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IFAC-PapersOnLine 48-11 (2015) 685–690

Sampled-Data Control of Nonlinear Systems Based on Fridman's Analysis and Passification Design ⋆

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Abstract: Stability conditions for sampled-data nonlinear control system with linear output feedback are studied. The case of sector bounded nonlinearities and uncertain sampling with the known upper bound on the sampling intervals is considered. Stability analysis is performed by input delay approach based on time-dependent Lyapunov-Krasovskii functionals refined by Emilia Fridman in 2010 (Fridman's method) and extended by Seifullaev and Fradkov in 2013 to nonlinear control systems with sector bounded nonlinearity (Lurie systems). Based on classical results of V.A. Yakubovich about S-procedure the problem is reduced to feasibility analysis of linear matrix inequalities. Linear output feedback controller design is based on the passification method. The results are illustrated by example showing that both bounds on the sampling interval and accuracy of those bounds depend essentially on the controller choice.

Keywords: nonlinear control systems, passivity, passification, sampled-data control, LMI

1. INTRODUCTION

An important problem for design of computer controlled systems is the proper choice of the sampling interval providing stability and the desired performance of the control system. This problem is by no means trivial even for linear systems if one needs to evaluate nonconservative bounds for maximum admitted value of sampling interval. As for nonlinear systems the problem is not well studied despite its importance.

Recently in the literature an interest has grown up in a novel approach to the sampling time evaluation based on transformation of discrete-continuous system models to continuous delayed system with time-varying delay. The origin of the idea can be traced back to Mikheev et al. (1988); Fridman (1992). However being combined with the descriptor method of delayed systems analysis proposed by Fridman (2001) the idea has become equipped with a powerful calculation tools based on LMI and has become a powerful design method allowing one to dramatically reduce conservativity of the design Fridman et al. (2004); Fridman (2010). However, the input delay method was developed only for linear systems until recently. The only exception is the paper Teel et al. (1998) where the existence of a nonzero sampling interval for ISS nonlinear systems is shown. In Seifullaev and Fradkov (2013) the Fridman's method was extended to nonlinear systems with sector-bounded nonlinearities.

Existing results on sampled-data control by Fridman's (input-delay) method are dealing only with the analysis of the hybrid system stability. However, the estimate for sampling interval apparently depends on the controller. Therefore, quality of estimate depends on quality of controller design. In this paper we make an attempt to combine analysis and design for sampled-data control based on input-delay method. The passification based linear controller design is used. The estimates for sampling interval optimized over possible choice of linear output feedback are provided.

2. PRELIMINARIES

Below we introduce some results (presented in Fradkov (2008)) related to passification of linear systems with respect to given output. Consider the system

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t) + \mathcal{B}u(t), \\ y(t) &= \mathcal{C}x(t), \quad y_1(t) = \mathcal{C}_1 x(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}$ and $y_1(t) \in \mathbb{R}$ are the output variables, $u(t) \in \mathbb{R}$ is the control input, $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B} \in \mathbb{R}^{n\times 1}$, $\mathcal{C} \in \mathbb{R}^{1\times n}$, $\mathcal{C}_1 \in \mathbb{R}^{1\times n}$ are constant matrices.

Definition 1. System (1) is called *strictly passive*, if there exist a nonnegative function $V(x)$ (storage function) and a scalar function $\mu(x)$, where $\mu(x) > 0$ for $x \neq 0$, such that

$$V(x) \leqslant V(x_0) + \int_0^t [u(t)y(t) - \mu(x(t))]\, dt \tag{2}$$

⋆ The work is supported by Saint Petersburg State University, (grant 6.38.230.2015). The procedure and conditions for controller design were obtained in IPME RAS under support of Russian Scientic Foundation (grant 14-29-00142)

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2015.09.267

for any solution of system (1) satisfying $x(0) = x_0, x(t) = x$.

