Backwards waves in a cylindrical shell: comparison of 2D shell theories with 3D theory of elasticity

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Abstract

Dispersion of elastic waves in an infinite circular cylindrical shell is studied with special attention to backwards waves. Two types of boundary conditions on the surfaces of the shell are considered: free faces and the spring-type boundary condition on the outer surface, analogous to Winkler foundation for a plate. For each fixed wavenumber in the circumferential direction three lowest modes are investigated both on the basis of 3D theory of elasticity and on the basis of 2D approximate shell theories. For the shell with spring-type boundary condition the results of analytical and numerical investigations of backwards waves are presented. The limits of applicability of 2D theories are illustrated by comparison with the 3D solution. For the shell with free faces it is shown that there are no backwards waves in the range of applicability of long-wave asymptotic approximations.

1 Introduction

Dispersion of elastic waves in a cylindrical shell has been investigated in numerous publications. The three-dimensional theory of elasticity [1] and approximate shell theories [2] were used to govern the motion of the shell. A number of papers are concerned with analysis of the accuracy of the classical Kirchhoff–Love theory of shells and its refinements [3, 4]. But in such a rich system as a shell the peculiar phenomena may occur which require a special study. In this paper the accuracy of approximate shell theories in the case of an anomalous dispersion characterized by opposite signs of the phase and group velocities (backwards waves) is investigated. For a shell with free faces, it is shown that this phenomenon does not occur in the range of applicability of 2D theories. The existing of a backward wave in the framework of the classical Kirchhoff–Love theory is not confirmed by 3D theory of elasticity. But it suggest that this wave can exist in a shell with some other material properties or under other boundary conditions on the faces, if the changes lead to increasing of the flexural stiffness of the shell. As an example of such a situation we consider a shell with spring-type boundary condition on the outer surface.

2 Statement of the problem

Consider an infinite cylindrical shell occupying in cylindrical coordinate system (r, θ, z) the domain $\{R - h/2 < r < R + h/2, 0 \leqslant \theta < 2\pi, -\infty < z < \infty\}$, where R is the radius of the midsurface, h is the thickness of the shell. Let us assume that the action of environment can be modeled as a normal load proportional to the transverse displacement with the coefficient K_0 (this model can be considered as an analogue of a plate on the Winkler foundation [5]). The harmonic vibrations with the circular frequency ω_0 are considered, the factor $\exp(-\mathrm{i}\omega_0 t)$ is omitted everywhere. Let us introduce curvilinear coordinates (θ, z) on the midsurface $r = R$. The equations of motion according to the Kirchhoff–Love-type theory of shells [6] can be written in the form

$$\begin{aligned}
\left[\alpha_1\left(\partial_\theta{}^2 + \nu_-\widetilde{\partial}_z^2\right) + \omega^2\right] u_1 + \nu_+\widetilde{\partial}_z\partial_\theta u_2 + \partial_\theta\left(1 + 2\alpha^2[1 - \partial_\theta^2 - \widetilde{\partial}_z^2]\right) u_n = 0,\\
\nu_+\widetilde{\partial}_z\partial_\theta u_1 + \left(\nu_-\partial_\theta^2 + \widetilde{\partial}_z^2 + \omega^2\right) u_2 + \nu\widetilde{\partial}_z u_n = 0,\\
-\partial_\theta(1 + 2\alpha^2[1 - \partial_\theta^2 - \widetilde{\partial}_z^2]) u_1 - \nu\widetilde{\partial}_z u_2\\
+\left[\alpha^2(2\partial_\theta^2 - 1 + 2\nu\widetilde{\partial}_z^2 - [\partial_\theta^2 + \widetilde{\partial}_z^2]^2) - 1 + \omega^2 - 4\nu_+\nu_- h_*^{-1} K\right] u_n = 0,
\end{aligned} \tag{1}$$

where u_1, u_2, u_n are components of the midsurface displacement in the circumferential, longitudinal and radial directions, respectively, $\widetilde{\partial}_z = R\partial_z$, $\alpha^2 = \frac{1}{12}\left(\frac{h}{R}\right)^2$, $\alpha_1 = 1 + 4\alpha^2$, $\nu_\pm = (1 \pm \nu)/2$, $h_* = h/R$ is the relative thickness of the shell, $K = K_0 R/E$, $\omega = \omega_0 R/c_s$ is the dimensionless frequency, $c_s = \sqrt{E/((1-\nu^2)\rho)}$, E, ν, ρ are Young's modulus, Poisson's ratio and the density of the material.

