Chaotic dynamics of interacting pendulums (the decision of the synchronization problem)

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Abstract

It's consider as an example of two coplanar metal pendulums, suspension points of which are at the same horizontal level and the same fixed distance of b from each other. It's shown that the principle of possibility of synchronization due to two main factors. The first one is the effect of electromagnetic interaction between the pendulums. The second one is taking into account of the power of EM radiation coming to the nonlinear attenuation.

1 Introduction

In this paper, we would not research internal structure of the clock mechanism. It was obtained the system of nonlinear dynamic equations of motion and it was given analytical estimates of synchronization time of t_{synch}, supported by numerical solution of the equations obtained, which has not bad agreement with the experimental results (see below).
The problem to which this paper is devoted is not new one, because dated back to Huygens time, who first turned attention to the effect of synchronization of physical metal pendulums, hanging at some distance from each other. Later the effect of synchronization have been searched in other papers (for example [1] – [4]) and in monographs (for example, [5] – [7]). We should note that in some sources (for example, [7] and [8]), the model of synchronization is based on condition of small adjustment of pendulums swing connected with "dry friction" which proportional to the velocity of pendulums motion and in the opinion of the authors, the internal structure of the clock mechanism is determined only. In this paper, we would not research internal structure of the clock mechanism, but we approach the solution of the problem from a fundamentally different physical point of view.
We should note that none of the mentioned sources estimated the time of synchronization t_{synch} was not estimated in any mentioned sources, what is more, it was not offer physically grounded interaction between pendulums, leading exactly to the effect of synchronization.
What is more, the effect of EM radiation had never been taken into account, however, as we are going to prove now, it plays a pivotal role in this interesting phenomenon. Also, in the previous papers it was not mentioned at all.

We should notice that the arguments given below can easily be transferred to any other similar problems that are somehow related to the effects of the radiation of the currently known physical fields (gravitational, electromagnetic and acoustic) and are an attribute of any moving and interacting objects. As an example we choose two absolutely identical physical pendulums, points of suspension are at the same distance b from each other. For the sake of concretization the calculations below, we should take the pendulums are coplanar as in Fig. 1. We should notice that the explained algorithm for calculation is trivially genereliazed even if the pendulums are suspended in parallel planes, however, the substance of the issue is identical in both cases.
In the general, if lengths of suspension are different or equal to l_1 and l_2 , then according to the geometry of Fig. 1, we get the following expression for the distance between the centers of the pendulums:

$$R = \sqrt{l_1^2 + l_2^2 + b^2 + 2bl_1 \sin\varphi_1 - 2bl_2 \sin\varphi_2 - 2l_1 l_2 \cos(\varphi_1 - \varphi_2)} \qquad (1)$$

In our case, when both pendulums are identical, i.e. $l_1 = l_2 = l$, $m_1 = m_2 = m$from the formula (1) synchronization condition is trivially written, i.e.$R = b$ equality should be realized $R = b$. This automatically leads to the equation:

$$l + b\sin\varphi_1 - b\sin\varphi_2 - l\cos(\varphi_1 - \varphi_2) \qquad (2)$$

Solving this equation in φ_2 , we obtain that:

$$\sin\varphi_2 = \sin\varphi_1 \qquad (3)$$

I.e. the synchronization condition is

$$\varphi_1 = \varphi_2, \dot{\varphi}_1 = \dot{\varphi}_2 \qquad (4)$$

2 Setting of the problem

Before proceeding to the direct formulation and solution of the problem, it is necessary to say a few words about the physical side of the problem. For identical pendulums when $l_1 = l_2 = l$, $m_1 = m_2 = m$ the potential energy should be represented in a symmetrical form as a half sum

$$U(R) = \frac{U_{12}(R) + U_{21}(R)}{2} \qquad (5)$$

The total energy of the system should be like this

$$E = T + U = U_0 + \frac{ml_c^2\dot{\varphi}_1^2}{2} + \frac{ml_c^2\dot{\varphi}_2^2}{2} - mgl_c\left(\cos\varphi_1 + \cos\varphi_2\right) + \\ + \frac{U_{12}(R) + U_{21}(R)}{2} = const, \qquad (6)$$

where $U_0 = mgH$, l_c the distance of suspension point of the pendulum from its center of attraction, H height of suspension of pendulums above the Earth. In contrast to the dependence (1), the distance between the centers of the pendulums is conveniently introduced in vector form, which automatically allow for the curve trajectory of its motion. Indeed, since at the initial instant time $t = 0$, the distance is R_0 , then at any moment of the time it can be represented as

$$\mathbf{R} = \mathbf{R_0} - \int_{\mathbf{0}}^{\mathbf{t}} (\mathbf{v_1(t)} + \mathbf{v_2(t)})\, \mathbf{dt}, \tag{7}$$

where $\mathbf{v_1}, \mathbf{v_2}$ the velocities of both balls. As a result, the potential energy of interaction can be represented in the following form

$$U = U_G \left(\left| \mathbf{R_0} - \int_{\mathbf{0}}^{\mathbf{t}} (\mathbf{v_1(t)} + \mathbf{v_2(t)})\, \mathbf{dt} \right| \right) + U_{EM} \left(\left| \mathbf{R_0} - \int_{\mathbf{0}}^{\mathbf{t}} (\mathbf{v_1(t)} + \mathbf{v_2(t)})\, \mathbf{dt} \right| \right). \tag{8}$$

