Non-smooth first integrals of dissipative systems with four degrees of freedom

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Abstract

In this study, we show the integrability of certain classes of dynamic systems on the tangent bundle to a four-dimensional manifold. In this case, the force fields have so-called variable dissipation and generalize the cases considered previously.

1 Introduction

In many problems of dynamics, mechanical systems arise with the space of positions — a four-dimensional manifold. Their phase spaces naturally become the tangent bundles to these manifolds. Thus, for example, the study of a five-dimensional generalized spherical pendulum in a nonconservative force field leads to a dynamic system on the tangent bundle to a four-dimensional sphere, while the special metric on it is induced by an additional symmetry group. In this case, the dynamic systems describing the motion of such a pendulum have alternating dissipation and the complete list of first integrals consists of transcendental functions expressed through a finite combination of elementary functions.

We also single out the class of problems on the motion of a point along a four-dimensional surface, the metric on it being induced by the Euclidean metric of a comprehensive space. In a number of cases, the complete list of first integrals consisting of transcendental functions can also be found in systems with dissipation. The results obtained are especially important in the sense of the presence of a precisely nonconservative force field in the system.

2 Equations of geodesic lines under a change of coordinates and its first integrals

It is well known that, in the case of a four-dimensional Riemannian manifold M^4 with coordinates (α, β), $\beta = (\beta_1, \beta_2, \beta_3)$, and affine connection $\Gamma^i_{jk}(x)$ the equations of geodesic lines on the tangent bundle $T_*M^4\{\dot\alpha, \dot\beta_1, \dot\beta_2, \dot\beta_3; \alpha, \beta_1, \beta_2, \beta_3\}$, $\alpha = x^1$, $\beta_1 = x^2$, $\beta_2 = x^3$, $\beta_3 = x^4$, $x = (x^1, x^2, x^3, x^4)$, have the following form (the derivatives are taken with respect to the natural parameter):

$$\ddot{x}^i + \sum_{j,k=1}^{4} \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k = 0, \ i = 1, \ldots, 4. \tag{1}$$

Let us study the structure of Eqs. (1) under a change of coordinates on the tangent bundle T_*M^4. Consider a change of coordinates of the tangent space:

$$\dot{x}^i = \sum_{j=1}^{4} R^{ij}(x)z_j, \tag{2}$$

which can be inverted:

$$z_j = \sum_{i=1}^{4} T_{ji}(x)\dot{x}^i,$$

here $R^{ij}, T_{ji}, i, j = 1, \ldots, 4$, are functions of x^1, x^2, x^3, x^4, and

$$RT = E,$$

where

$$R = (R^{ij}), \ T = (T_{ji}).$$

We also call Eqs. (2) new kinematic relations, i.e., relations on the tangent bundle T_*M^4.

The following equalities are valid:

$$\dot{z}_j = \sum_{i=1}^{4} \dot{T}_{ji}\dot{x}^i + \sum_{i=1}^{4} T_{ji}\ddot{x}^i, \ \dot{T}_{ji} = \sum_{k=1}^{4} T_{ji,k}\dot{x}^k, \tag{3}$$

where

$$T_{ji,k} = \frac{\partial T_{ji}}{\partial x^k}, \ j, i, k = 1, \ldots, 4.$$

We also have:

$$\dot{z}_i = \sum_{j,k=1}^{4} T_{ij,k}\dot{x}^j\dot{x}^k - \sum_{j,p,q=1}^{4} T_{ij}\Gamma^j_{pq}\dot{x}^p\dot{x}^q. \tag{4}$$

Otherwise, we can rewrite Eq. (4) in the form

$$\dot{z}_i + \sum_{j,k=1}^{4} Q_{ijk}\dot{x}^j\dot{x}^k|_{(2)} = 0, \tag{5}$$

where

$$Q_{ijk}(x) = \sum_{s=1}^{4} T_{is}(x)\Gamma^s_{jk}(x) - T_{ij,k}(x). \tag{6}$$

Proposition 2.1. *System (1) is equivalent to compound system (2), (4) in a domain where* $\det R(x) \neq 0$.

Therefore, the result of the passage from equations of geodesic lines (1) to an equivalent system of equations (2), (4) depends both on the change of variables (2) (i.e., introduced kinematic relations) and on the affine connection $\Gamma^i_{jk}(x)$.

