Coherent structures and localized modes in collective models of accelerator physics

Antonina N. Fedorova, Michael G. Zeitlin
zeitlin@math.ipme.ru, anton@math.ipme.ru

Abstract

We present applications of our multiresolution approach to various popular models of accelerated physics. Mostly we are interested in the description of complex beam motions with the internal collective behaviour generated by background electromagnetic fields: Vlasov-Maxwell-Poisson systems, envelope/momentum RMS approximation, a beam-beam interaction model. We obtain the representation for all dynamical variables as multiresolution expansion via high-localized nonlinear eigenmodes in the base of various compactly supported wavelet-like functions. Numerical modelling demonstrates the creation of coherent structures, generated by internal hidden symmetry on the level of the underlying functional spaces and appearance of (meta)stable patterns.

1 Introduction

In this paper we consider the applications of a new numerical-analytical technique which is based on the methods of local nonlinear harmonic analysis (LNHA) a.k.a. wavelet analysis in the case of affine group as a group of internal symmetries, to three (nonlinear) beam/accelerator physics problems which can be characterized by the collective type behaviour: some forms of Vlasov-Maxwell-Poisson equations[1], RMS envelope dynamics[2], the model of beam-beam interactions [3]. Such an approach may be useful in all models in which it is possible and reasonable to reduce all complicated problems related with statistical distributions to the problems described by systems of nonlinear ordinary/partial differential equations with or without some (functional)constraints. LHNA is a relatively novel set of mathematical methods, which gives us the possibility to work with well-localized bases in functional spaces and gives the maximum sparse forms for the general type of operators (differential, integral, pseudodifferential) in such bases. Our approach is based on the variational-multiscale approach developed by us [4]-[15] and allows to consider the polynomial and rational type of nonlinearities. Multiscale representation for the solutions has the following multiresolution decomposition via nonlinear high-localized eigenmodes

$$
\begin{aligned}
u(t,x) &= \sum_{(i,j)\in Z^2} a_{ij} U^i(x) V^j(t), && (1)\\
V^k(t) &= V_N^{k,slow}(t) + \sum_{i\ge N} V_i^k(\omega_i^1 t), \quad \omega_i^1 \sim 2^i && (2)\\
U^k(x) &= U_M^{k,slow}(x) + \sum_{j\ge M} U_j^k(\omega_j^2 x), \quad \omega_j^2 \sim 2^j, && (3)
\end{aligned}
$$

which corresponds to the full multiresolution expansion in all underlying time/space scales (x are the generalized space coordinates or phase space coordinates, t is time coordinate). The representation (1) provides the expansion into the slow part $u_{N,M}^{slow}$ and fast oscillating parts for arbitrary N, M. So, we may move from coarse scales of resolution to the finest one for obtaining more detailed information about our dynamical process. The first terms in the RHS of formulas (1)-(3) correspond, on the global level of function space decomposition, to the resolution space and the second ones to detail space. In this way we give contribution to our full solution from each scale of resolution or each time/space scale or from each high-localized nonlinear eigenmode (Fig.1). The same is correct for the contribution to power spectral density (energy spectrum): we can take into account contributions from each level/scale of resolution. In all these models, numerical modelling demonstrates the appearance of coherent high-localized structures and (meta)stable patterns formation. In part 2 we start from the description of Vlasov-Maxwell-Poisson equations, root-mean-square (RMS) envelope dynamics and beam-beam interaction model, after that in part 3 we consider our generic approach based on variational-multiresolution formulation. We give explicit representation for all dynamical variables in the base of compactly supported wavelets or nonlinear eigenmodes. Our solutions are parametrized by solutions of a number of reduced algebraical problems, one from which is nonlinear with the same degree of nonlinearity and the rest are the linear problems which correspond to a particular method of calculation of scalar products of functions from wavelet bases and their derivatives. In part 4, we consider numerical modelling based on our analytical approach.

