Feedback control for some solutions of the sine-Gordon equation

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Аннотация

Evolution of an initial localized bell-shaped state for the sine-Gordon equation is considered. It is obtained numerically that variation in the parameters of the localized input gives rise to different propagating waves as time goes. The speed gradient feedback control method is employed to achieve unified wave profile weakly dependent on initial conditions. Two speed-gradient like algorithms are developed and compared. It is shown that the algorithm using coefficient at the second spatial derivative term in the sine- Gordon equation allows one to generate the same wave with prescribed energy from different initial states having different energies.

Keywords: feedback control; nonlinear waves; sine-Gordon equation; numerical solution

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Introduction

It is known that solutions to nonlinear equations are sensitive to initial conditions, and even moderate variations in them result in a qualitatively different wave evolution. One of them is the sine Gordon equation that describes many interesting phenomena, e.g., dynamics of coupled pendulums, Josephson junction arrays, interaction of atomic chains [1, 2, 3], it also accounts for continuum limits of the crystalline lattices [4, 5].

One possibility to recover the unified shape of the wave may be in applications of the control methods [6, 7, 8]. Methods of control theory (cybernetics) attract a growing interest of physicists for more than two decades [8, 9, 10]. Among many effects achievable by means of control is reducing sensitivity on initial conditions. Previously these methods were mainly used for oscillation control problems for systems governed by ordinary differential equations [8, 11]. Use of the control mechanism in the wave processes concern envelope wave equations [6, 12], reaction-diffusion equations [13, 14], and the sine- Gordon equation [7, 15, 16]. Some control related methods for sin-Gordon equation use feedforward (nonfeedback) controlling actions [6, 12]. The other ones apply control changing the equation completely, and do not using measurement of the current system state [13, 14]. A method for asymptotic stabilization of the sine-Gordon equation without damping by high-gain output boundary feedback is proposed in [15, 16]. However efficient methods to control oscillatory modes in the sine-Gordon equation were not proposed, according to the best authors' knowledge. Therefore it is interesting to study a possibility to control oscillatory modes of the sine- Gordon equation by means of the speed-gradient method that has been successfully applied to ODE oscillatory systems [8].

The control may be established by different ways. For example, the control of dispersive terms in the equations may be performed [6]. However, the feedback control methods look more promising. A successive attempt to extend the control methods to the wave problems has been done in [7, 15] where the boundary conditions were controlled to achieve significant difference in the wave behavior of the sine Gordon equation and some of its generalization.

In this paper, we develop an algorithm of control with a feedback for evolution of the bell-shaped input for the sine- Gordon equation and compare it with the previous algorithm suggested in [8]. The difference between the algorithms is that they control coefficients at different terms of the sine- Gordon equation. Control of one or another coefficient may have a physical reason of it may be in variation of a spring rigidity if application to the dynamics of coupled pendulums is considered [1, 3].

The paper is organized as follows. First section is devoted to development of the speed gradient algorithms with a feedback based on a control of different coefficients of the sine Gordon equation. Next section considers evolution of an initial localized bell-shaped input without control. Application of the control algorithms is studied in Sec. 3, while Conclusions summarize the results and discuss future work.

1 Speed-gradient algorithms for the control with a feedback of the sine- Gordon equation

Consider a solution $U(x,t)$ of the sine- Gordon equation,

$$U_{tt} - U_{xx} + \sin(U) = 0, \tag{1}$$

One of the speed-gradient control algorithms with a feedback has been developed for the sine- Gordon equation in [8]. It was suggested there to include an external action,$F = F(t)$, in the equation yielding

$$U_{tt} + F \sin U - U_{xx} = 0, \quad t \geq 0, \tag{2}$$

Then it was assumed that $F = 1 + u(t)$, where $u(t)$ is a control action. The aim of the control is supposed to achieve the basic energy of system (4) equal to the defined value H^*,

$$H(t) \to H^*. \tag{3}$$

The Hamiltonian for Eq. (1) is

$$H = \frac{1}{2} \int\limits_{-\infty}^{+\infty} \left(U_t^2 + U_x^2 + (1 - \cos U) \right) \, dx.. \tag{4}$$

Then the control of the speed-gradient $u = -\gamma \dfrac{\partial \omega}{\partial u}$ was obtained in [8]

$$u(t) = \gamma \big(H(t) - H^* \big) \int\limits_{-\infty}^{+\infty} U_t \cdot \sin U \, dx, \tag{5}$$

where $\gamma > 0$ is a parameter of amplification of the algorithm.