In paper [7] the classical theory with modified inertia is proposed, which is constructed as higher order long-wave asymptotic approximation of 3D equations of elasticity. Let us present the equations of this theory in the form

$$\begin{aligned}
\left[\partial_\theta{}^2\left(1 + \alpha^2\right) + \nu_-\widetilde{\partial}_z^2\left(1 + 4\alpha^2\right)\right] u_1 + \nu_+\widetilde{\partial}_z\partial_\theta u_2\\
+\partial_\theta\left(1 - \alpha^2[\partial_\theta^2 + (2-\nu)\widetilde{\partial}_z^2]\right) u_n + (I_{tg}\mathrm{u}_{tg})_1 = 0,\\
\nu_+\widetilde{\partial}_z\partial_\theta u_1 + \left(\nu_-\partial_\theta^2 + \widetilde{\partial}_z^2\right) u_2 + \nu\widetilde{\partial}_z u_n + (I_{tg}\mathrm{u}_{tg})_2 = 0,\\
-\partial_\theta(1 - \alpha^2[\partial_\theta^2 + (2-\nu)\widetilde{\partial}_z^2]) u_1 - \nu\widetilde{\partial}_z u_2\\
+\left[-\alpha^2[\partial_\theta^2 + \widetilde{\partial}_z^2]^2 - 1 + I_{tr} - 4\nu_+\nu_- h_*^{-1} K\right] u_n = 0
\end{aligned} \tag{2}$$

with operators of modified inertia

$$I_{tg}\mathrm{u}_{tg} = \omega^2\left[\mathrm{u}_{tg} + \eta^2\omega^2\sum_{k=1}^{3} d_k\left(-\nu_-^{-1}\eta^2\omega^2\right)^{k-1}\mathrm{grad}_{tg}\left(\partial_\theta u_1 + \widetilde{\partial}_z u_2\right)\right],$$

$$I_{tr} = \omega^2\sum_{k=0}^{1}\left(-\nu_-^{-1}\eta^2\omega^2\right)^k\left[a_k + \eta^2 b_k\left(\partial_\theta^2 + \widetilde{\partial}_z^2\right)\right],$$

where $\mathrm{u}_{tg} = (u_1, u_2)^{\mathrm{T}}$, $\eta = h/(2R)$, $\mathrm{grad}_{tg} = (\partial_\theta, \widetilde{\partial}_z)^{\mathrm{T}}$, $a_0 = 1$, a_1, b_k, d_k are coefficients depending on ν, which can be found in [7]. The term $4\nu_+\nu_- h_*^{-1} K$ should also

be refined in order to retain the asymptotic error of equations (2), but it requires an additional asymptotic analysis of the corresponding 3D equations. In this paper we restrict ourselves with refinements in describing the motion of the shell, which are expressed through operators of modified inertia.
To analyze the accuracy of equations (1) and (2) we use three-dimensional theory of elasticity. Let us introduce dimensionless variables

$$\begin{gathered} r = R\tilde{r}, z = R\tilde{z}, \{u_r, u_\theta, u_z\} = R\{\tilde{u}_r, \tilde{u}_\theta, \tilde{u}_z\}, \tilde{\omega} = R\omega c_2^{-1} = \omega/\sqrt{\nu_-}, \\ \{\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{r\theta}, \sigma_{rz}, \sigma_{\theta z}\} = E\,[2(1+\nu)]^{-1}\{\tilde{\sigma}_{rr}, \tilde{\sigma}_{\theta\theta}, \tilde{\sigma}_{zz}, \tilde{\sigma}_{r\theta}, \tilde{\sigma}_{rz}, \tilde{\sigma}_{\theta z}\}, \end{gathered} \tag{3}$$