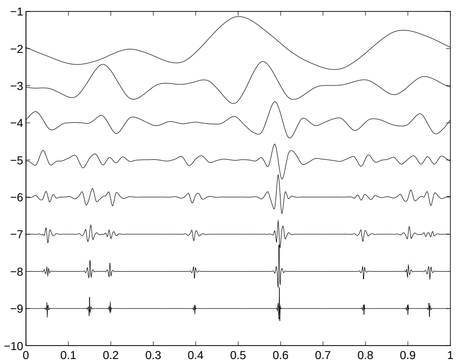

Figure 1: Multiscale/high-localized eigenmodes decomposition.

2 Collective models

2.1 Vlasov-Maxwell-Poisson equations

Analysis based on the full form of nonlinear Vlasov-Maxwell-Poisson equations leads to more clear understanding of the collective effects and nonlinear beam dynamics of high intensity beam propagation in the periodic-focusing and uniform-focusing transport systems. We consider the following form of equations ([1],[2] for setup and designation):

$$\Big\{\frac{\partial}{\partial s}+p_x\frac{\partial}{\partial x}+p_y\frac{\partial}{\partial y}-\Big[k_x(s)x+\frac{\partial\psi}{\partial x}\Big]\frac{\partial}{\partial p_x}-$$

$$\Big[k_y(s)y+\frac{\partial\psi}{\partial y}\Big]\frac{\partial}{\partial p_y}\Big\}f_b(x,y,p_x,p_y,s)=0, \tag{4}$$

$$\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)\psi=-\frac{2\pi K_b}{N_b}\int \mathrm{d}p_x\mathrm{d}p_y f_b, \tag{5}$$

$$\int \mathrm{d}x\mathrm{d}y\mathrm{d}p_x\mathrm{d}p_y f_b=N_b \tag{6}$$

The corresponding Hamiltonian for transverse single-particle motion is given by

$$H(x,y,p_x,p_y,s)=\frac{1}{2}(p_x^2+p_y^2)+\frac{1}{2}[k_x(s)x^2 \tag{7}$$

$$+k_y(s)y^2]+H_1(x,y,p_x,p_y,s)+\psi(x,y,s),$$

where H_1 is nonlinear (polynomial/rational) part of the full Hamiltonian. In case of Vlasov-Maxwell-Poisson system we may transform (4) into invariant form

$$\frac{\partial f_b}{\partial s}+[f,H]=0. \tag{8}$$

2.2 RMS equations

We consider an approach based on the second moments of distribution functions for calculation of the evolution of RMS envelope of a beam. The RMS envelope equations are the most useful for analysis of the beam self–forces (space–charge) effects and also allow to consider both transverse and longitudinal dynamics of space-charge-dominated relativistic high–brightness axisymmetric/asymmetric beams, which under short laser pulse–driven radio-frequency photoinjectors have fast transition from nonrelativistic to relativistic regime [2]. Analysis of halo growth in beams, appeared as result of bunch oscillations in the particle-core model, is also based on three-dimensional envelope equations [2]. We can consider the different forms of RMS envelope equations, which are not more than nonlinear differential equations with rational nonlinearities and variable coefficients from the formal point of view. Let $f(x_1,x_2)$ be the distribution function which gives full information about noninteracting ensemble of beam particles regarding to trace space or transverse phase coordinates (x_1,x_2). Then we may extract the first nontrivial effects of collective dynamics from the second moments

$$\sigma^2_{x_ix_j} \;=\; <x_ix_j>=\int\int x_ix_jf(x_i,x_j)\mathrm{d}x_i\mathrm{d}x_j. \tag{9}$$

