{56} = \frac{hJ}{20} + \frac{11ha^2\rho}{105},$$

$$M_{13} = M_{17} = -M_{35} = -M_{57} = \frac{hJ}{4},$$

$$M_{14} = -M_{18} = -M_{23} = M_{27} = M_{36} = -M_{45} = M_{58} = -M_{67} = \frac{ahJ}{20},$$

$$M_{15} = -\frac{3hJ}{5a} + \frac{9}{35} ah\rho, \quad M_{16} = -M_{25} = \frac{h(21J - 26a^2\rho)}{420},$$

$$M_{22} = M_{66} = \frac{h(7aJ + 2a^3\rho)}{105}, \quad M_{24} = M_{68} = 0,$$

$$M_{26} = \frac{h(-14aJ - 12a^3\rho)}{840}, \quad M_{28} = -M_{46} = -\frac{a^2hJ}{120},$$
$$M_{33} = M_{77} = \frac{13}{210}ah(3J + 4h^2\rho), \quad M_{34} = -M_{78} = 11a^2h(\frac{J}{420} + \frac{h^2\rho}{315}),$$
$$M_{37} = \frac{3}{140}ah(3J + 4h^2\rho), \quad M_{38} = -M_{47} = -\frac{13}{2520}a^2h(3J + 4h^2\rho),$$
$$M_{44} = M_{88} = \frac{ha^3J}{210} + \frac{2}{315}h^3a^3\rho, \quad M_{48} = -\frac{3ha^3J}{840} - \frac{1}{210}h^3a^3\rho.$$

We formulate the equation to determine the frequencies of free oscillations:

$$|K^{-1}M - \frac{1}{\omega^2}E| = 0.$$

The results of numerical calculations (when the beam is divided into two finite elements) we given for the case, when physical constants of the beam have the values of the previous problem, and $\rho = 7700kg/m^3$, $J = 5,3 \cdot 10^{-6}kg/m$.

Table 12: The lowest frequency of free oscillation ω.

		Micropolar beam(sec^{-1})		Classical beam(sec^{-1})	
$a(m)$	$h(m)$	Exact value	2 finite elements	Exact value	2 finite elements
$8 \cdot 10^{-3}$	$0,2 \cdot 10^{-3}$	$0,848 \cdot 10^5$	$0,849 \cdot 10^5$	$0,7181 \cdot 10^5$	$0,7184 \cdot 10^5$
10^{-7}	$0,5 \cdot 10^{-9}$	$0,1965 \cdot 10^{11}$	$0,1969 \cdot 10^{11}$	$0,1404 \cdot 10^{10}$	$0,1408 \cdot 10^{10}$
10^{-8}	$0,5 \cdot 10^{-10}$	$0,1965 \cdot 10^{12}$	$0,1969 \cdot 10^{12}$	$0,1404 \cdot 10^{11}$	$0,1408 \cdot 10^{11}$

As can be seen from the tables above, the micropolarity of the beam material increases the frequency of oscillations, and in the nanosized region, the frequencies are in the terahertz range.

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