3 A fairly general case

Consider next a sufficiently general case of specifying kinematic relations in the following form:

$$\begin{aligned}
&\dot{\alpha} = -z_4,\\
&\dot{\beta}_1 = z_3 f_1(\alpha),\\
&\dot{\beta}_2 = z_2 f_2(\alpha) g_1(\beta_1),\\
&\dot{\beta}_3 = z_1 f_3(\alpha) g_2(\beta_1) h(\beta_2),
\end{aligned} \tag{7}$$

where $f_k(\alpha)$, $k = 1, 2, 3$, $g_l(\beta_1)$, $l = 1, 2$, $h(\beta_2)$ are smooth functions on their domain of definition. Such coordinates z_1, z_2, z_3, z_4 in the tangent space are introduced when the following equations of geodesic lines are considered [1, 2, 3] (in particular, on surfaces of revolution):

$$\begin{cases}
\ddot{\alpha} + \Gamma^{\alpha}_{11}(\alpha,\beta)\dot{\beta}_1^2 + \Gamma^{\alpha}_{22}(\alpha,\beta)\dot{\beta}_2^2 + \Gamma^{\alpha}_{33}(\alpha,\beta)\dot{\beta}_3^2 = 0,\\
\ddot{\beta}_1 + 2\Gamma^{1}_{\alpha 1}(\alpha,\beta)\dot{\alpha}\dot{\beta}_1 + \Gamma^{1}_{22}(\alpha,\beta)\dot{\beta}_2^2 + \Gamma^{1}_{33}(\alpha,\beta)\dot{\beta}_3^2 = 0,\\
\ddot{\beta}_2 + 2\Gamma^{2}_{\alpha 2}(\alpha,\beta)\dot{\alpha}\dot{\beta}_2 + 2\Gamma^{2}_{12}(\alpha,\beta)\dot{\beta}_1\dot{\beta}_2 + \Gamma^{2}_{33}(\alpha,\beta)\dot{\beta}_3^2 = 0,\\
\ddot{\beta}_3 + 2\Gamma^{3}_{\alpha 3}(\alpha,\beta)\dot{\alpha}\dot{\beta}_3 + 2\Gamma^{3}_{13}(\alpha,\beta)\dot{\beta}_1\dot{\beta}_3 + 2\Gamma^{3}_{23}(\alpha,\beta)\dot{\beta}_2\dot{\beta}_3 = 0,
\end{cases} \tag{8}$$

i.e., other connection coefficients are zero. In case (7) Eqs. (4) take the form

$$\begin{aligned}
\dot{z}_1 = &\left[2\Gamma^{3}_{\alpha 3}(\alpha,\beta) + \frac{d\ln|f_3(\alpha)|}{d\alpha}\right] z_1 z_4 - \left[2\Gamma^{3}_{13}(\alpha,\beta) + \frac{d\ln|g_2(\beta_1)|}{d\beta_1}\right] f_1(\alpha) z_1 z_3 -\\
&- \left[2\Gamma^{3}_{23}(\alpha,\beta) + \frac{d\ln|h(\beta_2)|}{d\beta_2}\right] f_2(\alpha) g_1(\beta_1) z_1 z_2,\\
\dot{z}_2 = &\left[2\Gamma^{2}_{\alpha 2}(\alpha,\beta) + \frac{d\ln|f_2(\alpha)|}{d\alpha}\right] z_2 z_4 - \left[2\Gamma^{2}_{12}(\alpha,\beta) + \frac{d\ln|g_1(\beta_1)|}{d\beta_1}\right] f_1(\alpha) z_2 z_3 -\\
&- \Gamma^{2}_{33}(\alpha,\beta)\frac{f_3^2(\alpha)}{f_2(\alpha)}\frac{g_2^2(\beta_1)}{g_1(\beta_1)} h^2(\beta_2) z_1^2,\\
\dot{z}_3 = &\left[2\Gamma^{1}_{\alpha 1}(\alpha,\beta) + \frac{d\ln|f_1(\alpha)|}{d\alpha}\right] z_3 z_4 - \Gamma^{1}_{22}(\alpha,\beta)\frac{f_2^2(\alpha)}{f_1(\alpha)} g_1^2(\beta_1) z_2^2 -\\
&- \Gamma^{1}_{33}(\alpha,\beta)\frac{f_3^2(\alpha)}{f_1(\alpha)} g_2^2(\beta_1) h^2(\beta_2) z_1^2,\\
\dot{z}_4 = &\,\Gamma^{\alpha}_{11} f_1^2(\alpha) z_3^2 + \Gamma^{\alpha}_{22} f_2^2(\alpha) g_1^2(\beta_1) z_2^2 + \Gamma^{\alpha}_{33} f_3^2(\alpha) g_2^2(\beta_1) h^2(\beta_2) z_1^2,
\end{aligned} \tag{9}$$