Denote the characteristic polynomial and transfer function (from u to y) of (1) as $\tilde{\alpha}(s)$ and $\mathcal{W}(s)$, respectively, i.e.

$$\tilde{\alpha}(s) = \det(sI - \mathcal{A}) = s^n + \tilde{\alpha}_{n-1}s^{n-1} + \dots,$$
$$\mathcal{W}(s) = \mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B}.$$

Definition 2. System (1) is called *minimum phase* if the polynomial $\tilde{\beta}(s) = \tilde{\alpha}(s)\mathcal{W}(s)$ is Hurwitz stable (all its zeros belong to the open left half-plane). It is called *hyper-minimum-phase* if it is minimum phase and $\mathcal{CB} > 0$.

It is required to find the conditions of the existence of scalar gain $\mathcal{K}$ such that system (1) with the output feedback

$$u = -\mathcal{K}y_1 + v, \tag{3}$$

where $v \in \mathbb{R}$ is a new input, is strictly passive with respect to the output y.

Introduce the following notation

$$\mathcal{A}_\mathcal{K} = \mathcal{A} - \mathcal{B}\mathcal{K}\mathcal{C}_1.$$

Consider quadratic storage function $V(x) = \frac{1}{2}x^\mathrm{T}\mathcal{H}x$, where $\mathcal{H} = \mathcal{H}^\mathrm{T} > 0$ is some symmetric positive matrix, and function $\mu(x) = \mu\,|x|^2, \mu > 0$. Then the integral inequality (2) can be rewritten as follows:

$$x^\mathrm{T}\left(\mathcal{H}\mathcal{A}_\mathcal{K} + \mathcal{A}_\mathcal{K}{}^\mathrm{T}\mathcal{H}\right)x + x^\mathrm{T}\mathcal{H}\mathcal{B}v \leqslant vy - \mu\,|x|^2 \tag{4}$$

for some $\mu > 0$ and all $x \in \mathbb{R}^n, v \in \mathbb{R}$.

In its turn, dissipation inequality (4) is equivalent to the following matrix relations:

$$\begin{cases} \mathcal{H}\mathcal{B} = \mathcal{C}^\mathrm{T}, \\ \mathcal{H}\mathcal{A}_\mathcal{K} + \mathcal{A}_\mathcal{K}{}^\mathrm{T}\mathcal{H} < 0. \end{cases} \tag{5}$$

Denote the transfer function (from u to y_1) of (1) as $\mathcal{W}_1(s) = \mathcal{C}_1(sI - \mathcal{A})^{-1}\mathcal{B}$. Define $\tilde{\psi}(s) = \dfrac{\tilde{\beta}_1(s)}{\tilde{\beta}(s)}$, where $\tilde{\beta}_1(s) = \tilde{\alpha}(s)\mathcal{W}_1(s)$. The function $\tilde{\psi}(s)$ can be interpreted as relative gain $\tilde{\psi}(s) = \dfrac{\mathcal{W}_1(s)}{\mathcal{W}(s)}$.

Definition 3. We say that system (1), (3) possesses *F-property* if either $\mathrm{Re}\,\tilde{\psi}(is) \geqslant 0 \quad \forall s \in \mathbb{R}$ and $\mathrm{Re}\,\dfrac{\tilde{\alpha}(is)}{\tilde{\beta}(is)} > 0$ for $\mathrm{Re}\,\tilde{\psi}(is) = 0$, or $\mathrm{Re}\,\tilde{\psi}(is) \leqslant 0 \quad \forall s \in \mathbb{R}$ and $\mathrm{Re}\,\dfrac{\tilde{\alpha}(is)}{\tilde{\beta}(is)} < 0$ for $\mathrm{Re}\,\tilde{\psi}(is) = 0$.

The solvability conditions for (5) (the existence of appropriate $\mathcal{H}$ and $\mathcal{K}$) were given in Fradkov (2008) based on the "semi-singular" version of the Yakubovich–Kalman–Popov or Kalman–Yakubovich–Popov (KYP) lemma established by V.A. Yakubovich (1966). The main result of Fradkov (2008) on hight gain passifiability can be formulated in the following theorem (Theorem 4 in Fradkov (2008)):

Theorem 1. The system (1) is strictly passifiable by output feedback (3) with all sufficiently large $\mathcal{K}$ if and only if the transfer function $\mathcal{W}(s)$ is hyper-minimum-phase (polynomial $\tilde{\beta}(s)$ is Hurwitz stable of degree $n-1$) and either (A) $\deg(\tilde{\beta}_1(s)) = n-1$ and system possesses F-property, or (B) $\deg(\tilde{\beta}_1(s)) = n-2$, system possesses F-property and $\tilde{\alpha}_{n-1} > 0$.

