Solution of nonlinear non-autonomous Klein-Fock-Gordon equation

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Abstract

This paper presents methods of obtaining of functionally-invariant solutions $U(x, y, z, t)$ of nonautonomous nonlinear Klein-Fock-Gordon equation. Solutions $U(x, y, z, t)$ are obtained in the form of an arbitrary function which depends on one $\tau(x, y, z, t)$ or two $\alpha(x, y, z, t)$, $\beta(x, y, z, t)$ specially constructed functions. These functions are called ansatzes. Ansatzes (τ, α, β) are defined as solutions of separate equations (algebraic or mixed forms — algebraic and partial differential equations). Equations defining ansatzes contain arbitrary functions depending on (τ, α, β). Proposed methods allow to find $U(x, y, z, t)$ for the special class of nonautonomous nonlinear Klein-Fock-Gordon equations. General methods of solution are illustrated by examples of finding particular exact analytical solutions of nonautonomous Liouville equation.

1 Introduction

Nonlinear Klein-Fock-Gordon equation (NKFG)

$$U_{xx} + U_{yy} + U_{zz} - \frac{U_{tt}}{v^2} = F(U), \tag{1}$$

where $F(U)$ is an arbitrary nonlinear function U and lower symbolic index denotes partial derivative by the corresponding variable, plays a fundamental role in the modern natural sciences.

Equations (1) for particular forms of function $F(U)$ are well known in the mathematical physics. Eq. (1) with $F(U) = \exp U$ first came in the theory of surfaces with constant negative curvature. It was solved by Liouville [1]. Eq. (1) with $F(U)$ which equals to the sum of exponents describes oscillations of chain of nonlinear pendulums [2], and with $F(U)$ in the form of truncated series by functions $\sinh nU$, $\cosh nU$ $(n = 1, 2, \ldots)$ modells orientational structure of ferromagnetic media in the magnec field [3]. Many papers in mathematics, applied and theoretical physics are devoted to the analysis of nonlinear equations [4] including the equation (1) with $F(U)$ in the form of truncated Fourier series (sine-Gordon, double sine-Gordon etc.) and truncated Taylor series (Ginzburg-Landau equation). Outlined equations describe various physical phenomena and model numerous technological procecces [5].

Though it is necessary to idealize both physical phenomena and technological procecces making assumptions about uniformity of media and fields of external actions. Various physical phenomena and technological procecces are described by nonautonomous nonlinear Klein-Fock-Gordon equation

$$U_{xx} + U_{yy} + U_{zz} - \frac{U_{tt}}{v^2} = p(x,y,z,t)\,F(U). \tag{2}$$

Here $p(x,y,z,t)$ is some function.
Analytical methods of solving equation (2) are practically absent in the literature. This paper presents an approach to finding of exact analytical solutions of nonautonomous NKFG equation based on the methods of building of functionally-invariant solutions of partial differential equations.

2 Methods of obtaining of analytical solutions of nonautonomous Klein-Fock-Gordon equation

Solution of the differential equation is called functionally-invariant if it is in the form of an arbitrary function $U = f(W)$ depending on another definite function W called ansatz. The ansatz is a solution of one or several equations. Equations can be algebraic or differential or a mixed type. There are functionally invariant solutions depending on two or more ansatzes.
The idea of the existence of functionally invariant solutions was suggested by C.Jacobi [6]. A.Forsyth [7] found functionally invariant solutions of the Laplace equation, wave equation, and of the Helmholtz equation. In studying electromagnetic waves, Bateman [8] fundamentally and consistently developed the Jacobi idea as applied to the wave equation. S.L.Sobolev and V.I.Smirnov [9]–[12] successfully used the method to construct functionally invariant solutions to solve problems of diffraction and sound wave propagation in uniform and layered solid media. N.P.Erugin [13] made a large contribution to developing the theory of this method. Functionally-invariant solutions of both autonomous and nonautonomous NKFG equation in particular sine-Gordon equation were obtained by authors of [14]–[18].
We will find solutions of nonautonomous NKFG equation (2) in the form of composite function $U = f(W)$. Then Eq. (2) is as follows

$$f''\left[W_x^2 + W_y^2 + W_z^2 - \frac{W_t^2}{v^2}\right] + f'\left[W_{xx} + W_{yy} + W_{zz} - \frac{W_{tt}}{v^2}\right] = p\,F[f(W)]. \tag{3}$$