where $\mathrm{u} = (u_r, u_\theta, u_z)^{\mathrm{T}}$ is the displacement vector, $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{r\theta}, \sigma_{rz}, \sigma_{\theta z}$ are components of the stress tensor, $c_2 = \sqrt{E/(2(1+\nu)\rho)}$ is transverse wave speed. The displacement vector can be presented in the terms of wave potentials φ, $\boldsymbol{\psi}$ as

$$\widetilde{\mathrm{u}} = \operatorname{grad}\varphi + \operatorname{rot}\boldsymbol{\psi} \tag{4}$$

with additional condition $\operatorname{div}\boldsymbol{\psi} = 0$. After introducing the dimensionless variables (3) the equations for the potentials take the form

$$\Delta\varphi + \kappa^2\tilde{\omega}^2\varphi = 0, \quad \Delta\boldsymbol{\psi} + \tilde{\omega}^2\boldsymbol{\psi} = 0 \tag{5}$$

with $\kappa = \sqrt{(1-2\nu)(2\,(1-\nu))^{-1}}$. The expressions of the stresses in terms of φ, $\boldsymbol{\psi}$ can be found in [8]. Boundary conditions on the faces of the shell $\tilde{r} = 1 \pm \eta$ are

$$\tilde{\sigma}_{rr}|_{\tilde{r}=1+\eta} = -K\,\tilde{u}_r|_{\tilde{r}=1+\eta}, \quad \tilde{\sigma}_{rr}|_{\tilde{r}=1-\eta} = 0, \quad \tilde{\sigma}_{r\theta}|_{\tilde{r}=1\pm\eta} = \tilde{\sigma}_{rz}|_{\tilde{r}=1\pm\eta} = 0. \tag{6}$$

Further we will investigate the normal modes that propagate along the z-axis. The dependence on z is assumed to be in the form $\exp(\mathrm{i}\lambda z/R)$, where λ is the dimensionless wavenumber. The dispersion equations corresponding to approximate theories (1) and (2) can be derived by the method described in [9]. In the case of 3D problem (4)–(6) we use approach proposed in [8]. The variable θ is separated by assuming the law $\sin(m\theta)$ for functions u_1, u_θ and $\cos(m\theta)$ for u_2, u_n, u_z, u_r $(m = 0, 1, 2, \ldots)$. In the case of 3D theory the infinite series of modes exists for each m, but only the lowest three of them can be described on the basis of equations (1) and (2). The real branches of dispersion curves $\omega(\lambda)$ start from cut-off frequencies ω_k $(k = 1, 2, \ldots)$. We are interested in the backwards waves arising in the vicinity of a cut-off frequency, when it coincides with some other one. For the sake of convenience we will refer this point as coincidence frequency.

3 Shell with free surfaces

In the case of free faces $(K = 0)$ the asymptotic behavior of cut-off frequencies in the framework of the Kirchhoff–Love theory can be presented as [10]

$$\omega_1 = \frac{\eta m(m^2-1)}{\sqrt{3(m^2+1)}}\left(1 + O\left(\eta^2\right)\right), \; \omega_2 = m\sqrt{\nu_-}, \; \omega_3 = \sqrt{m^2+1}\left(1 + O\left(\eta^2\right)\right). \tag{7}$$

It can be easily verified that the frequencies ω_2 and ω_3 cannot coincide. The coincidence of frequencies ω_1 and ω_2 requires $m \sim \eta^{-1}$ or $q = 1$, where q is the variability