RMS emittance ellipse is given by $\varepsilon^2_{x,rms} =< x_i^2 >< x_j^2 > - < x_i x_j >^2$ $(i \neq j)$. Expressions for twiss parameters are also based on the second moments. We will consider the following particular cases of RMS envelope equations, which describe the evolution of moments (9) ([2] for full designation): for asymmetric beams we have the system of two envelope equations of the second order for σ_{x_1} and σ_{x_2}:

$$\begin{aligned} \sigma''_{x_1} + \sigma'_{x_1}\frac{\gamma'}{\gamma} + \Omega^2_{x_1}\left(\frac{\gamma'}{\gamma}\right)^2\sigma_{x_1} &= I/(I_0(\sigma_{x_1}+\sigma_{x_2})\gamma^3) + \varepsilon^2_{nx_1}/\sigma^3_{x_1}\gamma^2, \\ \sigma''_{x_2} + \sigma'_{x_2}\frac{\gamma'}{\gamma} + \Omega^2_{x_2}\left(\frac{\gamma'}{\gamma}\right)^2\sigma_{x_2} &= I/(I_0(\sigma_{x_1}+\sigma_{x_2})\gamma^3) + \varepsilon^2_{nx_2}/\sigma^3_{x_2}\gamma^2. \end{aligned} \tag{10}$$

The envelope equation for an axisymmetric beam is a particular case of preceding equations. Also we have the related Lawson equation for evolution of RMS envelope in the paraxial limit, which governs evolution of cylindrical symmetric envelope under external linear focusing channel of strenght K_r:

$$\sigma'' + \sigma'\left(\frac{\gamma'}{\beta^2\gamma}\right) + K_r\sigma = \frac{k_s}{\sigma\beta^3\gamma^3} + \frac{\varepsilon_n^2}{\sigma^3\beta^2\gamma^2}, \tag{11}$$

where $K_r \equiv -F_r/r\beta^2\gamma mc^2$, $\beta \equiv \nu_b/c = \sqrt{1-\gamma^{-2}}$. According to [2] we have the following form for envelope equations in the model of halo formation by bunch oscillations:

$$\begin{aligned} \ddot{X} + k_x^2(s)X - \frac{3K}{8}\frac{\xi_x}{YZ} - \frac{\varepsilon_x^2}{X^3} &= 0, \\ \ddot{Y} + k_y^2(s)Y - \frac{3K}{8}\frac{\xi_y}{XZ} - \frac{\varepsilon_y^2}{Y^3} &= 0, \\ \ddot{Z} + k_z^2(s)Z - \gamma^2\frac{3K}{8}\frac{\xi_z}{XY} - \frac{\varepsilon_z^2}{Z^3} &= 0, \end{aligned} \tag{12}$$

where X(s), Y(s), Z(s) are bunch envelopes, ξ_x, ξ_y, $\xi_z = F(X,Y,Z)$.
After transformations to the Cauchy form we can see that all these equations from the formal point of view are not more than ordinary differential equations with rational nonlinearities and variable coefficients Also, we may consider regimes in which γ, γ' are not fixed functions/constants but satisfy some additional differential constraints/equations, but this case does not change our general approach of the next part.

2.3 Beam-beam modelling

In A. Chao e.a. model [3] for simulation of beam-beam interaction, the initial collective description for distribution function $f(s,x,p)$

$$\frac{\partial f}{\partial s} + p\frac{\partial f}{\partial x} - (k(s)x - F(x,s))\frac{\partial f}{\partial p} = 0 \tag{13}$$

is reduced to Fockker-Planck (FP) equations on the first stage and after that to a very nontrivial dynamical system with complex behaviour:

$$\begin{aligned} &\frac{\mathrm{d}^2\sigma_k}{\mathrm{d}s^2} + \Gamma_k\frac{\mathrm{d}\sigma_k}{\mathrm{d}s} + F_k\sigma_k = \frac{1}{\beta_k^2 a_k^2\sigma_k^3}, \\ &\frac{\mathrm{d}a_n}{\mathrm{d}s} = \Gamma_k a_k(1 - a_k^2\sigma_k^2). \end{aligned} \tag{14}$$

The solutions of dynamical system (14) provides the parameters of enveloping gaussian anzatz for solution of initial FP equations on the second stage of this global reduction which encodes stochastic collective motion into the data describing behaviour of nonlinear dynamic system.