Here and elsewhere prime denotes ordinary derivative with respect to the argument. Two obvious propositions could be made on the basis of (3).
Proposition 1. If function W satisfies to equations

$$W_x^2 + W_y^2 + W_z^2 - \frac{W_t^2}{v^2} = 0, \qquad W_{xx} + W_{yy} + W_{zz} - \frac{W_{tt}}{v^2} = p(x,y,z,t), \tag{4}$$

then solution of equation (2) is given by inversion of the integral

$$\int \frac{df}{F(f)} = W(x,y,z,t). \tag{5}$$

Proposition 2. If function W satisfies to equations

$$W_x^2 + W_y^2 + W_z^2 - \frac{W_t^2}{v^2} = p(x, y, z, t), \qquad W_{xx} + W_{yy} + W_{zz} - \frac{W_{tt}}{v^2} = 0, \tag{6}$$

then solution of equation (2) is given by inversion of the integral

$$\int \frac{df}{\sqrt{E + V}} = \pm\sqrt{2}W(x, y, z, t). \tag{7}$$

Here $F(U) = V'(U)$ and E is constant of integration.
For analytical expressions $F(U)$ listed in the Introduction integral (5) by corresponding substitution of variable reduces to the integration of rational fraction. Function f defined by (7) is obtained by inversion of elliptical or hyperelliptical integral with genre defined by number of summands in the function $F(U)$. Because problem of finding function f from equations (5) and (7) is solved in literature the key problem of solving nonautonomous NKFG equation reduces to finding ansatz W satisfying to equations (4) and (6). This problem could be solved by application of methods of building functinally-invariant solutions of partial differential equations.
First method. We will seek the soutions of equations (4) in the form

$$W = \varphi(\tau). \tag{8}$$

Here $\varphi(\tau)$ is arbitrary function τ and $\tau(x, y, z, t)$ is root of algebraic equation

$$x\,\xi(\tau) + y\,\eta(\tau) + z\,\zeta(\tau) - v^2 t\tau = \frac{s^2 + q^2}{2}, \tag{9}$$
$$s^2 = x^2 + y^2 + z^2 - v^2 t^2, \qquad q^2 = \xi^2(\tau) + \eta^2(\tau) + \zeta^2(\tau) - v^2\tau^2,$$

and $\xi(\tau), \eta(\tau), \zeta(\tau)$ are arbitrary functions τ.
Equation (9) implicitly defines dependence τ from time and space coordinates. Therefrom using rules of implicit functions differentiation we obtain partial derivatives τ of first and second order and make sure that τ satisfies to equations

$$\tau_x^2 + \tau_y^2 + \tau_z^2 - \frac{\tau_t^2}{v^2} = 0, \qquad \tau_{xx} + \tau_{yy} + \tau_{zz} - \frac{\tau_{tt}}{v^2} = \frac{2}{\nu}, \tag{10}$$

$$\nu = \xi_\tau(x - \xi) + \eta_\tau(y - \eta) + \zeta_\tau(z - \zeta) - v^2(t - \tau). \tag{11}$$

In deriving equations (10) the following relations were taken into account

$$\xi_\tau\tau_x + \eta_\tau\tau_y + \zeta_\tau\tau_z + \tau_t = 1, \tag{12}$$

$$\nu_x\tau_x + \nu_y\tau_y + \nu_z\tau_z - \frac{\nu_t\tau_t}{v^2} = 1. \tag{13}$$