3. PROBLEM STATEMENT

Consider the nonlinear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + q\xi(t) + Bu(t), \\ \sigma(t) &= r^\mathrm{T}x(t), \quad \xi(t) = \varphi(\sigma(t), t), \end{aligned} \tag{6}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $\sigma(t) \in \mathbb{R}$ is the output variable, $u(t) \in \mathbb{R}$ is the control input, $A \in \mathbb{R}^{n\times n}$ is the constant matrix, $B \in \mathbb{R}^n, q \in \mathbb{R}^n, r \in \mathbb{R}^n$ are the constant vectors.

Assume that $\xi(t) = \varphi(\sigma(t), t)$ is the nonlinear function (see Fig.1) satisfying

$$\mu_1\,\sigma^2 \leqslant \sigma\,\xi \leqslant \mu_2\,\sigma^2 \tag{7}$$

for all $t \geqslant 0$ where $\mu_1 < \mu_2 \leqslant 0$ are real numbers.

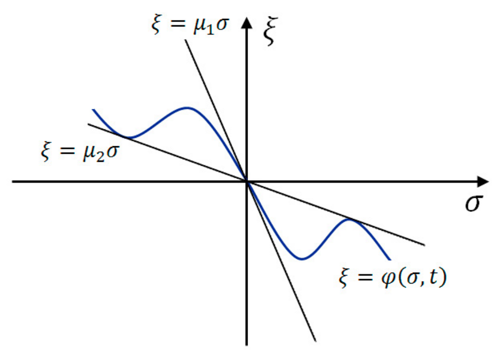

Fig. 1. Sector bounded nonlinearity

Problem 1. Consider the output feedback control function

$$u(t) = -K\sigma(t), \tag{8}$$

where $K \in \mathbb{R}$. It is required to find the conditions of the existence of gain K such that closed-loop system (6), (8) is exponentially stable.

Given a sequence of sampling times $0 = t_0 < t_1 < \ldots < t_k < \ \ldots$, where $\lim_{k\to\infty} t_k = \infty$. Assume that $h \in \mathbb{R}\,(h > 0)$ and

$$t_{k+1} - t_k \leqslant h, \quad \forall k \geqslant 0 \tag{9}$$

and consider a sampled-time control law

$$u(t) = -K\sigma(t_k), \ \ t_k \leqslant t < t_{k+1}. \tag{10}$$

The law (10) can be rewritten as follows:

$$u(t) = -K\sigma(t - \tau(t)), \tag{11}$$

where $\tau(t) = t - t_k$, $t_k \leqslant t < t_{k+1}$.

Problem 2. It's required to analyze the influence of the upper bound h of sampling intervals on the closed-loop system exponential stability:

$$\begin{aligned} &\dot{x}(t) = Ax(t) - BKr^\mathrm{T}x(t - \tau(t)) + q\xi(t), \\ &\sigma(t) = r^\mathrm{T}x(t), \quad \xi(t) = \varphi(\sigma(t), t), \\ &\tau(t) = t - t_k, \ \ t \in [t_k, t_{k+1}). \end{aligned} \tag{12}$$

4. MAIN RESULT

4.1 Passifiability of nonlinear system with respect to given input

Add an additional input $\omega(t) \in \mathbb{R}$ to system (6):

$$\begin{aligned} \dot{x}(t) &= Ax(t) + q\xi(t) + Bu(t) + q\omega(t), \\ \sigma(t) &= r^{\mathrm{T}}x(t), \quad \xi(t) = \varphi(\sigma(t), t). \end{aligned} \tag{13}$$

Find the conditions of the existence of gain K such that closed-loop system (13), (8) is strictly passive from input ω to output σ.