3 Rational dynamics

After some anzatzes ([4]-[15]) all problems above may be reduced to the dynamical systems (cases 2.2 and 2.3 (system (14)) above):

$$Q_i(x)\frac{\mathrm{d}x_i}{\mathrm{d}t} = P_i(x,t), \quad x = (x_1, ..., x_n), \tag{15}$$
$$i = 1, ..., n, \quad \max_i deg\ P_i = p, \quad \max_i deg\ Q_i = q$$

or a set of such systems (cases 2.1, 2.3 (full equation (13)) above) corresponding to each independent coordinate in phase space. They have the fixed initial (or boundary) conditions $x_i(0)$, where P_i, Q_i are not more than polynomial functions of dynamical variables x_j and have arbitrary dependence of time. Because of time dilation we can consider only next time interval: $0 \leq t \leq 1$. Let us consider a set of functions

$$\Phi_i(t) = x_i\frac{\mathrm{d}}{\mathrm{d}t}(Q_i y_i) + P_i y_i \tag{16}$$

and a set of functionals

$$F_i(x) = \int_0^1 \Phi_i(t)dt - Q_i x_i y_i \mid_0^1, \tag{17}$$

where $y_i(t)$ $(y_i(0) = 0)$ are dual (variational) variables. It is obvious that the initial system and the system

$$F_i(x) = 0 \tag{18}$$

are equivalent. Of course, we consider regular $Q_i(x)$ at $t = 0$ or $t = 1$, i.e. $Q_i(x(0)), Q_i(x(1)) \neq \infty$. Now we consider formal expansions for x_i, y_i:

$$x_i(t) = x_i(0) + \sum_k \lambda_i^k \varphi_k(t) \quad y_j(t) = \sum_r \eta_j^r \varphi_r(t), \tag{19}$$

where $\varphi_k(t)$ are basis functions for proper functional space (L^2, L^p, Sobolev, etc.), which corresponds to concrete particular problem. It should be noted that initial conditions demand only $\varphi_k(0) = 0$ and for $r = 1, ..., N, \quad i = 1, ..., n$, we collect the "generalized Fourier coefficients" in the following data set:

$$\lambda = \{\lambda_i\} = \{\lambda_i^r\} = (\lambda_i^1, \lambda_i^2, ..., \lambda_i^N), \tag{20}$$

where the lower index i corresponds to expansion of dynamical variable with index i, i.e. x_i and the upper index r correspond to the numbers of terms in the expansion of dynamical variables in the formal series. Then we put (19) into the functional equations (18) and as a result we have the following reduced algebraical system of equations on the set of unknown coefficients λ_i^k of expansions (19):

$$L(Q_{ij}, \lambda, \alpha_I) = M(P_{ij}, \lambda, \beta_J), \tag{21}$$

where operators L and M are algebraization of RHS and LHS of initial problem (15) and the data set λ (20) are unknowns of Reduced System of Algebraical Equations (RSAE)(21). Here Q_{ij} are the coefficients (with possible time dependence) of LHS of initial system of differential equations (15) and as a consequence the coefficients of RSAE, while P_{ij} are coefficients (with possible time dependence) of RHS of initial system of differential equations (15) and as a consequence the coefficients of reduced RSAE too. $I = (i_1, ..., i_{q+2})$, $J = (j_1, ..., j_{p+1})$ are multiindexes by which are labelled α_I and β_I, which are the other coefficients of RSAE (21). So, we have:

$$\beta_J = \{\beta_{j_1 \ldots j_{p+1}}\} = \int \prod_{1 \le j_k \le p+1} \varphi_{j_k}, \tag{22}$$

where p is the degree of polinomial operator P (15)

$$\alpha_I = \{\alpha_{i_1} ... \alpha_{i_{q+2}}\} = \sum_{i_1, \ldots, i_{q+2}} \int \varphi_{i_1} ... \dot{\varphi_{i_s}} ... \varphi_{i_{q+2}}, \tag{23}$$