On the basis of the Proposition **1** and equations (10) we obtain that (8) is the solution of equation (2) if function $f(W)$ is obtained from equation (5) and

$$p(x, y, z, t) = \frac{2}{\nu}\varphi_\tau. \tag{14}$$

We note that in spite of the simplicity of the analytic dependence (14) computed solution of nonautunomous NKFG equation is sufficiently general. Note first that $\varphi(\tau)$ is arbitrary function τ and as well we seek anzats τ from the equation (9) which contains three arbitrary functions $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$.
We seek function W in the form

$$W = \Psi(\nu, \tau). \tag{15}$$

For this anzats equations (4), (6) takes the form

$$W_x^2 + W_y^2 + W_z^2 - \frac{W_t^2}{v^2} = \Psi_\nu^2(2\sigma - q^2) + 2\Psi_\nu\Psi_\tau + \Psi_\tau\frac{2}{\nu}, \tag{16}$$

$$W_{xx} + W_{yy} + W_{zz} - \frac{W_{tt}}{v^2} = \frac{1}{\nu}(2\sigma - q_1^2)(\nu\Psi_\nu + \Psi)_\nu + \frac{2}{\nu}\Psi_\nu(\nu\Psi_\nu + \Psi). \tag{17}$$

Here

$$\sigma = \xi_{\tau\tau}(x - \xi) + \eta_{\tau\tau}(y - \eta) + \zeta_{\tau\tau}(z - \zeta), \tag{18}$$

$$q_1^2 = \xi_\tau^2 + \eta_\tau^2 + \zeta_\tau^2 - v^2. \tag{19}$$

In deriving equations (16), (17) account must be taken of (13) and equations which are satisfied by function $\nu(x, y, z, t)$

$$\nu_x^2 + \nu_y^2 + \nu_z^2 - \frac{\nu_t^2}{v^2} = 2\sigma - q_1^2, \tag{20}$$

$$\nu_{xx} + \nu_{yy} + \nu_{zz} - \frac{\nu_{tt}}{v^2} = \frac{2}{\nu}(2\sigma - q_1^2). \tag{21}$$

From (17) it is seen that $\Psi(\nu, \tau)$ is the solution of the second equation (6) i.e. wave function if

$$\nu\Psi_\nu + \Psi = 0, \quad \Psi(\nu, \tau) = \frac{\varphi(\tau)}{\nu}. \tag{22}$$

For this solution

$$W_x^2 + W_y^2 + W_z^2 - \frac{W_t^2}{v^2} = \frac{\varphi^2}{\nu^4}\left[2\sigma - q_1^2 - 2\nu\frac{\varphi_\tau}{\varphi}\right]. \tag{23}$$

On the basis of Proposition **2** we arrive at conclusion that (22) is solution of the equation (2) if

$$p(x, y, z, t) = \frac{\varphi^2}{\nu^4}\left[2\sigma - q_1^2 - 2\nu\frac{\varphi_\tau}{\varphi}\right]. \tag{24}$$

More general solution of nonautonomous NKFG equation could be obtained assuming that

$$W = \Psi(\tau, \lambda, \nu), \tag{25}$$

where

$$\lambda = l(\tau)(x-\xi) + m(\tau)(y-\eta) + n(\tau)(z-\zeta) - v^2 w(\tau)(t-\tau). \tag{26}$$

Here $l(\tau)$, $m(\tau)$, $n(\tau)$, $w(\tau)$ are arbitrary functions. We impose on them the following relationships

$$l\xi_\tau + m\eta_\tau + n\zeta_\tau - v^2 w = 0, \tag{27}$$

$$l^2 + m^2 + n^2 = v^2 w^2. \tag{28}$$

For anzats W defined by (25), equations (4), (6) are reduced to the form

$$W_x^2 + W_y^2 + W_z^2 - \frac{W_t^2}{v^2} = \frac{2}{\nu}\left(\Psi_\tau + \omega\Psi_\lambda + \sigma\Psi_\nu\right)\left(\lambda\Psi_\lambda + \nu\Psi_\nu\right) - \frac{q_1^2}{\nu}\Psi_\nu(2\lambda^2\Psi_\lambda + \nu\Psi_\nu), \tag{29}$$