Introduce the following notations:

$$\alpha(s) = \det(sI - A) = s^n + \alpha_{n-1}s^{n-1} + \dots,$$
$$W(s) = r^{\mathrm{T}}(sI - A)^{-1}B, \quad W_q(s) = r^{\mathrm{T}}(sI - A)^{-1}q,$$
$$\beta(s) = \alpha(s)W(s), \quad \beta_q(s) = \alpha(s)W_q(s), \quad \psi(s) = \frac{\beta(s)}{\beta_q(s)}.$$

Theorem 2. Let the following conditions be fulfilled:

(i) The polynomial $\beta_q(s)$ is Hurwitz stable (i.e. all its zeros belong to the open left half-plane), and $r^{\mathrm{T}}q > 0$.

(ii) Either $\mathrm{Re}\,\psi(is) \geqslant 0 \quad \forall s \in \mathbb{R}$ and $\mathrm{Re}\,\dfrac{\alpha(is)}{\beta_q(is)} > 0$ for $\mathrm{Re}\,\psi(is) = 0$, or $\mathrm{Re}\,\psi(is) \leqslant 0 \quad \forall s \in \mathbb{R}$ and $\mathrm{Re}\,\dfrac{\alpha(is)}{\beta_q(is)} < 0$ for $\mathrm{Re}\,\psi(is) = 0$.

(iii) Either $\deg\beta(s) = n - 1$, or $\deg\beta(s) = n - 2$ and $\alpha_{n-1} > 0$.

Then system (13) is strictly passifiable by output feedback (8) with all sufficiently large K.

Proof. Consider the function $V(x(t)) = \frac{1}{2}x^{\mathrm{T}}(t)Hx(t)$, where $H \in \mathbb{R}^{n\times n}$ is some symmetric positive matrix. Then

$$\dot{V}(x(t), t) = \frac{1}{2}x(t)^{\mathrm{T}}H\dot{x}(t) + \frac{1}{2}\dot{x}(t)^{\mathrm{T}}Hx(t)$$
$$= x^{\mathrm{T}}(t)(HA_K + A_K^{\mathrm{T}}H)x(t) + x^{\mathrm{T}}(t)Hq\xi(t) + x^{\mathrm{T}}(t)Hq\omega(t),$$

where $A_K = A - BKC$.

Taking into account (7), we obtain that the strict passivity of system (13) from ω to σ (i.e. $\dot{V} < \omega\,\sigma$) is equivalent to the following matrix relations:

$$\begin{cases} Hq = r, \\ HA_K + A_K^{\mathrm{T}}H < 0. \end{cases} \tag{14}$$

Introduce new notations:

$$\mathcal{H} = H^{-1},\ \mathcal{A} = A^{\mathrm{T}},\ \mathcal{K} = K,\ \mathcal{B} = r,\ \mathcal{C} = q^{\mathrm{T}},\ \mathcal{C}_1 = B^{\mathrm{T}}.$$

Then (14) can be rewritten as follows:

$$\begin{cases} \mathcal{H}\mathcal{B} = \mathcal{C}^{\mathrm{T}}, \\ \mathcal{H}\mathcal{A}_{\mathcal{K}} + \mathcal{A}_{\mathcal{K}}{}^{\mathrm{T}}\mathcal{H} < 0, \\ \mathcal{A}_{\mathcal{K}} = \mathcal{A} - \mathcal{B}\mathcal{K}\mathcal{C}_1. \end{cases} \tag{15}$$

Comparing (15) and (5), we obtain that the fulfillment of (14) is equivalent to strict passivity of system (1) from input v to output y.

Since

$$\tilde{\alpha}(s) = \det(sI - \mathcal{A}) = \det(sI - A^{\mathrm{T}}) = \det(sI - A) = \alpha(s),$$

$$\mathcal{W}(s) = \mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B}$$
$$= q^{\mathrm{T}}(sI - A^{\mathrm{T}})^{-1}r = r^{\mathrm{T}}(sI - A)^{-1}q = W_q(s),$$

$$\mathcal{W}_1(s) = \mathcal{C}_1(sI - \mathcal{A})^{-1}\mathcal{B}$$
$$= B^{\mathrm{T}}(sI - A^{\mathrm{T}})^{-1}r = r^{\mathrm{T}}(sI - A)^{-1}B = W(s),$$

it is easy to see that condition (i) is equivalent to the fact that system (1) is hyper-minimum-phase, and condition (ii) suggests that system (1) possesses F-property.