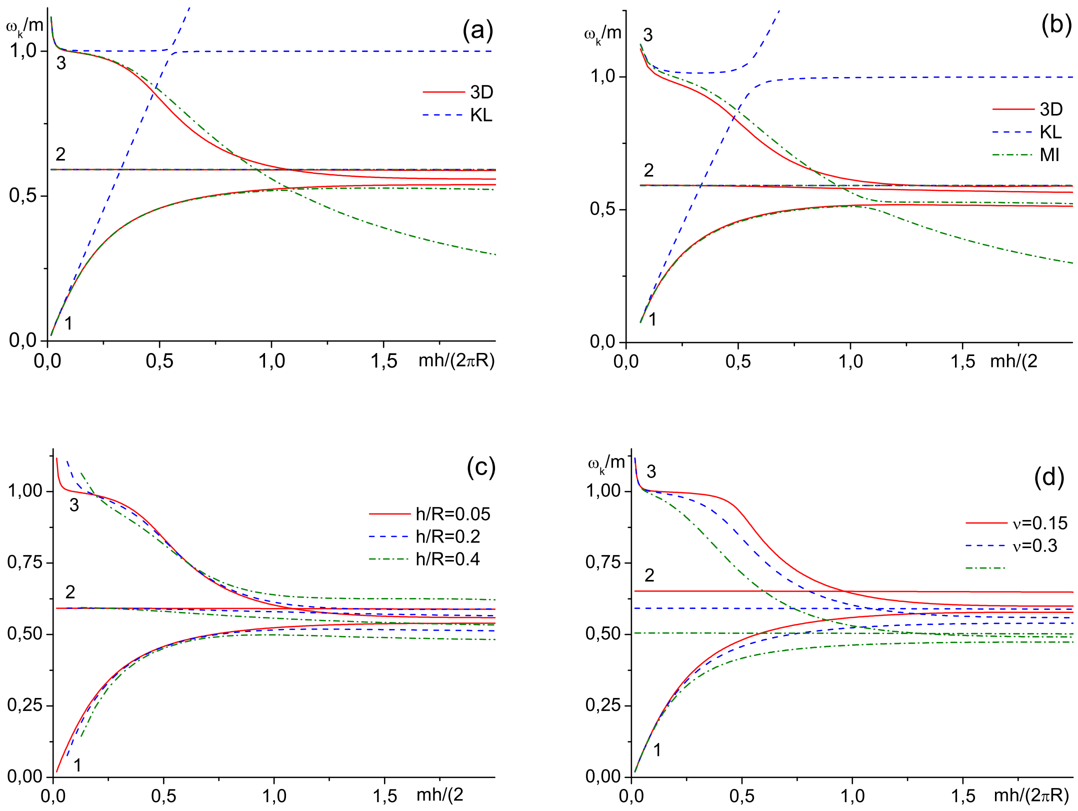

Figure 1: Numerical results for cut-off frequencies: $\nu = 0.3, h/R = 0.05$ (a), $\nu = 0.3, h/R = 0.2$ (b), $\nu = 0.3$ (c), $h/R = 0.05$ (d)

index in terms of [7]. The range of applicability of Kirchhoff–Love theory was defined in [7] as $q < 1$, thus the coincidence frequency does not belong to it. But this reason does not exclude the existence of the coincidence frequency and, consequently, the backwards waves by large values of m or in a thick shell. To investigate this matter a series of numerical experiments was performed, in which the cut-off frequency were calculated on the basis of each of the three theories formulated in Sec. 2. The most interesting results are shown in Fig. 1. As one can see, the coincidence of ω_1 and ω_2 predicted by Kirchhoff–Love theory is not confirmed by 3D theory of elasticity. The theory with modified inertia has much greater range of applicability and predicts the behavior of cut-off frequencies ω_1 and ω_2 with a good accuracy (see Fig. 1,a,b). The investigation on the basis of 3D theory for different values of h/R and ν has shown that apparently the coincidence of ω_1 and ω_2 is not possible (see Fig. 1,c,d). But for some not very great values of h/R there is a coincidence frequency $\omega_2 = \omega_3$. For example, in Fig. 1,c we can see it at $h/R = 0.05$ (red lines) but cannot see it at $h/R = 0.2, 0.4$ (blue and green lines). Since parameter m for this point corresponds to $q \geqslant 1$, it cannot be described by 2D theories of shells. Calculations show that the backward wave arises in the vicinity of ω_2, but the domain of its existence on the axis ω is very narrow ($\lesssim 10^{-5}$ for the dimensionless frequency introduced as in (1)). The investigation of that backward wave in more detail does not match the goal of this paper.