$$W_{xx} + W_{yy} + W_{zz} - \frac{W_{tt}}{v^2} =$$
$$= \frac{2}{\nu}\left(\frac{\partial}{\partial\tau} + \omega\frac{\partial}{\partial\lambda} + \sigma\frac{\partial}{\partial\nu}\right)\left(\lambda\Psi_\lambda + \nu\Psi_\nu + \Psi\right) - \frac{q_1^2}{\nu}\left(4\lambda\Psi_{\lambda\nu} + \nu\Psi_{\nu\nu}\right). \tag{30}$$

From (29) it is seen that $\Psi(\tau,\lambda,\nu)$ will be a solution of equations (4) if

$$q_1^2 = \xi_\tau^2 + \eta_\tau^2 + \zeta_\tau^2 - v^2 = 0, \tag{31}$$

$$\lambda\Psi_\lambda + \nu\Psi_\nu = 0. \tag{32}$$

General solution of the equation (32) has the form

$$\Psi(\tau,\lambda,\nu) = g\left(\frac{\lambda}{\nu}\right)\varphi(\tau). \tag{33}$$

Here $g(\lambda/\nu)$ is arbitrary homogenous function of zero order. For the solution (33)

$$p(x,y,z,t) = -\frac{2}{\nu^2}g^2\varphi^2\left[\frac{\varphi_\tau}{\varphi} - \frac{\sigma}{\nu} + \frac{g'}{\nu g}(\omega + \sigma\lambda)\right]. \tag{34}$$

Hence (25), (33) will be solutions of the equation (2) if $f(W)$ is the inversion of integral (5) and function p is given by (34).
It follows from (30) that $W = \Psi(\tau,\lambda,\nu)$ will be a wave function if condition (31) is satisfied and function $\Psi(\tau,\lambda,\nu)$ satisfies the equation

$$\lambda\Psi_\lambda + \nu\Psi_\nu + \Psi = 0. \tag{35}$$

Solution of equation (35) will be arbitrary homogeneous function of negative first order

$$\Psi(\tau,\lambda,\nu) = \frac{1}{\nu}g\left(\frac{\lambda}{\nu}\right)\varphi(\tau). \tag{36}$$

According to the Proposition **2** anzats $W = \Psi(\tau,\lambda,\nu)$ given by (36) will be a solution of nonautonomous NKFG equation (2) if

$$p(x,y,z,t) = -\frac{2}{\nu}\Psi(\Psi_\tau + \omega\Psi_\lambda + \sigma\Psi_\nu). \tag{37}$$

3 Examples of construction of analytic solutions of nonautonomous Liouville equation

Proposed method of solutions of NKFG equation is applicable for obtaining of exact analytic solutions of nonautonomous Liouville wave equation

$$U_{xx} + U_{yy} + U_{zz} - \frac{U_{tt}}{v^2} = p(x, y, z, t)\, e^U. \tag{38}$$

However we will not attempt to find general solutions. Contrary having the aim to illustrate general methods we will represent simple particular solutions.
According to (5) solution of thq equation (38) is given by

$$U = \ln \frac{1}{C - W}, \tag{39}$$

if function $W(x, y, z, t)$ satisfies to equations (4). In the case then $W(x, y, z, t)$ satisfies equations (6) solution of (38) according to (7) will be

$$U = -2 \ln \sinh \frac{W + C}{\sqrt{2}}. \tag{40}$$

In solutions (39), (40) C is contant of integration. Function $W(x, y, z, t)$ as is explained above could be constructed in a variety of ways. If to find the solution of equation (38) according to the first method then $W(x, y, z, t)$ it is an arbitrary function of anzats $\tau(x, y, z, t)$ and anzats is a root of an algebraic equation (9). In order to find particular solutions it is necessary to define arbitrary functions $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$. Assume that