and Eqs. (8) are almost everywhere equivalent to compound system (7), (9) on the manifold $T_*M^4\{z_4, z_3, z_2, z_1; \alpha, \beta_1, \beta_2, \beta_3\}$.

To integrate system (7), (9) completely, it is necessary to know, generally speaking, seven independent first integrals.

Proposition 3.1. *If the system of equalities*

$$\begin{cases} 2\Gamma^1_{\alpha 1}(\alpha,\beta) + \dfrac{d\ln|f_1(\alpha)|}{d\alpha} + \Gamma^\alpha_{11}(\alpha,\beta)f_1^2(\alpha) \equiv 0, \\ 2\Gamma^2_{\alpha 2}(\alpha,\beta) + \dfrac{d\ln|f_2(\alpha)|}{d\alpha} + \Gamma^\alpha_{22}(\alpha,\beta)f_2^2(\alpha)g_1^2(\beta_1) \equiv 0, \\ \left[2\Gamma^2_{12}(\alpha,\beta) + \dfrac{d\ln|g_1(\beta_1)|}{d\beta_1}\right] f_1^2(\alpha) + \Gamma^1_{22}(\alpha,\beta)f_2^2(\alpha)g_1^2(\beta_1) \equiv 0, \\ 2\Gamma^3_{\alpha 3}(\alpha,\beta) + \dfrac{d\ln|f_3(\alpha)|}{d\alpha} + \Gamma^\alpha_{33}(\alpha,\beta)f_3^2(\alpha)g_2^2(\beta_1)h^2(\beta_2) \equiv 0, \\ \left[2\Gamma^3_{13}(\alpha,\beta) + \dfrac{d\ln|g_2(\beta_1)|}{d\beta_1}\right] f_1^2(\alpha) + \Gamma^1_{33}(\alpha,\beta)f_3^2(\alpha)g_2^2(\beta_1)h^2(\beta_2) \equiv 0, \\ \left[2\Gamma^3_{23}(\alpha,\beta) + \dfrac{d\ln|h(\beta_2)|}{d\beta_2}\right] f_2^2(\alpha)g_1^2(\beta_1) + \Gamma^2_{33}(\alpha,\beta)f_3^2(\alpha)g_2^2(\beta_1)h^2(\beta_2) \equiv 0, \end{cases} \tag{10}$$

is valid everywhere in its domain of definition, system (7), (9) has an analytic first integral of the form

$$\Phi_1(z_4,\ldots,z_1) = z_1^2 + \ldots + z_4^2 = C_1^2 = const. \tag{11}$$

One can prove a special existence theorem for the solution $f_k(\alpha)$, $k = 1,2,3$, $g_l(\beta_1)$, $l = 1,2$, $h(\beta_2)$ of system (10) for the presence of analytic integral (11) for system (7), (9) of equations of geodesic lines. Below, however, we do not need all conditions (10) in studying dynamic systems with dissipation. Nevertheless, in what follows, we suppose that the condition

$$f_1(\alpha) = f_2(\alpha) = f_3(\alpha) = f(\alpha), \tag{12}$$

is satisfied in Eqs. (7); the functions $g_l(\beta_1)$, $l = 1,2$, $h(\beta_2)$ must satisfy the transformed third equality from (10):

$$\begin{cases} 2\Gamma^2_{12}(\alpha,\beta) + \dfrac{d\ln|g_1(\beta_1)|}{d\beta_1} + \Gamma^1_{22}(\alpha,\beta)g_1^2(\beta_1) \equiv 0, \\ 2\Gamma^3_{13}(\alpha,\beta) + \dfrac{d\ln|g_2(\beta_1)|}{d\beta_1} + \Gamma^1_{33}(\alpha,\beta)g_2^2(\beta_1)h^2(\beta_2) \equiv 0, \\ 2\Gamma^3_{23}(\alpha,\beta) + \dfrac{d\ln|h(\beta_2)|}{d\beta_2} + \Gamma^2_{33}(\alpha,\beta)h^2(\beta_2) \equiv 0. \end{cases} \tag{13}$$

Thus, the functions $g_l(\beta_1)$, $l = 1,2$, $h(\beta_2)$ depend on the connection coefficients; as for restrictions on the function $f(\alpha)$ they are given below.