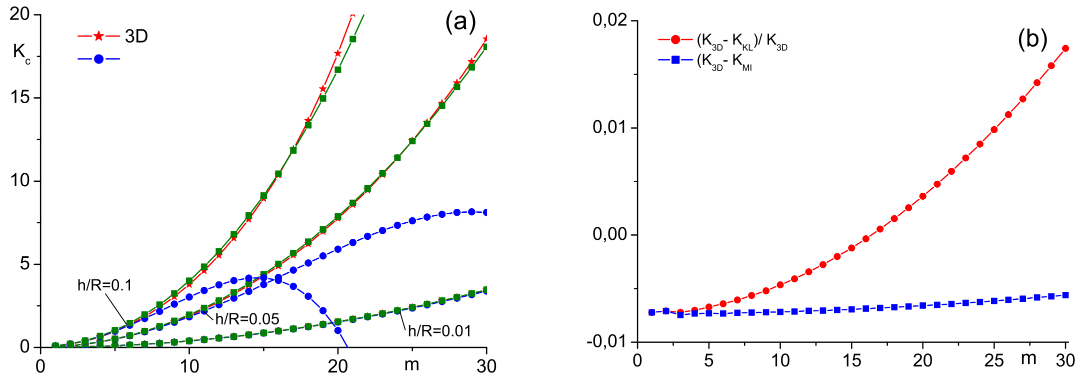

Figure 2: Comparison of three theories for coincidence parameter K_c (a); relative errors of 2D theories for $h/R = 0.01$ (b)

The numerical results presented in Fig. 1 show that the curves for ω_1 and ω_2 have a tendency of drawing closer to one another. In the next section we consider a shell model in which these curves intersect.

4 Shell with spring-type boundary conditions on the outer surface

In the case $K \neq 0$ there is one additional parameter of the problem. An analysis of the dispersion equations corresponding to the problems formulated in Sec. 2 shows that at $\lambda = 0$ all three of them are separated into two independent equations. One of the pair is an equation for the cut-off frequency ω_2, which is independent on K and is equal to that in the case of free surface. Substituting ω_2 in the second equation we obtain an explicit expression for the value of $K = K_c$, at which ω_1 coincides with ω_2. For the Kirchhoff–Love theory (1) the coincidence parameter $K_c = K_{KL}$ is

$$K_{KL} = \frac{4\nu_+\nu_- h_*}{2\nu_+ + 8\alpha^2}\left[2(\nu_- - \nu_+\alpha^2) + \left(2\nu_+\nu_- + 2\alpha^2(5-3\nu)\right)m^2 - 2\alpha^2\nu_+ m^4\right],$$

for the other theories the expressions are too cumbersome to be presented here. Further we denote them as K_{MI} for the theory with modified inertia and K_{3D} for the 3D theory of elasticity. The asymptotic behavior of K_{KL} at $m \sim 1$

$$K_{KL} = 4\nu_+\nu_- h_* \left[\frac{1-\nu}{1+\nu} + \frac{1-\nu}{2}m^2 + O\left(\alpha^2\right)\right]$$

shows that at the small values of m the coincidence parameter nearly proportional to relative thickness.

In Fig. 2,a the dependence $K_c(m)$, calculated on the basis of each of the three theories, is shown for the different values of h/R. Here the Kirchhoff–Love-type theory can give qualitatively correct results, but only for small m. As in the case of free faces, the range of applicability of the theory with modified inertia is much greater (see also the relative errors on Fig. 2,b).