The first term is the usual gravitational interaction of two material objects, but we will dwell on the second term in (8) in more detail. To find it we should recall some of the basic principles of electrodynamics (see, for example, [9]) and among other factors, Maxwell's equations. In the quasi static case from classical electrodynamics it follows that the vector potential $\mathbf{A}$ of the magnetic field should satisfy equation

$$\Delta \mathbf{A} = -\frac{4\pi}{c} \mathbf{j}, \tag{9}$$

where $\mathbf{j}$ is the inertial current of the density in moving balls, c is the velocity of the light. As it's known (see ref. [9]) the interaction determined by the motion of the electrons can be represented in the form:

$$U = -\frac{1}{c} \int_V \mathbf{j}\mathbf{A} dV. \tag{10}$$

In our case, a moving pendulum, conditionally denoted by index 1, induces a potential on the second pendulum $\mathbf{A_1}$. Therefore, in the accordance with the expr. (10) and eq. (9) we have for the potential interaction energy

$$U_{EM} = \frac{1}{c^2} \int_{V_1} \int_{V_2} \frac{\mathbf{j_1}\mathbf{j_2}}{\tilde{R}} dV_1 dV_2, \tag{11}$$

where the vector $\tilde{\mathbf{R}}$ is determined as $\tilde{\mathbf{R}} = \mathbf{R} + \mathbf{r_1} - \mathbf{r_2}$, where radius-vectors $\mathbf{r_1}, \mathbf{r_2}$ are current directions in each of the balls counted from their centers, for which integration is carried out in (11), V_1 and V_2 the volumes of these balls. Since the current density in the moving sphere is $\mathbf{j} = \rho_\mathbf{e} \mathbf{v}$, where ρ_e electric density in the

ball, its velocity at fluctuating motion is defined as $\mathbf{v} = l_c\dot{\varphi}\boldsymbol{\tau}$, where l_c the distance of suspension center from center of attraction the system of the ball + rod.
In the result with the accordance formula (11) we can obtain that the potential energy of interaction is

$$U_{EM} = \frac{\dot{\varphi}_1\dot{\varphi}_2\left(\mathbf{k_1}\cdot\mathbf{k_2}\right)\rho_e^2}{c^2}\int\limits_{V_1}\int\limits_{V_2}\frac{\left(l_c\boldsymbol{\tau}_1 + r_1\mathbf{v_1}\right)\left(l_c\boldsymbol{\tau}_2 + r_2\mathbf{v_2}\right)dV_1dV_2}{\tilde{R}} =$$
$$= \frac{\dot{\varphi}_1\dot{\varphi}_2\left(\mathbf{k_1}\cdot\mathbf{k_2}\right)\rho_e^2}{c^2}\left[l_c^2\left(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\right)\int\limits_{V_1}\int\limits_{V_2}\frac{dV_1dV_2}{\tilde{R}} + l_c\int\limits_{V_1}\int\limits_{V_2}\frac{r_1\left(\boldsymbol{\tau_1}\cdot\mathbf{v_2}\right)dV_1dV_2}{\tilde{R}} +\right.$$
$$\left. + l_c\int\limits_{V_1}\int\limits_{V_2}\frac{r_2\left(\boldsymbol{\tau_2}\cdot\mathbf{v_1}\right)dV_1dV_2}{\tilde{R}} + \int\limits_{V_1}\int\limits_{V_2}\frac{r_1r_2\left(\mathbf{v_1}\cdot\mathbf{v_2}\right)dV_1dV_2}{\tilde{R}}\right], \tag{12}$$

where $\mathbf{k_1}$ and $\mathbf{k_2}$ - unit vectors directed along the angular velocities ω_1 and ω_2 and along the axis$z$, perpendicular to the plane of the Fig. 1
As we can see from the Fig. 1, scalar product of unit vectors $\boldsymbol{\tau_1}$ and $\boldsymbol{\tau_2}$ is $\boldsymbol{\tau_1}\cdot\boldsymbol{\tau_2} = \cos\left(\pi - \varphi_1 + \varphi_2\right) = -\cos\left(\varphi_1 - \varphi\right)$. For other scalar products in (15), we are finding that $\boldsymbol{\tau_1}\cdot\mathbf{v_2} = \cos\left(\boldsymbol{\varphi_1} - \boldsymbol{\varphi_2}\right)$, $\boldsymbol{\tau_2}\cdot\mathbf{v_1} = \cos\left(\boldsymbol{\varphi_2} - \varphi'\right)$, $\mathbf{v_1}\cdot\mathbf{v_2} = \cos\left(\varphi - \varphi'\right)$, where angles φ and φ' are the current vector angles in the plane $x - y$ of the spherical coordinate system for which integration is carrying out, i.e.$dV_1 = r_1^2\sin\theta_1 dr_1 d\theta_1 d\varphi$ and $dV_2 = r_2^2\sin\theta_2 dr_2 d\theta_2 d\varphi'$.
After all calculation we are finding from the expr. (12)

$$U_{EM} = \frac{\dot{\varphi}_1\dot{\varphi}_2\left(\mathbf{k_1}\cdot\mathbf{k_2}\right)\rho_e^2}{c^2}\left[-l_c^2\cos\left(\varphi_1 + \varphi_2\right)\int\limits_{V}\int\limits_{V}\frac{dV_1dV_2}{\tilde{R}} + F\right], \tag{13}$$