Taking into account (iii),we obtain from Theorem 1 that for all sufficiently large $\mathcal{K}$ system (1) is strictly passive from v to y.

Corollary 1. (Solution to Problem 1). If conditions (i) – (iii) of Theorem 2 are fulfilled, then system (6), (8) is exponentially stable with all sufficiently large K.

Proof. Since the storage function $V(x)$ is positive and radially unbounded, it follows from Lyapunov theorem that (13) is exponentially stable for $\omega \equiv 0$.

4.2 Stability analysis of sampled-data nonlinear system based on Fridman's method and S-procedure

The key tools to obtain the results are application of Yakubovich's S-procedure Yakubovich et al. (2004) and Fridman's method for linear system Fridman (2010), based on the interpretation of a networked control system as a continuous-time delayed system with time-varying (sawtooth) delay ("Input-delay method"), general time-dependent Lyapunov-Krasovskii functionals, descriptor method, Jensen's inequality, and Schur complements. The following theorem is based on the results proposed in Seifullaev and Fradkov (2013).

Assume that P, Q are symmetric positive definite $n \times n$ matrices, P_2, P_3, X, X_1, Y_1, Y_2, Y_3, T are some $n \times n$ matrices, $\varkappa_0$, $\varkappa_1$ are positive real scalars.

Consider the following matrices :

$$\Theta = \begin{bmatrix} P + h\dfrac{X + X^T}{2} & hX_1 - hX \\ * & -hX_1 - hX_1^T + h\dfrac{X + X^T}{2} \end{bmatrix},$$

$$\Psi_0 = \begin{bmatrix} \Phi_{11}^- & \Phi_{12}^- & \Phi_{13}^- & \Phi_{14}^- \\ * & \Phi_{22}^- & \Phi_{23}^- & \Phi_{24} \\ * & * & \Phi_{33}^- & \Phi_{34} \\ * & * & * & \Phi_{44}^- \end{bmatrix},$$

$$\Psi_1 = \begin{bmatrix} \Phi_{11}^+ & \Phi_{12}^+ & \Phi_{13}^+ & \Phi_{14}^+ & hY_1^T \\ * & \Phi_{22}^+ & \Phi_{23}^+ & \Phi_{24} & hY_2^T \\ * & * & \Phi_{33}^+ & \Phi_{34} & hT^T \\ * & * & * & \Phi_{44}^+ & hq^TY_3^T \\ * & * & * & * & -hQe^{-2\alpha h} \end{bmatrix},$$