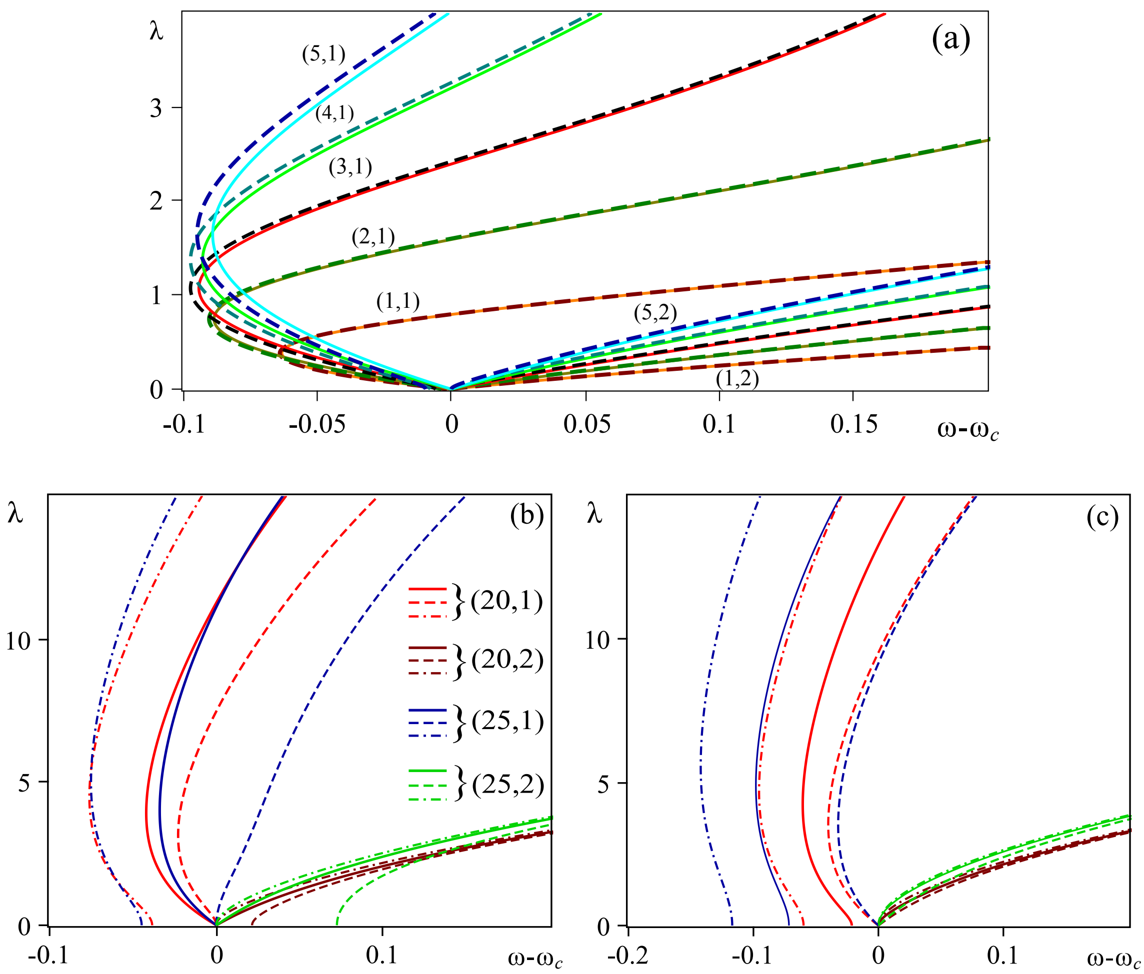

Figure 3: Dispersion curves for $\nu = 0.3$, $h/R = 0.01$, $K = K_{3D}$: $m = \overline{1,5}$ (a), $m = 20, 25$ (b), and the same as (b) for $K = K_{KL}$ (c). Solid lines: 3D theory, dashed lines: KL theory, dash-dotted lines: theory with MI

In Fig. 3 the comparison of dispersion curves in the vicinity of the coincidence frequency $\omega_c = \omega_1 = \omega_2$ is presented. Here the numbers of modes are indicated by pairs (m, k) with $k = 1, 2$. Because of the difference between parameters K_c calculated on the basis of 3D theory of elasticity and 2D shell theories there is no coincidence of cut-off frequencies for the latter. The error of Kirchhoff–Love theory at $m = 25$ leads to a qualitatively incorrect result: according to it, the backwards wave does not exist at $K = K_{3D}$, although the calculations on the basis of 3D theory of elasticity shows its existence (see Fig. 3,b, blue lines). On the other hand, at $K = K_{KL}$ there is the backwards wave according to Kirchhoff–Love theory (see Fig. 3,c). Notice, that at $n = 25$ the relative error of K_{KL} is less than 2% (see Fig. 2,b). But it reveals itself to be essential in the problem under consideration. The theory with modified inertia gives the qualitative correct results in both cases.