Another algorithm may be developed similarly for the control of the coefficient at U_{xx},

$$U_{tt} + \sin U - (1 + u(t)) U_{xx} = 0, \quad t \geq 0, \tag{6}$$

where $u = u(x,t)$ is a control action. Following the procedure from [8] one obtains that temporal variation of the energy (2) is not zero due to the control action,

$$\begin{aligned} \frac{dH}{dt} &= \int\limits_{-\infty}^{+\infty} \left(U_t \cdot U_{tt} - U_{xx} \cdot U_t + \sin U \cdot U_t\right)\, dx = \int\limits_{-\infty}^{+\infty} U_t \cdot \left(U_{tt} - U_{xx} + \sin U\right) dx \\ &= \int\limits_{-\infty}^{+\infty} U_t \cdot \big(- U_{xx} + (1+u) U_{xx} \big)\, dx = u(t) \int\limits_{-\infty}^{+\infty} U_t\, U_{xx}\, dx. \end{aligned} \tag{7}$$

Let us introduce the objective function $V(t) = \frac{1}{2}\big(H(t) - H^*\big)^2$, whose minimum corresponds to the satisfaction of the given condition $H(t) = H^*$. The control algorithm is developed using the speed-gradient method [8] relative to the function $V(t)$. Assuming $H^* = const_t$, one obtains $\omega(t) \equiv \frac{dV}{dt} = \big(H(t) - H^*\big)\frac{dH}{dt}$.

Taking partial derivative in u we obtain the speed-gradient control in the form

$$u(t) = -\gamma\big(H(t) - H^*\big) \int\limits_{-\infty}^{+\infty} U_t \cdot U_{xx}\, dx. \tag{8}$$

One can note that control (8) may be more physically reasonable than control (5). In particular, the control of the coefficient at U_{xx} in Eq. (1) concerns variation in a spring rigidity if the sine Gordon equation is used to describe the dynamics of coupled pendulums [1, 3].

2 Evolution of localized input without feedback

Consider the case when initial condition for U has the form of a localized bell-shaped initial state (for example, in the form of the Gaussian distribution),

$$U(x,0) = Q_1 \exp\left(-\frac{(x-x_0)^2}{Q_2}\right), \tag{9}$$

where $Q_1,\ Q_2, x_0$ are the constant parameters. The initial velocity is supposed to be zero, $U(x,0)_t = 0$. The boundary conditions assumed are: $U(-\infty,t) = 0$, $U(+\infty,t) = 0$ $(\forall t)$. Such input splits into a sequence of bell-shaped localized waves of permanent shape provided that the equation possesses an exact bell-shaped solution like for the Korteweg-de Vries equation or the Boussinesq equation. Each of generated bell-shaped wave is described by the single traveling solitary wave solution of the equation. The sine-