where the function

$$F = \int\limits_0^{r_0} r_1^3 dr_1 \int\limits_0^{r_0} r_2^3 dr_2 \int\limits_0^{\pi}\sin\theta_1 d\theta_1 \int\limits_0^{\pi}\frac{\left(\sqrt{A-B} - \sqrt{A+B}\right)\sin\theta_2}{2r_1r_2\sin\theta_1\sin\theta_2}d\theta_2 =$$
$$2\int\limits_0^{r_0} r_1^2 dr_1 \int\limits_0^{r_0} r_2^2 dr_2 \int\limits_0^{\pi} d\theta_1 \int\limits_0^{\pi}\left(\sqrt{R^2 + 2Rr_1\cos\theta_1 - 2Rr_2\cos\theta_2 + r_1^2 + r_2^2 - 2r_1r_2\left[\cos\left(\theta_1 - \theta_2\right) - \right.}\right.$$
$$\left. - \sqrt{R^2 + 2Rr_1\cos\theta_1 - 2Rr_2\cos\theta_2 + r_1^2 + r_2^2 - 2r_1r_2\cos\theta_1\cos\theta_2}\right)d\theta_2$$

and approximately we have that the interaction is

$$U_{EM} = \frac{\dot{\varphi}_1\dot{\varphi}_2\left(\mathbf{k_1}\cdot\mathbf{k_2}\right)\rho_e^2 V^2 l_c^2\cos\left(\varphi_1 + \varphi_2\right)}{c^2 b}\xi \tag{14}$$

where ξ – is a numerical dimensionless factor of the order of unity. It is not so important for our investigation. As we can easily understand, scalar product of unit vectors $\mathbf{k_1}$ and $\mathbf{k_2}$ should be written as $\mathbf{k_1} \cdot \mathbf{k_2} = \cos\psi$ and the sign of this expression at certain times must change. Essentially, such a fact could have significance, however, it does not carry a fundamental and profound meaning. Therefore for all analytic calculations carried out below the solution will be given at $\cos\psi = \pm 1$. So $\mathbf{v_1} = \dot{\varphi}_1 \mathbf{l} \boldsymbol{\tau_1}$ and $\mathbf{v_2} = \dot{\varphi}_2 \mathbf{l} \boldsymbol{\tau_2}$, where $\boldsymbol{\tau_{1,2}}$ is an unit vectors tangent to the trajectory of motion that can be represented as an expansion in a fixed two – dimensional basis $\mathbf{i}, \mathbf{j}$.

3 The power of the electromagnetic radiation of the moving pendulums

Herein we are going to pay attention on the most important moment of our theory and give a detailed calculation of the power of the EM radiation of pendulums, leading ultimately to their synchronization. For this goal we should recall some properties of the LiГкnard-Wiechert potentials. According to for example ref. [12] any moving charge creates a scalar potential and a vector potential $\mathbf{A}$ at some distance r from itself, which are given by the following symmetric formulas

$$
\begin{aligned}
\psi(\mathbf{r},t) &= \frac{e}{2}\left(\frac{1}{R-\dfrac{\mathbf{v}\cdot\mathbf{R}}{c}}+\frac{1}{R+\dfrac{\mathbf{v}\cdot\mathbf{R}}{c}}\right)=\frac{e}{R\left[1-\left(\dfrac{\mathbf{v}\cdot\mathbf{n}}{c}\right)^2\right]},\\
\mathbf{A}(\mathbf{r},t) &= \frac{e\mathbf{v}}{2c}\left(\frac{1}{R-\dfrac{\mathbf{v}\cdot\mathbf{R}}{c}}+\frac{1}{R+\dfrac{\mathbf{v}\cdot\mathbf{R}}{c}}\right)=\frac{e\mathbf{v}}{Rc\left[1-\left(\dfrac{\mathbf{v}\cdot\mathbf{n}}{c}\right)^2\right]}
\end{aligned}
\tag{15}
$$

where $\mathbf{n} = \dfrac{\mathbf{R}}{R}$ - is an unit vector, $R = |\mathbf{r} - \mathbf{r_0}(t)|$ is the distance, where $\mathbf{r_0}(t)$ is the trajectory of the charge and r is the point of observation. For our specific case these formulas we are completely trivially generalized to a moving metal pendulum and in the accordance of expr. (15), we obtain after the integration on volumes V_1 and V_2

$$
\begin{aligned}
\psi(\mathbf{r},t) &= \frac{enV}{R\left[1-\left(\dfrac{\mathbf{v}\cdot\mathbf{n}}{c}\right)^2\right]} \approx \frac{enV}{R}\left[1-\left(\frac{\mathbf{v}\cdot\mathbf{n}}{c}\right)^2\right],\\
\mathbf{A}(\mathbf{r},t) &= \frac{enV\mathbf{v}}{Rc\left[1-\left(\dfrac{\mathbf{v}\cdot\mathbf{n}}{c}\right)^2\right]} \approx \frac{enV\mathbf{v}}{R}\left[1-\left(\frac{\mathbf{v}\cdot\mathbf{n}}{c}\right)^2\right],
\end{aligned}
\tag{16}
$$

where n is the charge concentration, V is the volume of pendulum (see above). The potentials (16) give us a possible to calculate the distributions of EM fields outside pendulums. Indeed, in the accordance with the formulas which good known from the electrodynamics (see ref. [12]) we have for the electric and magnetic fields

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\psi, \mathbf{B} = rot\mathbf{A}. \tag{17}$$

Substituting here expr. (16), we are getting that

$$\begin{aligned}
\mathbf{E} &= -\frac{enV}{c^2}\left(\frac{\dot{\mathbf{v}}}{R} + \frac{\mathbf{v}\left(\mathbf{v}\cdot\mathbf{R}\right)}{R^3}\right) + \frac{enV\mathbf{R}}{R^3} + \frac{enV\mathbf{R}}{R^3}\left(\frac{\mathbf{v}\cdot\mathbf{n}}{c}\right)^2 + \frac{2enV\mathbf{v}}{R^3}\left(\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2 = \\
&= \frac{enV\mathbf{R}}{R^3} - \frac{enV}{c^2}\frac{\dot{\mathbf{v}}}{R} + \frac{enV\mathbf{R}}{R^3}\left(\frac{\mathbf{v}\cdot\mathbf{n}}{c}\right)^2 + \frac{2enV\mathbf{v}}{R^3}\left(\frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2, \\
\mathbf{B} &= \frac{enV}{cR^3}\left[\mathbf{v}\times\mathbf{R}\right],
\end{aligned} \tag{18}$$