where

$$
\begin{aligned}
\Phi_{11}^{-} &= A^{\mathrm{T}}P_2 + P_2^{\mathrm{T}}A + 2\alpha P - Y_1 - Y_1^{\mathrm{T}} - \\
&\quad - (1-2\alpha h)\,\frac{X+X^{\mathrm{T}}}{2} - \varkappa_0\mu_1\mu_2 r r^{\mathrm{T}},\\
\Phi_{11}^{+} &= A^{\mathrm{T}}P_2 + P_2^{\mathrm{T}}A + 2\alpha P - Y_1 - Y_1^{T} - \\
&\quad - \frac{X+X^{\mathrm{T}}}{2} - \varkappa_1\mu_1\mu_2 r r^{\mathrm{T}},\\
\Phi_{12}^{-} &= P - P_2^{\mathrm{T}} + A^{\mathrm{T}}P_3 - Y_2 + h\,\frac{X+X^{\mathrm{T}}}{2},\\
\Phi_{12}^{+} &= P - P_2^{\mathrm{T}} + A^{\mathrm{T}}P_3 - Y_2,\\
\Phi_{13}^{-} &= Y_1^{\mathrm{T}} - P_2^{\mathrm{T}}BKr^{\mathrm{T}} - T + (1-2\alpha h)(X - X_1),\\
\Phi_{13}^{+} &= Y_1^{\mathrm{T}} - P_2^{\mathrm{T}}BKr^{\mathrm{T}} - T + (X - X_1),\\
\Phi_{14}^{-} &= P_2^{\mathrm{T}} + \frac{1}{2}\varkappa_0(\mu_1+\mu_2)r, \qquad (16)\\
\Phi_{14}^{+} &= P_2^{\mathrm{T}} + \frac{1}{2}\varkappa_1(\mu_1+\mu_2)r,\\
\Phi_{22}^{-} &= -P_3 - P_3^{\mathrm{T}} + hQ, \quad \Phi_{22}^{+} = -P_3 - P_3^{\mathrm{T}},\\
\Phi_{23}^{-} &= Y_2^{\mathrm{T}} - P_3^{\mathrm{T}}BKr^{\mathrm{T}} - h(X - X_1),\\
\Phi_{23}^{+} &= Y_2^{\mathrm{T}} - P_3^{\mathrm{T}}BKr^{\mathrm{T}},\\
\Phi_{24} &= P_3^{\mathrm{T}}q, \quad \Phi_{34} = Y_3\,q,\\
\Phi_{33}^{-} &= T + T^{\mathrm{T}} - (1-2\alpha h)\,\frac{X+X^{\mathrm{T}}-2X_1-2X_1^{\mathrm{T}}}{2},\\
\Phi_{33}^{+} &= T + T^{\mathrm{T}} - \frac{X+X^{\mathrm{T}}-2X_1-2X_1^{\mathrm{T}}}{2},\\
\Phi_{44}^{-} &= -\varkappa_0, \quad \Phi_{44}^{+} = -\varkappa_1.
\end{aligned}
$$

Theorem 3. Given $\alpha > 0$, let there exist matrices $P \in \mathbb{R}^{n\times n}$ $(P > 0)$, $Q \in \mathbb{R}^{n\times n}$ $(Q > 0)$, $P_2 \in \mathbb{R}^{n\times n}$, $P_3 \in \mathbb{R}^{n\times n}$, $X \in \mathbb{R}^{n\times n}$, $X_1 \in \mathbb{R}^{n\times n}$, $T \in \mathbb{R}^{n\times n}$, $Y_1 \in \mathbb{R}^{n\times n}$, $Y_2 \in \mathbb{R}^{n\times n}$, Y_3 and positive real scalars $\varkappa_0$, $\varkappa_1$, such that the following LMIs

$$\Theta > 0, \quad \Psi_0 < 0, \quad \Psi_1 < 0$$

are feasible. Then system (12) is exponentially stable with decay rate α.

Corollary 2. In Seifullaev and Fradkov (2013) and Seifullaev and Fradkov (2015) it was shown that LMIs from Theorem 3 are convex on h: if they are feasible for h then they are feasible for any $\bar{h} < h$. Therefore, the necessary condition on K for the exponential stability with sampled-data control is the exponential stability of the system with continuous control $u = -K\sigma(t)$. Other words, the feasibility of Problem 1 is necessary condition for feasibility of Problem 2.

Corollary 3. Therefore, the following algorithm can be proposed:

(1) Check the conditions (i)–(iii) of Theorem 2 guaranteeing the existence of control K stabilizing continuous system.
(2) Choose a such gain K.
(3) Increasing the sampling interval bound h check the LMIs from Theorem 3 and find the maximum value of h, when the system is exponentially stable.

5. EXAMPLE

Let us illustrate the obtained results by an example.