Proposition 3.2. *If properties (12) and (13) are valid, and the equalities*

$$\Gamma^1_{\alpha 1}(\alpha,\beta) = \Gamma^2_{\alpha 2}(\alpha,\beta) = \Gamma^3_{\alpha 3}(\alpha,\beta) = \Gamma_1(\alpha), \tag{14}$$

are satisfied, system (7), (9) has a smooth first integral of the following form:

$$\Phi_2(z_3,z_2,z_1;\alpha) = \sqrt{z_1^2+z_2^2+z_3^2}\;\Phi_0(\alpha) = C_2 = const, \tag{15}$$

$$\Phi_0(\alpha) = f(\alpha)\exp\left\{2\int_{\alpha_0}^{\alpha}\Gamma_1(b)db\right\}.$$

Proposition 3.3. *If the properties in proposition 3.2 are valid, and also*

$$g_1(\beta_1) = g_2(\beta_1) = g(\beta_1), \tag{16}$$

herewith the equalities

$$\Gamma^2_{12}(\alpha, \beta) = \Gamma^3_{13}(\alpha, \beta) = \Gamma_2(\beta_1), \tag{17}$$

are valid, that system (7), (9) has a smooth first integral of the following form:

$$\Phi_3(z_2, z_1; \alpha, \beta_1) = \sqrt{z_1^2 + z_2^2}\Phi_0(\alpha)\Psi_1(\beta_1) = C_3 = const, \tag{18}$$

$$\Psi_1(\beta_1) = g(\beta_1)\exp\left\{2\int_{\beta_{10}}^{\beta_1}\Gamma_2(b)db\right\}.$$

Proposition 3.4. *If the properties in propositions 3.2, 3.3 are valid, herewith the equality*

$$\Gamma^3_{23}(\alpha, \beta) = \Gamma_3(\beta_2), \tag{19}$$

are valid, that system (7), (9) has a smooth first integral of the following form:

$$\Phi_4(z_1; \alpha, \beta_1, \beta_2) = z_1\Phi_0(\alpha)\Psi_1(\beta_1)\Psi_2(\beta_2) = C_4 = const, \tag{20}$$

$$\Psi_2(\beta_2) = h(\beta_2)\exp\left\{2\int_{\beta_{20}}^{\beta_2}\Gamma_3(b)db\right\}.$$

Proposition 3.5. *If the properties in propositions 3.2, 3.3, 3.4 are valid, that system (7), (9) has a first integral of the following form:*

$$\Phi_5(z_2, z_1; \alpha, \beta) = \beta_3 \pm \int_{\beta_{20}}^{\beta_2}\frac{C_4 h(b)}{\sqrt{C_3^2\Phi_2^2(b) - C_4^2}}\,db = C_5 = const. \tag{21}$$

Under the conditions listed above, system (7), (9) has a complete set (five) of independent first integrals of the form (11), (15), (18), (20), and (21).

4 Equations of motion on the tangent bundle of a three-dimensional manifold in a potential field of force and its first integrals

Let us now somewhat modify system (7), (9) under conditions (12)–(14), (16), (17), and (19), which yields a conservative system. Namely, the presence of the force field is characterized by the coefficient $F(\alpha)$ in the second equation of system (22). The system under consideration on the tangent bundle $T_*M^4\{z_4, z_3, z_2, z_1; \alpha, \beta_1, \beta_2, \beta_3\}$ takes the form