Where we have accounted that $\mathbf{R} = \mathbf{r} - \mathbf{r_0}(t)$ and accounted that $\dot{\mathbf{R}} = -\dot{\mathbf{r_0}} = -\mathbf{v}$. As it is known from the ref. [12] the radiation should be determined only by terms that include a derivative of the velocity with respect to time. This is due to the fact that when the emission intensity of the squared absolute value of the fields and and multiplied by the element of the spherical surface $R^2 dO$ where the solid angle element $dO = \sin\theta d\theta d\varphi$ and in the limit $R \to \infty$ of the sum of the squares of the fields defined in (18) will only the term of $\dot{\mathbf{v}}$. Therefore from the expr. (18) we can take only the radiation part of the electric field, i.e.

$$\mathbf{E}^{rad} = -\frac{enV}{c^2}\frac{\dot{\mathbf{v}}}{R}. \tag{19}$$

Since $\mathbf{v} = v\boldsymbol{\tau}$ for the acceleration we have $\dot{\mathbf{v}} = \dot{v}\boldsymbol{\tau} + \frac{v^2}{l}\mathbf{n}$, where $\mathbf{n}$ the unit vector of the normal to the trajectory of motion (in our case to a circle of radius l equal to the length of the suspension). Therefore following the definition of the emission intensity according I as it's shown in the ref. [12] and taking into account formula (19) above, we are getting that

$$I = \frac{c\mathbf{E}_{rad}^2}{8\pi} = \frac{c}{8\pi}\left(\frac{enV}{c^2}\frac{\dot{\mathbf{v}}}{R}\right)^2 = \frac{\left(enV\right)^2}{8\pi c^3}\frac{1}{R^2}\left(\dot{\mathbf{v}}^2 + \frac{\mathbf{v}^4}{l^2}\right). \tag{20}$$

As far as the tangential velocity is $v = l\dot{\varphi}$ we have, hence

$$I = \frac{\left(enV\right)^2}{8\pi c^3}\frac{l^2}{R^2}\left(\ddot{\varphi}^2 + \dot{\varphi}^4\right). \tag{21}$$

So as the power of radiation is define as $W = \int IR^2 dO = 4\pi IR^2$, from the expr. (21) we are obtain

$$W = \frac{\left(enVl\right)^2}{2c^3}\left(\ddot{\varphi}^2 + \dot{\varphi}^4\right) \tag{22}$$

4 Derivation of the motion equations in general form and the analysis

As it's shown in the ref. [11], in the general case we can write the following equation

$$\sum \dot{E} + \sum \dot{Q} + \sum W = 0 \tag{23}$$

where$\dot{Q}$ – is the dissipation function. Neglecting by the dissipative properties of the continuum from the expr. (32), we have

$$\sum \dot{E} + \sum W = 0. \tag{24}$$

After differentiating of the total energy over the time and using (6), (22) and (24), we are getting the following equation

$$\begin{aligned} & ml_c^2 \dot{\varphi}_1 \ddot{\varphi}_1 + ml_c^2 \dot{\varphi}_2 \ddot{\varphi}_2 + mgl_c \left(\dot{\varphi}_1 \sin \varphi_1 + \dot{\varphi}_2 \sin \varphi_2 \right) + \\ & + \frac{\partial U_{EM}}{\partial R} \frac{\partial R}{\partial \varphi_1} \dot{\varphi}_1 + \frac{\partial U_{EM}}{\partial R} \frac{\partial R}{\partial \varphi_2} \dot{\varphi}_2 + \frac{\left(enV \right)^2}{8\pi c^3} \frac{l^2}{R^2} \left(\ddot{\varphi}^2 + \dot{\varphi}^4 \right) = 0 \end{aligned} \tag{25}$$

Where the distance are

$$\begin{aligned} R = \Bigg(& \left(R_{0x} + l_c \left(\sin \varphi_1 - \sin \varphi_{01} + \sin \varphi_2 - \sin \varphi_{02} \right) \right)^2 + \\ & + \left(R_{0y} - l_c \left(\cos \varphi_1 - \cos \varphi_{01} + \cos \varphi_2 - \cos \varphi_{02} \right) \right)^2 \Bigg)^{\frac{1}{2}}. \end{aligned} \tag{26}$$

As we can see from the expr. (22) the emission power consists from two parts. The first part of the summand turns is much less than the second one and it's connected with the following simple reason. According to the numerical solutions of the resulting system of equations (25) (see below) the nonlinear "damping" due to the radiation leads to the fulfillment of the condition $|\ddot{\varphi}_{1,2}| \ll \left|\dot{\varphi}_{1,2}^2\right|$, which is confirmed by a graphic comparison of these two terms (see Fig. 2). This means that we can write the formula (22) in approximate form as $W \approx \dfrac{(enVl)^2}{2c^3}\dot{\varphi}^4$, as we written in the eq. (25).
After the substitution in the eq. (25), we obtain the following system of equations