Consider the following system:

$$
\begin{aligned}
\dot{x}_1(t) &= -2x_1(t) + \sin x_2(t),\\
\dot{x}_2(t) &= x_1(t) - x_2(t) + 2\sin x_2(t) + u(t)\\
u(t) &= -Kx_2(t-\tau(t)),\\
\tau(t) &= t - t_k, \;\; t \in [t_k, t_{k+1}), \;\; t_{k+1} - t_k \leqslant h.
\end{aligned} \qquad (17)
$$

System (17) can be rewritten as follows:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) - BKr^{\mathrm{T}}x(t-\tau(t)) + q\xi(t),\\
\sigma(t) &= r^{\mathrm{T}}x(t), \quad \xi(t) = \varphi(\sigma(t), t),\\
\tau(t) &= t - t_k, \;\; t \in [t_k, t_{k+1}), \;\; t_{k+1} - t_k \leqslant h,
\end{aligned} \qquad (18)
$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$q = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad r = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \xi(t) = \sin\sigma(t) - \sigma(t).$$

Note that the nonlinearity $\xi(t)$ satisfies the following sector inequalities (see Fig. 2):

$$\mu_1\,\sigma^2 \leqslant \sigma\,\xi \leqslant \mu_2\,\sigma^2$$

for all $t \geqslant 0$ where $\mu_1 = -1.2173, \mu_2 = 0$. Hence, condition (7) is fulfilled.

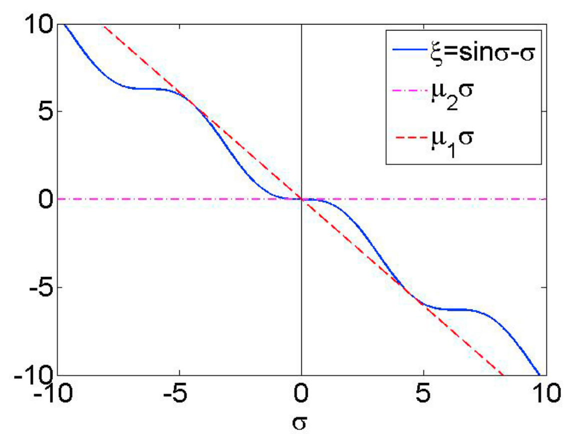

Fig. 2. Sector bounded nonlinearity

By direct calculations we obtain

$$\alpha(s) = \det(sI - A) = s^2 + s - 3,$$

$$W(s) = r^{\mathrm{T}}(sI - A)^{-1}B = \frac{s+2}{s^2+s-3},$$

$$W_q(s) = r^{\mathrm{T}}(sI - A)^{-1}q = \frac{2s+5}{s^2+s-3},$$

$$\beta(s) = \alpha(s)W(s) = s + 2, \quad \beta_q(s) = \alpha(s)W_q(s) = 2s + 5,$$

$$\psi(s) = \frac{\beta(s)}{\beta_q(s)} = \frac{s+2}{2s+5}.$$

It is easy to see that β_q is Hurwitz stable, $\deg\beta_q = n - 1 = 1$, and $r^{\mathrm{T}}q = 2 > 0$. Since

$$\psi(is) = \frac{is+2}{2is+5} = \frac{-s^2-3is-10}{-4s^2-25} = \frac{s^2+10}{4s^2+25} + i\,\frac{3s}{4s^2+25},$$

$\mathrm{Re}\,\psi(is) > 0\ \forall s \in \mathbb{R}$. Therefore, the conditions (i)–(iii) of Theorem 2 are fulfilled. Hence, there exists K stabilizing system (17) for $\tau \equiv 0$ (i.e. for continuous control).

Let $K = 3$. From Fig. 3 one can see that the system is stable (for $\tau \equiv 0$).

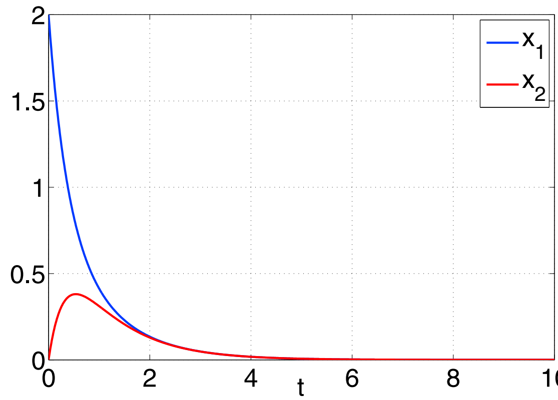

Fig. 3. Trajectories of closed-loop system for continuous control (i.e. $\tau \equiv 0$) and $K = 3$

The feasibility of LMIs from Theorem 3 is checked in Matlab with package Yalmip. The values of maximum upper bound h when (17) is exponentially stable with a small enough decay rate are given in Table 1. The estimates obtained by Theorem 3 is compared with those of simulation (see also Figs. 4 and 5).