where q is the degree of polynomial operator Q (15), $i_\ell = (1, ..., q+2)$, $\dot{\varphi_{i_s}} = \mathrm{d}\varphi_{i_s}/\mathrm{d}t$. Now we can solve RSAE (21) and determine unknown coefficients of the formal expansion (19), therefore we can obtain the solution for our initial problem (15). It should be noted that during modelling we consider only the truncated expansions (19) with N terms, so we have, from (21), the system of $N \times n$ algebraical equations with degree $\ell = max\{p, q\}$ and the degree of this algebraical system coincides with the degree of initial differential system. Finally, we have the solution of the initial nonlinear (rational) problem in the following form (it is a particular form of our general multiscale representation (1)-(3)):

$$x_i(t) = x_i(0) + \sum_{k=1}^{N} \lambda_i^k X_k(t), \tag{24}$$

where coefficients λ_i^k are roots of the corresponding reduced algebraical (polynomial) problem RSAE (21). Consequently, we have a parametrization of solution of initial problem by solution of reduced algebraical problem (21). The first main problem is a problem of computations of the coefficients α_I (23), β_J (22) of the reduced algebraical system. These problems may be explicitly solved inside multiresolution approach [4]-[15]. The obtained solutions are given in the form (24), where $X_k(t)$ are basis functions and λ_k^i are roots of reduced system of equations. In our case $X_k(t)$ are obtained via multiresolution expansions and represented by compactly supported wavelets and λ_k^i are roots of the corresponding general polynomial system (21). Because affine group of translation and dilations is inside the approach, this method resembles the action of a microscope. We have contribution to final result from each scale of resolution from the whole infinite scale of spaces. More exactly, the closed subspace $V_j (j \in \mathbf{Z})$ corresponds to level j of resolution, or to scale j.

We consider multiresolution analysis on $L^2(\mathbf{R}^n)$ [16] (of course, we may consider any different and proper functional space), which is a sequence of increasing closed subspaces V_j: $...V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset ...$ satisfying the following properties: let W_j be the orthonormal complement of V_j with respect to V_{j+1} : $V_{j+1} = V_j \bigoplus W_j$, then

$$L^2(\mathbf{R}) = \overline{V_0 \bigoplus_{j=0}^{\infty} W_j}. \qquad (25)$$

Such a functional space decomposition corresponds to the exact nonlinear (maximally) localized eigenmode decomposition (1)-(3). It should be noted that such representations give the best possible localization properties in the corresponding (phase)space/time coordinates. In contrast with different approaches formulas (1)-(3), (24) or, in general, (25) do not use perturbation technique or linearization procedures and represent exact dynamical evolution via generalized nonlinear localized eigenmodes. Finally, by using multiresolution decomposition which provides best localization in the underlying functional space, we can construct high-localized coherent structures in spatially–extended stochastic systems with collective behaviour. Definitely, coherence is a consequence of action of internal hidden symmetry which is a generic property of mutiresolution decomposition.

4 Modelling

Resulting multiresolution/multiscale representations for solutions of equations from part 2 in the high-localized bases/eigenmodes are demonstrated on Fig. 2–Fig. 7. Multiscale modelling [17] demonstrates the appearance of (meta)stable patterns formation from high-localized coherent structures. Fig. 2, Fig. 3 present contribution to the full expansion (1)-(3) from level 1 and level 4 of decomposition (25). Figures 4, 5 show the representations for full solutions, constructed from the first 6 eigenmodes (6 levels in formula (25)). Figures 6, 7 show (meta)stable patterns formation based on high-localized coherent structures.