$$\begin{cases} \varphi_1'' + \omega_0^2 \sin\varphi_1 + \omega_1^2 q_1 \left[\sin(\varphi_1-\varphi_2) - a\cos\varphi_2\right] - \lambda_2 q_2 (\varphi_2'' cos\psi - \varphi_2'\psi'\sin\psi)\cos(\varphi_1+\varphi_2) - \\ - 3\lambda_2 q_2 (\varphi_1'-\varphi_2')\sin(\varphi_1-\varphi_2)\cos(\varphi_1+\varphi_2)\varphi_2'\cos\psi + \kappa\varphi_1'^3 - \\ - \dfrac{3\lambda_2 a}{2}\cos(\varphi_1+\varphi_2)\left[\dfrac{\varphi_1'\cos\varphi_1-\varphi_2'\cos\varphi_2}{Q_{12}^5} + \dfrac{\varphi_2'\cos\varphi_2-\varphi_1'\cos\varphi_1}{Q_{21}^5}\right]\varphi_2'\cos\psi = 0, \\ \varphi_2'' + \omega_0^2 \sin\varphi_2 + \omega_1^2 q_1 \left[\sin(\varphi_1-\varphi_2) + a\cos\varphi_1\right] - \lambda_2 q_2 (\varphi_1'' cos\psi - \varphi_1'\psi'\sin\psi)\cos(\varphi_1+\varphi_2) - \\ - 3\lambda_2 q_2 (\varphi_1'-\varphi_2')\sin(\varphi_1-\varphi_2)\cos(\varphi_1+\varphi_2)\varphi_1'\cos\psi + \kappa\varphi_2'^3 - \\ - \dfrac{3\lambda_2 a}{2}\cos(\varphi_1+\varphi_2)\left[\dfrac{\varphi_1'\cos\varphi_1-\varphi_2'\cos\varphi_2}{Q_{12}^5} + \dfrac{\varphi_2'\cos\varphi_2-\varphi_1'\cos\varphi_1}{Q_{21}^5}\right]\varphi_1'\cos\psi = 0. \end{cases} \quad (27)$$

where the frequencies are

$$\omega_0^2 = \frac{g}{l_c}, \omega_1^2 = \frac{Gm}{l_c^3} \quad (28)$$

and the parameters are

$$\lambda_2 = \frac{\rho_e^2 V}{\rho_b c^2 b}\xi, \kappa = \frac{\rho_e^2 l^2 V}{2\rho_b c^3 l_c^2}\xi. \quad (29)$$

New functions are

$$q_1 = \frac{1}{2}\left(\frac{1}{Q_{12}^3} + \frac{1}{Q_{21}^3}\right), q_2 = \frac{1}{2}\left(\frac{1}{Q_{12}^5} + \frac{1}{Q_{21}^5}\right), \quad (30)$$

where the denominators are $Q_{12} = \sqrt{a^2 + 2\left(1-\cos(\varphi_1-\varphi_2)\right) + 2a\left(\sin\varphi_1 - \sin\varphi_2\right)}$ and $Q_{21} = \sqrt{a^2 + 2\left(1-\cos(\varphi_1-\varphi_2)\right) - 2a\left(\sin\varphi_1 - \sin\varphi_2\right)}$. Here $a = \frac{b}{l_c}$ – is a new dimensionless parameter. As we can see form the system (27) as it must be it is symmetric with respect to the change inversion operations $\varphi_1 \to -\varphi_1, \varphi_2 \to -\varphi_2$ and $\varphi_1 \to \varphi_2, \varphi_2 \to \varphi_1$. Introducing for convenience else one dimensionless parameter $\lambda_1 = \frac{\omega_1^2}{\omega_0^2}$, as well as dimensionless time $\tau = \omega_0 t$, we are finding in the result

$$\begin{cases} \varphi_1'' + \sin\varphi_1 + \lambda_1 q_1 \left[\sin(\varphi_1-\varphi_2) - a\cos\varphi_2\right] - \lambda_2 q_2 (\varphi_2'' cos\psi - \varphi_2'\psi'\sin\psi)\cos(\varphi_1+\varphi_2) - \\ - 3\lambda_2 q_2 (\varphi_1'-\varphi_2')\sin(\varphi_1-\varphi_2)\cos(\varphi_1+\varphi_2)\varphi_2'\cos\psi + k\varphi_1'^3 - \\ - \dfrac{3\lambda_2 a}{2}\cos(\varphi_1+\varphi_2)\left[\dfrac{\varphi_1'\cos\varphi_1-\varphi_2'\cos\varphi_2}{Q_{12}^5} + \dfrac{\varphi_2'\cos\varphi_2-\varphi_1'\cos\varphi_1}{Q_{21}^5}\right]\varphi_2'\cos\psi = 0, \\ \varphi_2'' + \sin\varphi_2 + \lambda_1 q_1 \left[\sin(\varphi_1-\varphi_2) + a\cos\varphi_1\right] - \lambda_2 q_2 (\varphi_1'' cos\psi - \varphi_1'\psi'\sin\psi)\cos(\varphi_1+\varphi_2) - \\ - 3\lambda_2 q_2 (\varphi_1'-\varphi_2')\sin(\varphi_1-\varphi_2)\cos(\varphi_1+\varphi_2)\varphi_1'\cos\psi + k\varphi_2'^3 - \\ - \dfrac{3\lambda_2 a}{2}\cos(\varphi_1+\varphi_2)\left[\dfrac{\varphi_1'\cos\varphi_1-\varphi_2'\cos\varphi_2}{Q_{12}^5} + \dfrac{\varphi_2'\cos\varphi_2-\varphi_1'\cos\varphi_1}{Q_{21}^5}\right]\varphi_1'\cos\psi = 0. \end{cases}$$