$$\begin{cases} \dot\alpha = -z_4, \\ \dot z_4 = F(\alpha) + \Gamma^\alpha_{11} f_1^2(\alpha) z_3^2 + \Gamma^\alpha_{22} f_2^2(\alpha) g_1^2(\beta_1) z_2^2 + \Gamma^\alpha_{33} f_3^2(\alpha) g_2^2(\beta_1) h^2(\beta_2) z_1^2, \\ \dot z_3 = \left[2\Gamma^1_{\alpha 1}(\alpha,\beta) + \dfrac{d\ln|f_1(\alpha)|}{d\alpha}\right] z_3 z_4 - \Gamma^1_{22}(\alpha,\beta)\dfrac{f_2^2(\alpha)}{f_1(\alpha)} g_1^2(\beta_1) z_2^2 - \\ - \Gamma^1_{33}(\alpha,\beta)\dfrac{f_3^2(\alpha)}{f_1(\alpha)} g_2^2(\beta_1) h^2(\beta_2) z_1^2, \\ \dot z_2 = \left[2\Gamma^2_{\alpha 2}(\alpha,\beta) + \dfrac{d\ln|f_2(\alpha)|}{d\alpha}\right] z_2 z_4 - \left[2\Gamma^2_{12}(\alpha,\beta) + \dfrac{d\ln|g_1(\beta_1)|}{d\beta_1}\right] f_1(\alpha) z_2 z_3 - \\ - \Gamma^2_{33}(\alpha,\beta)\dfrac{f_3^2(\alpha)}{f_2(\alpha)}\dfrac{g_2^2(\beta_1)}{g_1(\beta_1)} h^2(\beta_2) z_1^2, \\ \dot z_1 = \left[2\Gamma^3_{\alpha 3}(\alpha,\beta) + \dfrac{d\ln|f_3(\alpha)|}{d\alpha}\right] z_1 z_4 - \left[2\Gamma^3_{13}(\alpha,\beta) + \dfrac{d\ln|g_2(\beta_1)|}{d\beta_1}\right] f_1(\alpha) z_1 z_3 - \\ - \left[2\Gamma^3_{23}(\alpha,\beta) + \dfrac{d\ln|h(\beta_2)|}{d\beta_2}\right] f_2(\alpha) g_1(\beta_1) z_1 z_2, \\ \dot\beta_1 = z_3 f(\alpha), \\ \dot\beta_2 = z_2 f(\alpha) g(\beta_1), \\ \dot\beta_3 = z_1 f(\alpha) g(\beta_1) h(\beta_2), \end{cases} \tag{22}$$

and it is almost everywhere equivalent to the following system:

$$\begin{cases} \ddot\alpha + F(\alpha) + \Gamma^\alpha_{11}(\alpha,\beta)\dot\beta_1^2 + \Gamma^\alpha_{22}(\alpha,\beta)\dot\beta_2^2 + \Gamma^\alpha_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_1 + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_1 + \Gamma^1_{22}(\alpha,\beta)\dot\beta_2^2 + \Gamma^1_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_2 + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_2 + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_2 + \Gamma^2_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_3 + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_3 + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_3 + 2\Gamma_3(\beta_2)\dot\beta_2\dot\beta_3 = 0. \end{cases}$$

Proposition 4.1. *If the conditions of Proposition 3.1 are satisfied, system (22) has a smooth first integral of the following form:*

$$\Phi_1(z_4, \ldots, z_1; \alpha) = z_1^2 + \ldots + z_4^2 + F_1(\alpha) = C_1 = const, \; F_1(\alpha) = 2\int_{\alpha_0}^{\alpha} F(a)da. \tag{23}$$

Proposition 4.2. *If the conditions of Propositions 3.2, 3.3, and 3.4 are satisfied, system (22) has three smooth first integrals of form (15), (18), and (20).*

Proposition 4.3. *If the conditions of Proposition 3.5 are satisfied, system (22) has first integral of form (21).*

Under the conditions listed above, system (22) has a complete set of (five) independent first integrals of form (23), (15), (18), (20), and (21).