$$\xi = 0, \quad \eta = 0, \quad \zeta = 0. \tag{41}$$

Then

$$\tau = t \pm \frac{R}{v}, \quad R = \sqrt{x^2 + y^2 + z^2}, \tag{42}$$

$$p(x, y, z, t) = \frac{2}{\nu} W_\tau, \quad \nu = \pm vR. \tag{43}$$

Therefore for the case (41) solution of the Liouville equation (38) is given by formula (39) in which W is an arbitrary function τ and $\tau(x, y, z, t)$ and $p(x, y, z, t)$ are given by (42), (43).
Let

$$\xi = a_1 v\tau, \quad \eta = a_2 v\tau, \quad \zeta = a_3 v\tau, \quad a_1^2 + a_2^2 + a_3^2 = 1. \tag{44}$$

For this selection of functions ξ, η, ζ solution of (38) is (39) if

$$\tau = \frac{s^2}{2\nu}, \quad s^2 = x^2 + y^2 + z^2 - v^2 t^2, \tag{45}$$

$$p(x, y, z, t) = \frac{2}{\nu} W_\tau, \quad \nu = v(a_1 x + a_2 y + a_3 z - vt), \tag{46}$$

and W as in the case (41) is an arbitrary function τ.
New solutions of the equation (38) could be constructed if to introduce a function λ. According to the definition (26) it depends on functions $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$ which determine an anzats $\tau(x,y,z,t)$ and other arbitrary functions $l(\tau)$, $m(\tau)$, $n(\tau)$, and $w(\tau)$. Last functions are related to the first equations (27), (28). We find from them

$$\begin{aligned} l &= \frac{v\,w(\tau)}{\sqrt{q_1^2+v^2}}\left\{v\cos A+\sqrt{q_1^2}\,[\cos\delta\cos f(\tau)-\sin\delta\cos B\sin f(\tau)]\right\}, \\ m &= \frac{v\,w(\tau)}{\sqrt{q_1^2+v^2}}\left\{v\cos B+\sqrt{q_1^2}\sin C\sin f(\tau)\right\}, \\ n &= \frac{v\,w(\tau)}{\sqrt{q_1^2+v^2}}\left\{v\cos C-\sqrt{q_1^2}\,[\sin\delta\cos f(\tau)+\cos\delta\cos B\sin f(\tau)]\right\}. \end{aligned} \tag{47}$$

Here

$$\cos A=\frac{\xi_\tau}{\sqrt{q_1^2+v^2}},\ \cos B=\frac{\eta_\tau}{\sqrt{q_1^2+v^2}},\ \cos C=\frac{\zeta_\tau}{\sqrt{q_1^2+v^2}},\ \sin\delta=\frac{\xi_\tau}{\sqrt{\xi_\tau^2+\zeta_\tau^2}}, \tag{48}$$

and $f(\tau)$ is an arbitrary function τ.
If to define functions $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$ then from the equation (9) anzats could be find $\tau(x,y,z,t)$ and formula (26) helps with (47) compute the function λ.
Let

$$\xi=av\tau,\quad \eta=0,\quad \zeta=0. \tag{49}$$

Then

$$l=\frac{v}{a}w(\tau),\quad m=\frac{\sqrt{a^2-1}}{a}v\,w(\tau)\sin f(\tau),\quad n=-\frac{\sqrt{a^2-1}}{a}v\,w(\tau)\cos f(\tau). \tag{50}$$

On the basis of (49) and (50) we obtain

$$\begin{aligned} \tau &= \frac{1}{v(a^2-1)}\left[ax-vt\pm\sqrt{(x-avt)^2-(a^2-1)(y^2+z^2)}\right], \\ \nu &= \mp v\sqrt{(x-avt)^2-(a^2-1)(y^2+z^2)}, \\ \lambda &= \frac{v\,w(\tau)}{a}\left[x-avt+\sqrt{a^2-1}(y\sin f-z\cos f\right]. \end{aligned} \tag{51}$$