(31)

where the primes are means the differentiation over τ . The dimensionless parameter $k = \dfrac{\rho_e^2 l^2 V \omega_0}{2\rho_b c^3 l_c^2}$ (see expr. (29)), where $\rho_b = \dfrac{m}{V}$ – is a density of the metal balls. As we mentioned above, appearing in (27) the parameter $\cos\psi$ and it derivative for simplifying the analysis we are putting that $\cos\psi = \pm 1, (\cos\psi)' = 0$. Moreover, we take into account that $Q_{12} \approx Q_{21} = a$. In the result the equations (31) are simplified and we obtain the compact system of equations

$$\begin{cases} \varphi_1'' + \sin\varphi_1 + \gamma\varphi_2'' \cos(\varphi_1 + \varphi_2) + 3\gamma(\varphi_1' - \varphi_2') \sin(\varphi_1 - \varphi_2)\varphi_2' \cos(\varphi_1 + \varphi_2) + k\varphi_1'^3 = 0, \\ \varphi_2'' + \sin\varphi_2 + \gamma\varphi_1'' \cos(\varphi_1 + \varphi_2) + 3\gamma(\varphi_1' - \varphi_2') \sin(\varphi_1 - \varphi_2)\varphi_1' \cos(\varphi_1 + \varphi_2) + k\varphi_2'^3 = 0 \end{cases} \tag{32}$$

where the parameter $\gamma = \dfrac{\lambda_2}{a^5}$. To solve the equations (32), we should also define initial conditions, which we choose in the following form:

$$\varphi_1(0) = -\varphi_{01}, \varphi_2(0) = \varphi_{02}, \dot{\varphi}_1(0) = \dot{\varphi}_2(0) = 0. \tag{33}$$

The rigorous proof of the principle possibility of synchronization, given by us, is based only on two factors: 1. The potential energy of electromagnetic interaction between metallic spheres and 2. EM radiation. The numerical solution of the systems (32) at initial conditions (33) can be illustrated by the Figs. 3 – 6 (on this Fig. we are choice the parameters $\gamma = 10^{-2}$ and $k = 10^{-3}$).
So, as it's shown above analytical and numerical solution of the equations (32) and their analysis help us to realize that the eye of the synchronization problem is more understandable. The solution described above, explaining the mechanisms of interaction of pendulums and answers to the question about the physical nature of this interesting and very curious phenomenon. From the point of view of numerical analysis, graphical illustration of the solutions of the equations (32) is very important, because of its clarity, which allows us to show the entire synchronization stage in the figures, and numerically estimate the synchronization time t_{synchr} for different values of the parameters γ and k .

5 On physical nature of synchronization

We need to say that at first the issue of the synchronization of the pendulums, as a historical fact, applied to ordinary mechanical watches. Evidence of this is the very first experiment in this direction, conducted by Huygens with the aid of ship clocks, which for the first time established the fact of their synchronization. All subsequent studies, one way or another, were reduced to mechanical watches (see, for example, the monograph [7]), i.e. on the clockwork. The task posed in this article, as it appears from the previous text, was devoted to solving a purely physical problem, which is completely unrelated to the mechanics of the clockwork mechanism, and pursued only one goal. To show the principle possibility of synchronization due to taking into account the two most important physical factors, the nature of which is purely electromagnetic, i.e. 1. EM interaction between pendulums and 2. EM radiation. It is quite clear that out of all the many interactions currently known, this is gravitational, electromagnetic, Van der Waals and magnetic dipole. The last one is the most effective only for magnetic materials. This means that in case you want this to happen. However, in this case the physics becomes completely different because it will be too small in comparison with metals, and the main role will shift to the effect of EM wave emission due to the precession of the magnetization vector, as its shown in the papers [14], [15] (see also [16]). As it turned out, the greatest contribution to the attraction effect of metal pendulums is provided by electromagnetic interaction, accompanied by inhibition in the form of EM radiation, and that was described in some detail a little higher. At the moments of closest approach of pendulums, the interaction effect reaches a maximum, as a result of which the pendulums begin to intensely "feel" each other. One at the same time slightly slows down, and the other – slightly accelerated due to the more intense radiation coming from the opposite pendulum, but the total energy of both pendulums in the absence of dissipation should remain constant. It is rather subtle moment tells us only that the formal languages for describing any non-equilibrium phenomena with or without energy dissipation will be very different from the formalism of the description of phenomena in the language of radiation powers. The only essential condition for this is the movement of the object along the curvilinear trajectory.

6 Conclusion

1. Due to the assumption that the interaction between pendulums is of a long-range nature of electromagnetic interaction a system of symmetric relatively permutations and non-linear differential equations are obtained, which are the invariant relative to the transformations: $\varphi_1 \to \varphi_2, \varphi_2 \to \varphi_1$ and $\varphi_1 \to -\varphi_1, \varphi_2 \to -\varphi_2$. The EM radiation is a main factor of synchronization phenomenon. Based on these two physical factors, we have a possible to describe mathematically the entire synchronization process.

2. It is strictly analytically shown that in the approximation of small oscillations the synchronization effect occurs after a time t_{synchr}. The numerical values of which corresponds to the experimentally observed times.

3. Using numerical integration methods, we give a general solution of the non-linear system (32) of the differential equations illustrated by the Fig. 3 - 6.