	Theorem 3	Simulation	Quality of estimates
$K = 2$	$h = 0.68$	$h = h_*,\ 1.09 < h_* < 1.11$	62%
$K = 3$	$h = 0.53$	$h = h_*,\ 0.69 < h_* < 0.71$	75%
$K = 5$	$h = 0.35$	$h = h_*,\ 0.39 < h_* < 0.41$	89%
$K = 10$	$h = 0.187$	$h = h_*,\ 0.2 < h_* < 0.201$	94%

Table 1. Upper bounds for the variable sampling

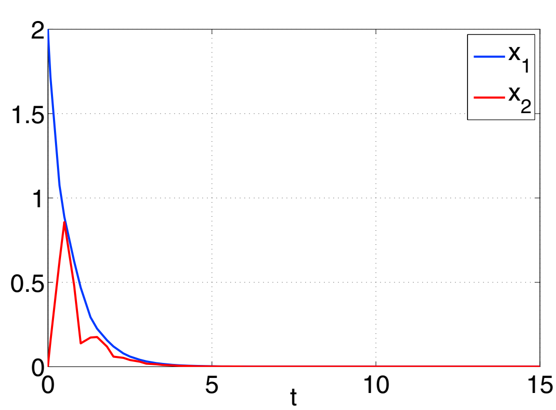

Fig. 4. Trajectories of closed-loop system for $K = 3, h = 0.5$

The dependence of the upper bound of sampling intervals obtained by Theorem 3 on controller gain K is shown in Fig 6. It is seen that the continuous system is stable for $K > 1.44$, and the optimal value of gain K, when h is maximal, is equal to 1.8.

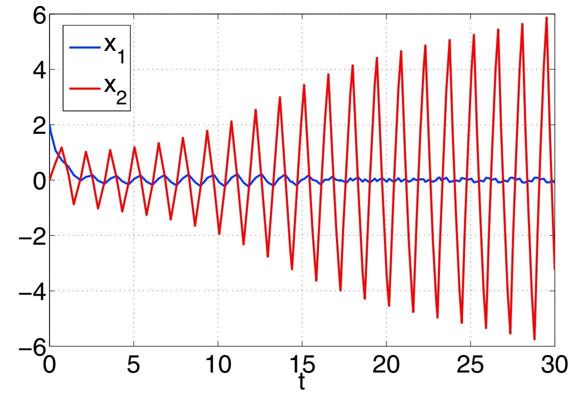

Fig. 5. Trajectories of closed-loop system for $K = 3, h = 0.71$

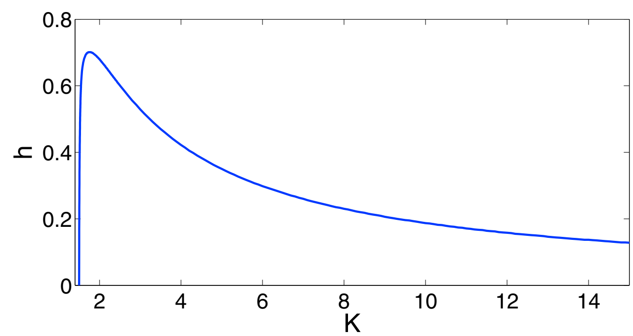

Fig. 6. The dependence of the upper bound h on gain K

6. CONCLUSIONS

The passification based linear controller design is described. Using Yakubovich's S-procedure and Fridman's method the problem of the maximum lower bound estimation for the sampling interval providing exponential stability of the closed loop system is reduced to feasibility analysis of linear matrix inequalities. The obtained results are illustrated by an example. It is shown that the proposed method provides estimates for sampling interval up to 94% of the value evaluated by simulation. Besides, both bounds on the sampling interval and accuracy of those bounds depend significantly on the controller choice.

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