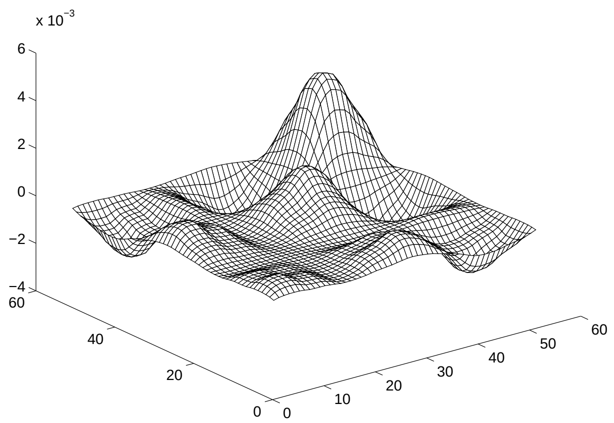

Figure 2: Base localized eigenmode

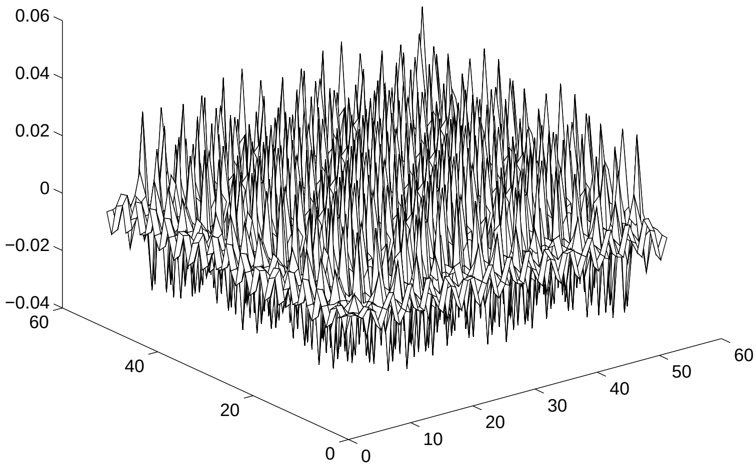

Figure 3: Four-eigenmodes decomposition

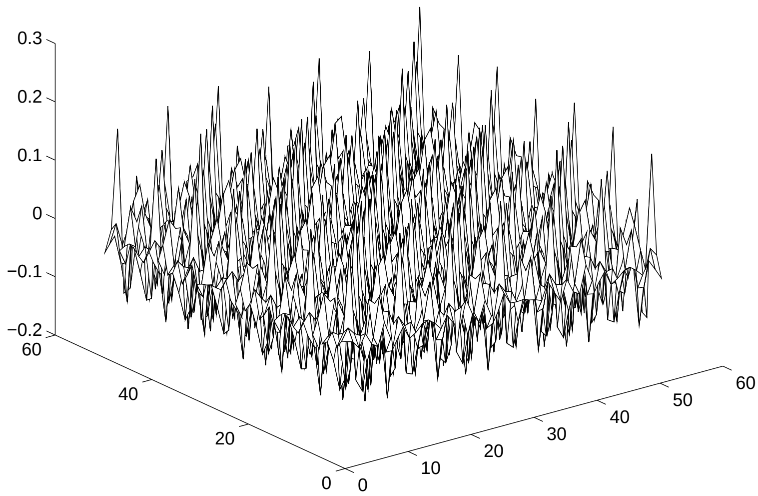

Figure 4: Appearence of coherent structure

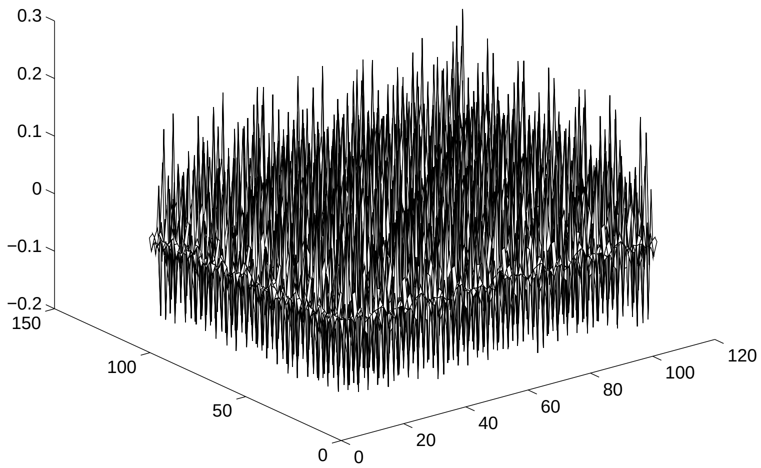

Figure 5: Six-eigenmodes decomposition

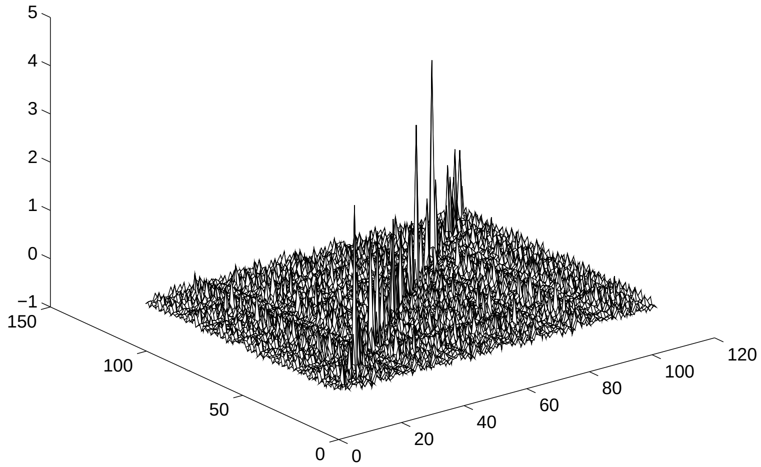

Figure 6: (Meta)Stable pattern 1

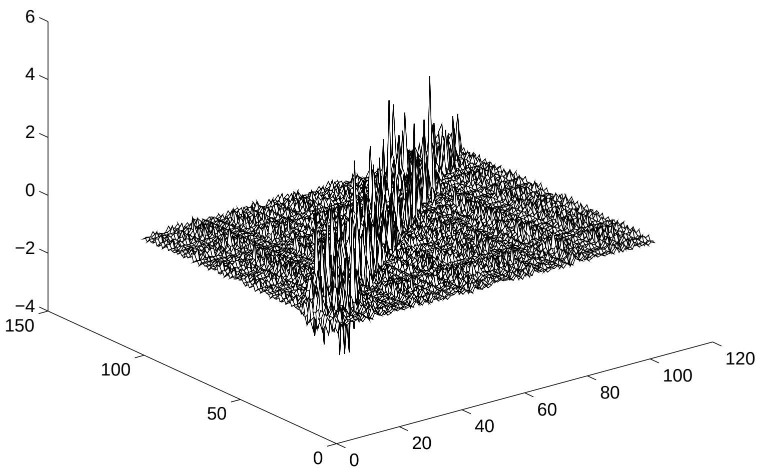

Figure 7: (Meta)Stable pattern 2

References

[1] R. Davidson, H. Qin, P. Channel, PRSTAB, 2, 074401, 1999.

[2] J.B. Rosenzweig, Fundamentals of Beam Physics, e-version: http://www.physics.ucla.edu/class/99F/250Rosenzweig/notes/ L. Serafini and J.B. Rosenzweig, *Phys. Rev. E* **55**, 7565, 1997; C. Allen, T. Wangler, papers in UCLA ICFA Proc., World Sci., 2000.

[3] A. Chao, e.a., Los Alamos preprint, physics/0010055.

[4] A.N. Fedorova and M.G. Zeitlin, Wavelets in Optimization and Approximations, *Math. and Comp. in Simulation*, **46**, p. 527, 1998.

[5] A.N. Fedorova and M.G. Zeitlin, Wavelet Approach to Mechanical Problems. Symplectic Group, Symplectic Topology and Symplectic Scales p.31; Wavelet Approach to Polynomial Mechanical Problems, p. 101, in *New Applications of Nonlinear and Chaotic Dynamics in Mechanics*, Ed. F. Moon, Kluwer, 1998.