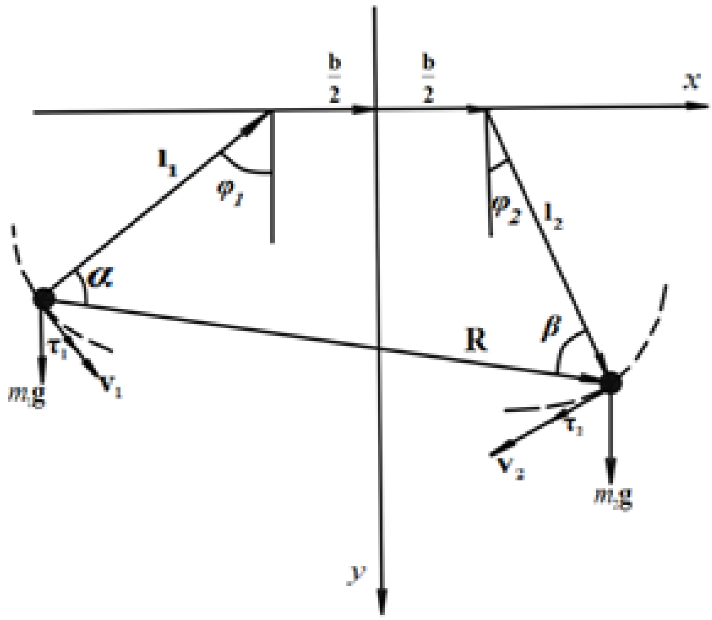
b
2
b
2
x
l1
φ1
φ2
l2
α
β
R
τ1
m1g
v1
τ1
v2
m2g
y

Fig. 1

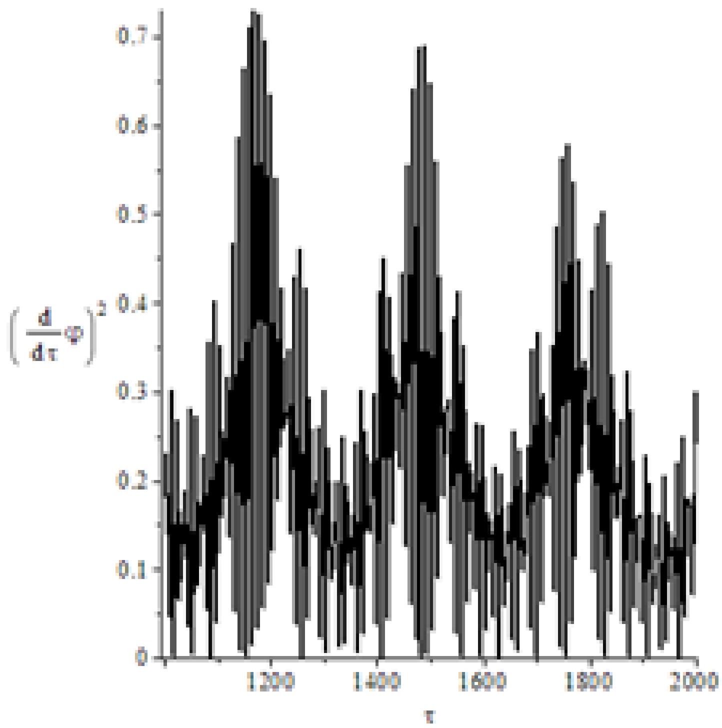
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1200
1400
1600
1800
2000
τ