Determinig functions τ,ν,λ by formulas (15), (22) we obtain an anzats W and by formula (40) obtain the solution of nonautonomous Liouviville equation (38). We note that for the case (49) formulas (33) and (36) will not give the solution of the equation (38) because condition (31) is not satisfied for it. It will be satisfied if $a=1$. In this case

$$\tau=\frac{s^2}{2\nu},\quad \nu=v(x-vt),\quad \lambda=\nu w(\tau). \tag{52}$$

Function $\varphi(\tau)$ satisfies the system of equations (4) and $\varphi(\tau)/\nu$ – system of equations (6). Threfore using functions (τ,ν,λ) (52) solution of nonautonomous Liouville equation (38) could be constructed using formulas obtained above. Therefore solution of (38) will be (40) if

$$W = \frac{1}{x^2+y^2+z^2-v^2t^2}, \quad p(x,y,z,t) = \frac{4}{(x^2+y^2+z^2-v^2t^2)^3}. \tag{53}$$

This Lorentz invariant solution is obtained if τ and ν defined by (52) and

$$W = \frac{\varphi(\tau)}{\nu}, \quad \varphi(\tau) = \frac{1}{2\tau}.$$

Examples of particular solutions of the equation (38) are given below. They are defined by the formula (39) in which W is an arbitrary function of an anzats τ and τ and function $p(x,y,z,t)$ are defined by a particular form of functions (ξ,η,ζ,θ).

1. $\xi = v\tau\cos\alpha\cos\beta, \quad \eta = v\tau\cos\alpha\sin\beta, \quad \zeta = v\tau\sin\alpha, \quad \theta = \tau,$

$$\tau = t \pm \frac{R}{v}, \quad p(x,y,z,t) = \pm\frac{2}{vR}W_\tau,$$

2. $\xi = v\tau\cos\alpha, \quad \eta = v\tau\sin\alpha, \quad \zeta = 0, \quad \theta = \tau,$

$$\tau = \frac{s^2}{\sqrt{x^2+y^2}-vt}, \quad p(x,y,z,t) = \frac{1}{\sqrt{x^2+y^2}-vt}\left[4 - \frac{\tau}{\sqrt{x^2+y^2}}\right]W_\tau,$$

3. $\xi = \tau\cos\alpha, \quad \eta = \tau\sin\alpha, \quad \zeta = \tau\sinh\beta, \quad \theta = \frac{\tau}{v}\cosh\beta,$

$$\tau = \sqrt{x^2+y^2} - \sqrt{v^2t^2-z^2}, \quad p(x,y,z,t) = \left[\frac{1}{\sqrt{x^2+y^2}} + \frac{1}{\sqrt{v^2t^2-z^2}}\right]W_\tau,$$

4. $\xi = \tau\cos\alpha\sinh\beta, \quad \eta = \tau\sin\alpha\sinh\beta, \quad \zeta = \tau, \quad \theta = \frac{\tau}{v}\cosh\beta,$

$$\tau = z + \sqrt{v^2t^2-x^2-y^2}, \quad p(x,y,z,t) = -\frac{2}{\sqrt{v^2t^2-x^2-y^2}}W_\tau.$$

In Fig. 1. spatial image of the solution **3** is given for the case $W(\tau) = \sin\tau$. Solutions of two-dimensional nonautonomous Liouville equations for some particular cases of function $p(x,y,t)$ have been obtained in papers [19, 20].