[6] A.N. Fedorova and M.G. Zeitlin, Nonlinear Dynamics of Accelerator via Wavelet Approach, in **CP405**, p. 87, American Institute of Physics, 1997. Los Alamos preprint, physics/9710035.

[7] A.N. Fedorova, M.G. Zeitlin and Z. Parsa, Wavelet Approach to Accelerator Problems, parts I, II, III, in Proc. PAC97 **2**, pp. 1502, 1505, 1508, American Physical Society/IEEE, 1998.

[8] A.N. Fedorova, M.G. Zeitlin and Z. Parsa, Nonlinear Effects in Accelerator Physics: from Scale to Scale via Wavelets, p. 930; Wavelet Approach to Hamiltonian, Chaotic and Quantum Calculations in Accelerator Physics, p. 933, in Proc. EPAC98, Institute of Physics, 1998.

[9] A.N. Fedorova, M.G. Zeitlin and Z. Parsa, Variational Approach in Wavelet Framework to Polynomial Approximations of Nonlinear Accelerator Problems, p. 48, in **CP468**, American Institute of Physics, 1999. Los Alamos preprint, physics/990262.

[10] A.N. Fedorova, M.G. Zeitlin and Z. Parsa, Symmetry, Hamiltonian Problems and Wavelets in Accelerator Physics, p. 69, in **CP468**, American Institute of Physics, 1999. Los Alamos preprint, physics/990263.

[11] A.N. Fedorova, M.G. Zeitlin, Nonlinear Accelerator Problems via Wavelets, Parts 1-8, in Proc. PAC99, pp. 1614, 1617, 1620, 2900, 2903, 2906, 2909, 2912, American Institute of Physics/IEEE, New York, 1999. Los Alamos preprints: physics/9904039,physics/9904040, physics/9904041, physics/9904042, physics/9904043, physics/9904045, physics/9904046, physics/9904047.

[12] A.N. Fedorova, M.G. Zeitlin, Variational-Wavelet Approach to RMS Envelope Equations, p. 235, in The Physics of High Brightness Beams, World Scientific, 2000. Los Alamos preprint: physics/0003095.

[13] A.N. Fedorova, M.G. Zeitlin, Spin-Orbital Motion: Symmetry and Dynamics, p. 415; Local Analysis of Nonlinear RMS Envelope Dynamics, p. 872; Multiresolution Representation for Orbital Dynamics in Multipolar Fields, p. 1101; Quasiclassical Calculations in Beam Dynamics, p. 1190; Multiscale Representations for Solutions of Vlasov-Maxwell Equations for Intense Beam Propagation, p. 1339; Nonlinear Beam Dynamics and Effects of Wigglers, p. 2325, in Proc. EPAC00, Austrian Acad. Sci., 2000. Los Alamos preprints: physics/0008045, physics/0008046, physics/0008047, physics/0008048, physics/0008049, physics/0008050.

[14] A.N. Fedorova, M.G. Zeitlin, Multiscale Analysis of RMS Envelope Dynamics, p. 300; Multiresolution Representations for Solutions of Vlasov - Maxwell - Poisson Equations,p. 303, in Proc. 20 International Linac Conf., SLAC, Stanford, 2000. Los Alamos preprints: physics/0008043, physics/0008200.

[15] A.N. Fedorova, M.G. Zeitlin, Localized Coherent Structures and Patterns Formation in Collective Models of Beam Motion, p. 527; Quasiclassical Calculations for Wigner Functions via Multiresolution, p. 539, in Quantum Aspects of Beam Physics, ed. P. Chen, World Scientific, 2002. Los Alamos preprints:physics/0101006, physics/0101007.

[16] Y. Meyer, Wavelets and Operators, CUP, 1990.

[17] D. Donoho, WaveLab, Stanford, 1998.

Antonina N. Fedorova, Michael G. Zeitlin

Mathematical Methods in Mechanics Group, IPME RAS, V.O. Bolshoj pr., 61, 199178, St. Petersburg, Russia

http://math.ipme.ru/zeitlin.html, http://mp.ipme.ru/zeitlin.html