5 Equations of motion on the tangent bundle of a two-dimensional manifold in a force field with dissipation and its first integrals

Let us now consider system (24). In doing this, we obtain a system with dissipation. Namely, the presence of dissipation (generally speaking, sign-alternating) is characterized by the coefficient $b\delta(\alpha)$ in the first equation of system (24):

$$\begin{cases} \dot\alpha = -z_4 + b\delta(\alpha), \\ \dot z_4 = F(\alpha) + \Gamma^\alpha_{11} f_1^2(\alpha) z_3^2 + \Gamma^\alpha_{22} f_2^2(\alpha) g_1^2(\beta_1) z_2^2 + \Gamma^\alpha_{33} f_3^2(\alpha) g_2^2(\beta_1) h^2(\beta_2) z_1^2, \\ \dot z_3 = \left[2\Gamma^1_{\alpha 1}(\alpha,\beta) + \dfrac{d\ln|f_1(\alpha)|}{d\alpha}\right] z_3 z_4 - \Gamma^1_{22}(\alpha,\beta)\dfrac{f_2^2(\alpha)}{f_1(\alpha)} g_1^2(\beta_1) z_2^2 - \\ - \Gamma^1_{33}(\alpha,\beta)\dfrac{f_3^2(\alpha)}{f_1(\alpha)} g_2^2(\beta_1) h^2(\beta_2) z_1^2, \\ \dot z_2 = \left[2\Gamma^2_{\alpha 2}(\alpha,\beta) + \dfrac{d\ln|f_2(\alpha)|}{d\alpha}\right] z_2 z_4 - \left[2\Gamma^2_{12}(\alpha,\beta) + \dfrac{d\ln|g_1(\beta_1)|}{d\beta_1}\right] f_1(\alpha) z_2 z_3 - \\ - \Gamma^2_{33}(\alpha,\beta)\dfrac{f_3^2(\alpha)}{f_2(\alpha)}\dfrac{g_2^2(\beta_1)}{g_1(\beta_1)} h^2(\beta_2) z_1^2, \\ \dot z_1 = \left[2\Gamma^3_{\alpha 3}(\alpha,\beta) + \dfrac{d\ln|f_3(\alpha)|}{d\alpha}\right] z_1 z_4 - \left[2\Gamma^3_{13}(\alpha,\beta) + \dfrac{d\ln|g_2(\beta_1)|}{d\beta_1}\right] f_1(\alpha) z_1 z_3 - \\ - \left[2\Gamma^3_{23}(\alpha,\beta) + \dfrac{d\ln|h(\beta_2)|}{d\beta_2}\right] f_2(\alpha) g_1(\beta_1) z_1 z_2, \\ \dot\beta_1 = z_3 f(\alpha), \\ \dot\beta_2 = z_2 f(\alpha) g(\beta_1), \\ \dot\beta_3 = z_1 f(\alpha) g(\beta_1) h(\beta_2), \end{cases} \tag{24}$$

which is almost everywhere equivalent to the following system

$$\begin{cases} \ddot\alpha - b\dot\alpha\delta'(\alpha) + F(\alpha) + \Gamma^\alpha_{11}(\alpha,\beta)\dot\beta_1^2 + \Gamma^\alpha_{22}(\alpha,\beta)\dot\beta_2^2 + \Gamma^\alpha_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_1 - b\dot\beta_1\delta(\alpha)W(\alpha) + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_1 + \Gamma^1_{22}(\alpha,\beta)\dot\beta_2^2 + \Gamma^1_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_2 - b\dot\beta_2\delta(\alpha)W(\alpha) + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_2 + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_2 + \Gamma^2_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_3 - b\dot\beta_3\delta(\alpha)W(\alpha) + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_3 + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_3 + 2\Gamma_3(\beta_2)\dot\beta_2\dot\beta_3 = 0, \\ W(\alpha) = 2\Gamma^1_{\alpha 1}(\alpha,\beta) + \dfrac{d\ln|f_1(\alpha)|}{d\alpha}. \end{cases}$$

Now we pass to integration of the sought six-order system (24) under condition (13), as well as under the equalities

$$\Gamma^\alpha_{11}(\alpha,\beta) = \Gamma^\alpha_{22}(\alpha,\beta) g^2(\beta_1) = \Gamma^\alpha_{33}(\alpha,\beta) g^2(\beta_1) h^2(\beta_2) = \Gamma_4(\alpha), \tag{25}$$

hold.