In Fig. 4 the behavior of eigenfunctions calculated on the basis of 3D and Kirchhoff–Love theories is investigated near the coincidence frequency ω_2 at $m = 5$, $h/R = 0.01$ and the values of K indicated on the graphs. Since the dependence of displacements along the thickness coordinate is very close to linear one at these values of parame-

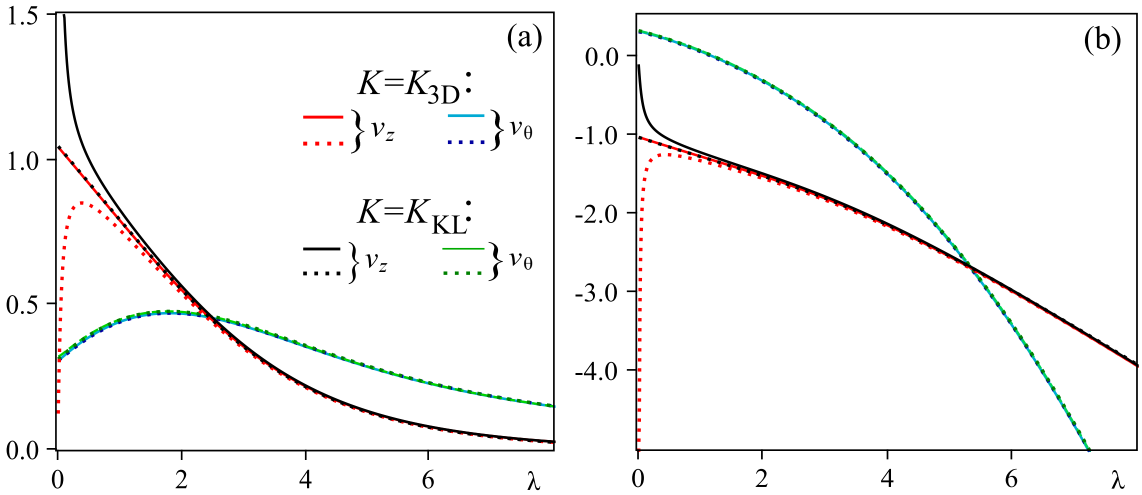

Figure 4: Evolution of normalized tangential displacements for modes $(5, 1)$ (a) and (5.2) (b) for $\nu = 0.3$, $h/R = 0.01$ and indicated values of K. Solid lines: 3D theory, dotted lines: KL theory

ters, it can be characterized by quantities v_θ, v_z representing normalized amplitudes of displacements $\mathrm{Re}(u_\theta/\ u_r|_{r=R+h/2})$, $\mathrm{Im}(u_z/\ u_r|_{r=R+h/2})$ averaged in respect to the coordinate $r \in [R - h/2, R + h/2]$. When using the Kirchhoff–Love theory, we put $v_\theta = \mathrm{Re}(u_1/u_n)$, $v_z = \mathrm{Im}(u_2/u_n)$. In the case of 3D theory, for both modes $(5, 1)$ and $(5, 2)$ the displacement u_r is distributed nearly uniformly along the thickness coordinate in the whole domain shown in Fig. 4, so the normalized and averaged displacement u_r is nearly equal to unity for any λ. Taking this into account, we can see from Fig. 4 that mode $(5, 1)$ becomes a flexural one ($|u_\theta|\,, |u_z| \ll |u_r|$) when the wavenumber grows, while mode $(5, 2)$ tends to torsional one ($|u_z|\,, |u_r| \ll |u_\theta|$). In the domain of the backwards wave (cf. Fig. 3,a) both modes have mixed character. The Kirchhoff–Love theory describes the behavior of displacements with good accuracy except the displacement v_z in the vicinity of $\lambda = 0$. This effect is apparently caused by the coupling between the tangential and flexural motions of the shell, which is absent at $\lambda = 0$ if there is no coincidence of the cut-off frequencies. The accuracy of Kirchhoff–Love theory seems to be improved after setting $K = K_{KL}$, but is still has a great error comparing with the solution of 3D problem calculated at the same value of K.

5 Conclusion

The investigation presented above shows that the using of 2D shell theories requires a careful analysis of their range of applicability, when some special effects are studied. The classical theory with modified inertia proposed in [7] can be effectively used in such an analysis. It was also noticed that the errors of approximate shell theories increase in the domains characterized by presence of backwards waves. For an isotropic shell with free faces it was shown that there are no backwards waves which could be described by long-wave approximations of the 3D equations of elasticity. But such an effect is still possible in some anisotropic or laminated shell.

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