4 Conclusion

Methods of obtaining exact analytic solutions of nonautonomous NKFG equation are divised. They are based on the ideas and methods of construction of functionally-invariant solutions of partial differential equations. Proposed methods allow to construct solutions of NKFG equation in the form of an arbitrary function depending on one or several anzatses. Equations for determining of anzatses are given and methods of their solution are discussed. Developped methods allow to find analytic solutions of the equation (2) for functions $p(x,y,z,t)$ of the special form. Distinctive feature consists in simultaneous obtaining of solutions and analytic form of the function $p(x,y,z,t)$. Solutions are particular but the have sufficiently general form

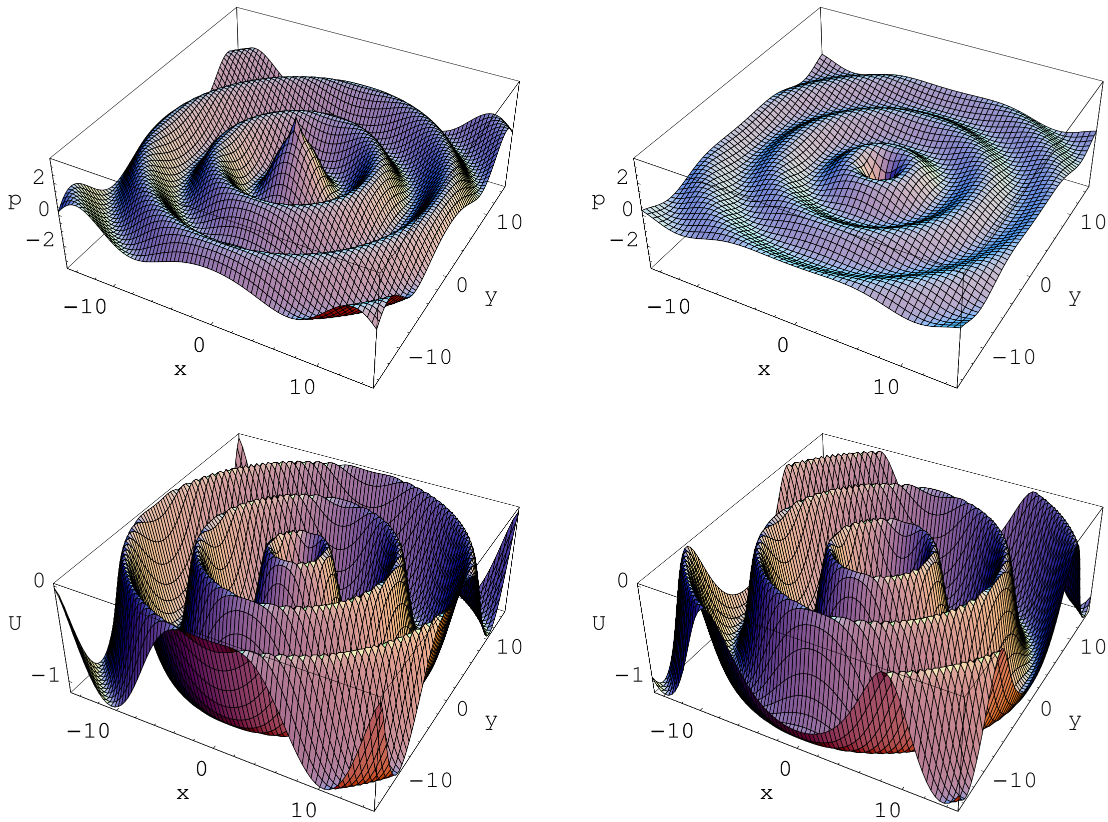

Figure 1: Function p and solution **3** for $z = 0$, $t = 1$ (left) and $t = 3$ (right).

because the include arbitrary functions. Untill now the problem of their selection in order to get given beforehand function $p(x, y, z, t)$ is not solved. This task needs further studies.

General methods of construction of functionally-invariant solutions of NKFG equation are illustrated by examples of finding particular exact analytical solutions of nonautonomous Liuoville equation.

Methods of solution of nonautonomous NKFG equation are described for the three-dimesional space. However they could be easily extended for the space of any dimension. We expect that proposed methods could be useful for the realization of nonlinear models which describe real physical phenomena and technological processes more adequately.

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