We also introduce (by analogy with (13)) a restriction on the function $f(\alpha)$. It must satisfy the transformed first equality from (10):

$$2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha} + \Gamma_4(\alpha)f^2(\alpha) \equiv 0. \tag{26}$$

To integrate it completely, one should know, generally speaking, seven independent first integrals. However, after the following change of variables,

$$w_4 = z_4,\ w_3 = \sqrt{z_1^2+z_2^2+z_3^2},\ w_2 = \frac{z_2}{z_1},\ w_1 = \frac{z_3}{\sqrt{z_1^2+z_2^2}},$$

system (24) decomposes as follows:

$$\begin{cases} \dot\alpha = -w_4 + b\delta(\alpha), \\ \dot w_4 = F(\alpha) + \Gamma_4(\alpha)f^2(\alpha)w_3^2, \\ \dot w_3 = \left[2\Gamma_1(\alpha) + \dfrac{d\ln|f(\alpha)|}{d\alpha}\right] w_3 w_4, \end{cases} \tag{27}$$

$$\begin{cases} \dot w_2 = \pm w_3\sqrt{1+w_2^2}f(\alpha)g(\beta_1)\left[2\Gamma_3(\beta_2) + \dfrac{d\ln|h(\beta_2)|}{d\beta_2}\right], \\ \dot\beta_2 = \pm\dfrac{w_2w_3}{\sqrt{1+w_2^2}}f(\alpha)g(\beta_1), \end{cases} \tag{28}$$

$$\begin{cases} \dot w_1 = \pm w_3\sqrt{1+w_1^2}f(\alpha)\left[2\Gamma_2(\beta_1) + \dfrac{d\ln|g(\beta_1)|}{d\beta_1}\right], \\ \dot\beta_1 = \pm\dfrac{w_1w_3}{\sqrt{1+w_1^2}}f(\alpha), \end{cases} \tag{29}$$

$$\dot\beta_3 = \pm\frac{w_3}{\sqrt{1+w_2^2}}f(\alpha)g(\beta_1)h(\beta_2). \tag{30}$$

It is seen that to integrate system (27)–(30) completely, it is sufficient to determine two independent first integrals of system (27), by one integral of systems (28) and (29), and an additional first integral "attaching" Eq. (30) (i.e., five integrals in total).

Theorem 5.1. *Let the equalities*

$$\Gamma_4(\alpha)f^2(\alpha) = \kappa\frac{d}{d\alpha}\ln|\delta(\alpha)|,\ F(\alpha) = \lambda\frac{d}{d\alpha}\frac{\delta^2(\alpha)}{2} \tag{31}$$

be valid for some $\kappa, \lambda \in \mathbf{R}$. Then system (24) under equalities (12), (13), (16), (25), and (26) has a complete set of (five) independent, generally speaking, transcendental first integrals.

6 Conclusions

By analogy with low-dimensional cases, we pay special attention to two important cases for the function $f(\alpha)$ defining the metric on a sphere:

$$f(\alpha) = \frac{\cos\alpha}{\sin\alpha}, \tag{32}$$

$$f(\alpha) = \frac{1}{\cos\alpha \sin\alpha}. \tag{33}$$

Case (32) forms a class of systems corresponding to the motion of a dynamically symmetric five-dimensional solid body at zero levels of cyclic integrals, generally speaking, in a nonconservative field of forces [3, 4, 5]. Case (33) forms a class of systems corresponding to the motion of a material point on a four-dimensional sphere also, generally speaking, in a nonconservative field of forces. In particular, at $\delta(\alpha) \equiv F(\alpha) \equiv 0$ the system under consideration describes a geodesic flow on a four-dimensional sphere. In case (32), if

$$\delta(\alpha) = \frac{F(\alpha)}{\cos\alpha},$$

the system describes the spatial motion of a five-dimensional solid body in the force field $F(\alpha)$ under the action of a tracking force [6, 7, 8]. In particular, if

$$F(\alpha) = \sin\alpha\cos\alpha,\ \delta(\alpha) = \sin\alpha,$$

the system also describes a generalized five-dimensional spherical pendulum in a nonconservative force field and has a complete set of transcendental first integrals that can be expressed in terms of a finite combination of elementary functions [9, 10, 11].

If the function $\delta(\alpha)$ is not periodic, the dissipative system under consideration is a system with variable dissipation with a zero mean (i.e., it is properly dissipative). Nevertheless, an explicit form of transcendental first integrals that can be expressed in terms of a finite combination of elementary functions can be obtained even in this case. This is a new nontrivial case of integrability of dissipative systems in an explicit form [12].

Acknowledgements

This work was supported by the Russian Foundation for Basic Research, project no. 15-01-00848-a.

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