Fig. 2

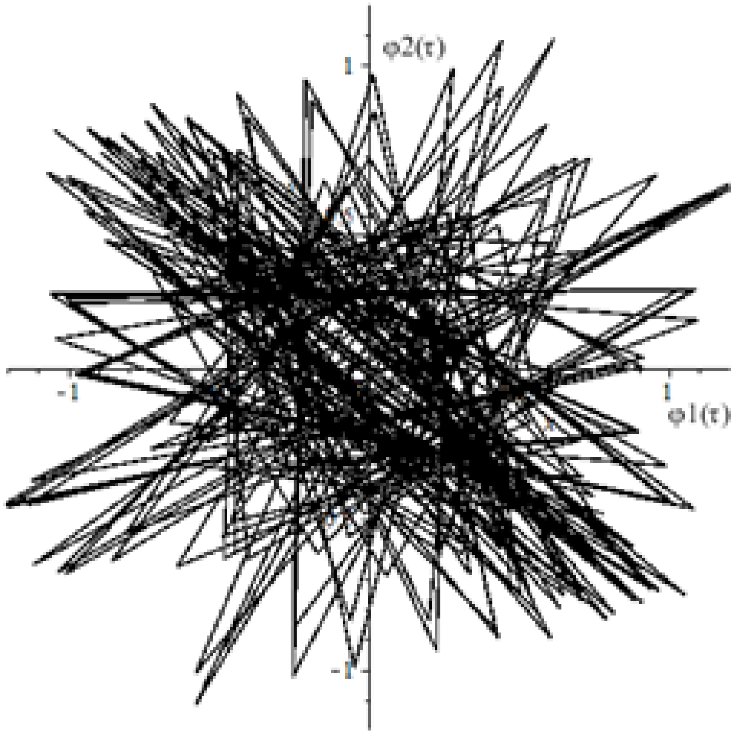

Fig. 3

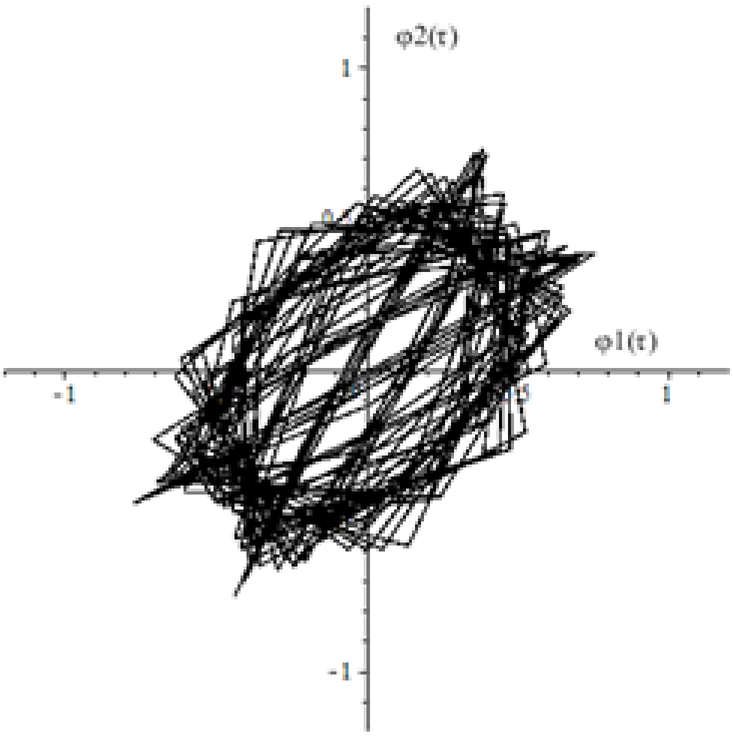

Fig. 4

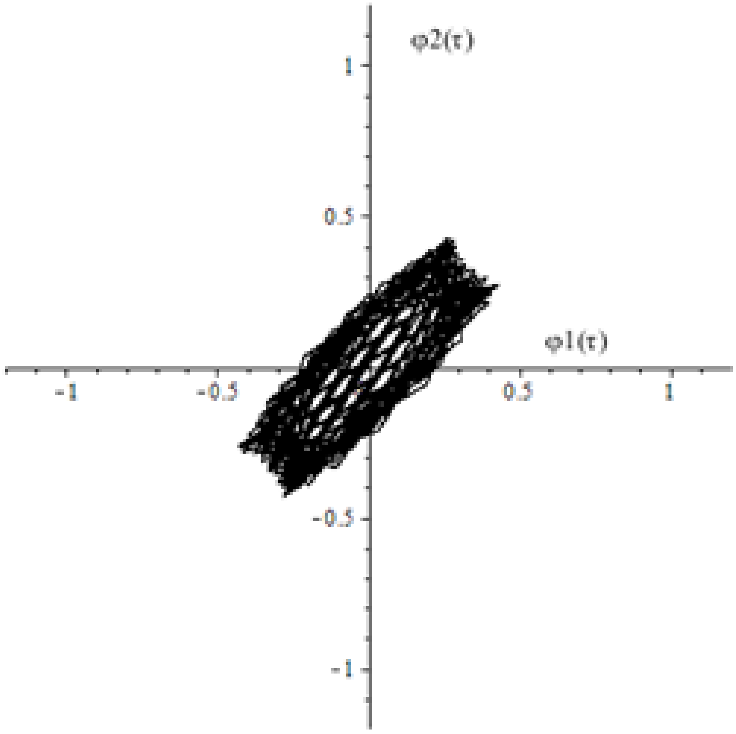

Fig. 5

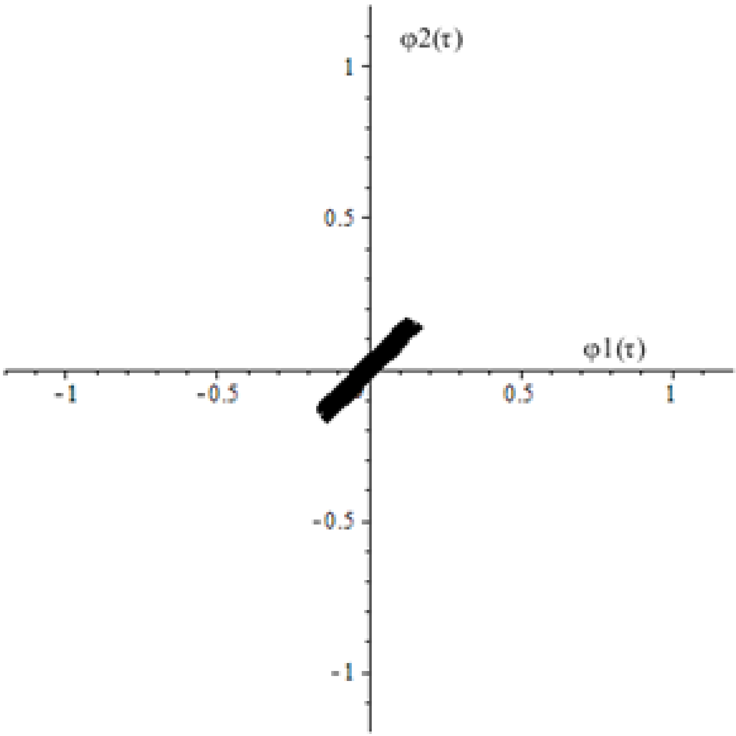

Fig. 6

Caption

Fig. 1 Schematic geometry of the problem
Fig. 2 Dependence $\dot{\varphi}^2_{1,2}(\tau)$ and $\ddot{\varphi}_{1,2}(\tau)$. In the given metric scale essentially, function $\ddot{\varphi}_{1,2}$ merges with abscissa τ . I.e., condition $|\ddot{\varphi}_{1,2}| \ll \left|\dot{\varphi}^2_{1,2}\right|$ realizes.
Fig. 3 Dependence $\varphi_2(\varphi_1)$ on the interval time $\tau \in [0, 500]$
Fig. 4 Dependence $\varphi_2(\varphi_1)$ on the interval time $\tau \in [2000, 3000]$
Fig. 5 Dependence $\varphi_2(\varphi_1)$ on the interval time $\tau \in [8000, 9000]$
Fig. 6 Dependence $\varphi_2 \approx \varphi_1$ on the interval time $\tau \in [48000, 49000]$

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