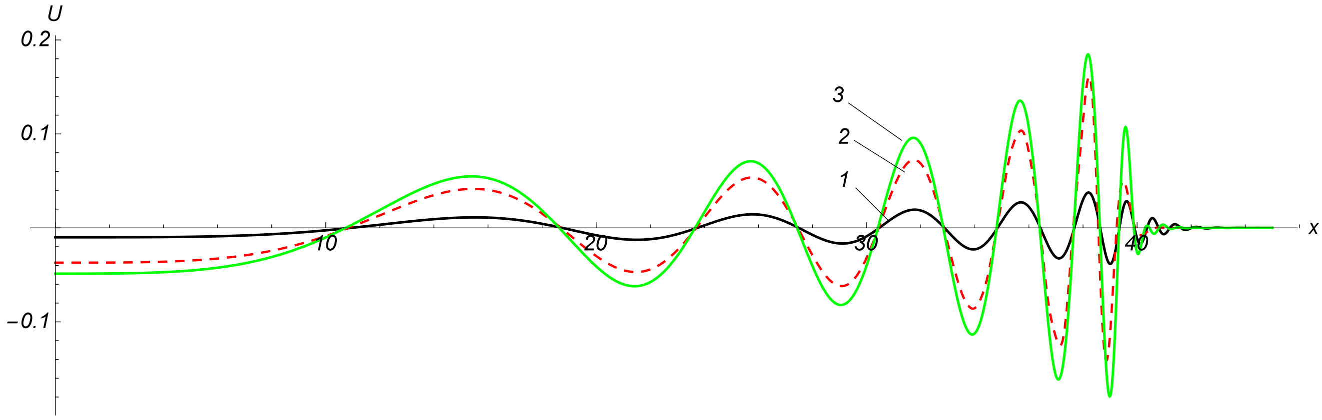

Рис. 1: Wave arising from the Gaussian input at $t = 40$ with the value of $Q_2 = 0.2$ and of Q_1 equal to 1. $Q_1 = 0.2$; 2. $Q_1 = 0.75$; 3. $Q_1 = 1$

Gordon equation does not possess exact bell-shaped traveling wave solution [1, 2, 3]. The equation is integrable, and the solution might be obtained using the Inverse Scattering Transform method. However, analytical solution in an explicit form is unlikely to obtain, and numerical tools are used to get the solution. Certainly, the solution with a control may be obtained only numerically.

Numerical study of evolution of the input (9) for Eq. (1) without control has been implemented using numerical tools of the Wolfram Mathematica, in particular, using its command NDSolve [17]. It is shown, that initial profile (9) splits into two waves moving in the opposite directions. Due to the symmetry of the problem, only the wave moving to the right will be considered further. Shown in Fig. 1 are the profiles of the waves obtained for various values of Q_2 but with fixed values of Q_1. One can see that the number of maxima/minima is the same for all waves, however, the shapes are rather different. Similar features are seen in Fig. 2 for the waves arising from input (9) for fixed values of Q_1 and at different values of Q_2.

3 Control of the solution with feedbacks

Numerical results in the previous section demonstrate us arising of different waves from initial state (9) with different parameters. Independence on the initial conditions may be provided by inclusion of a control with a feedback using algorithms (5) and (8).

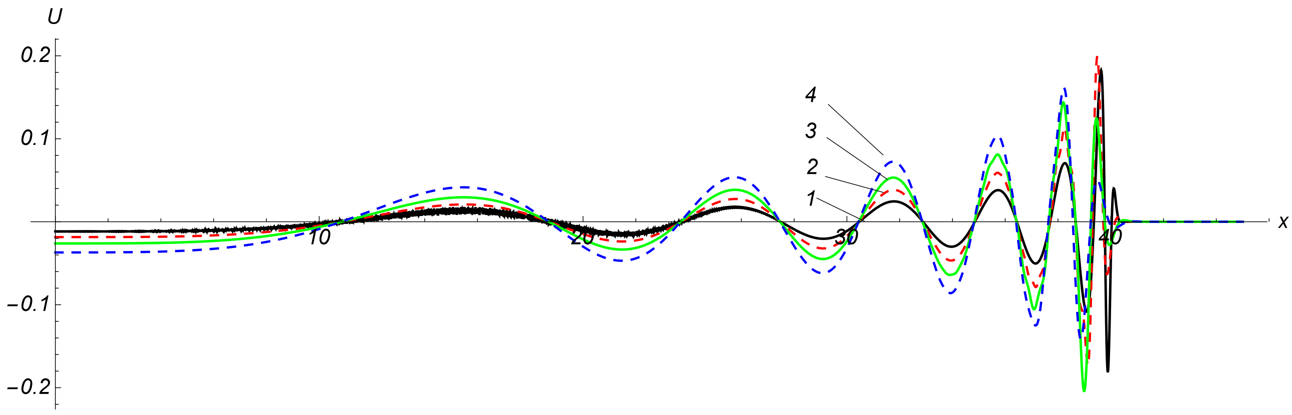

Рис. 2: Wave arising from the Gaussian input at $t = 40$ with the value of $Q_1 = 0.75$ and of Q_2 equal to 1. $Q_2 = 0.02$; 2. $Q_2 = 0.05$; 3. $Q_2 = 0.1$; 4. $Q_2 = 0.2$

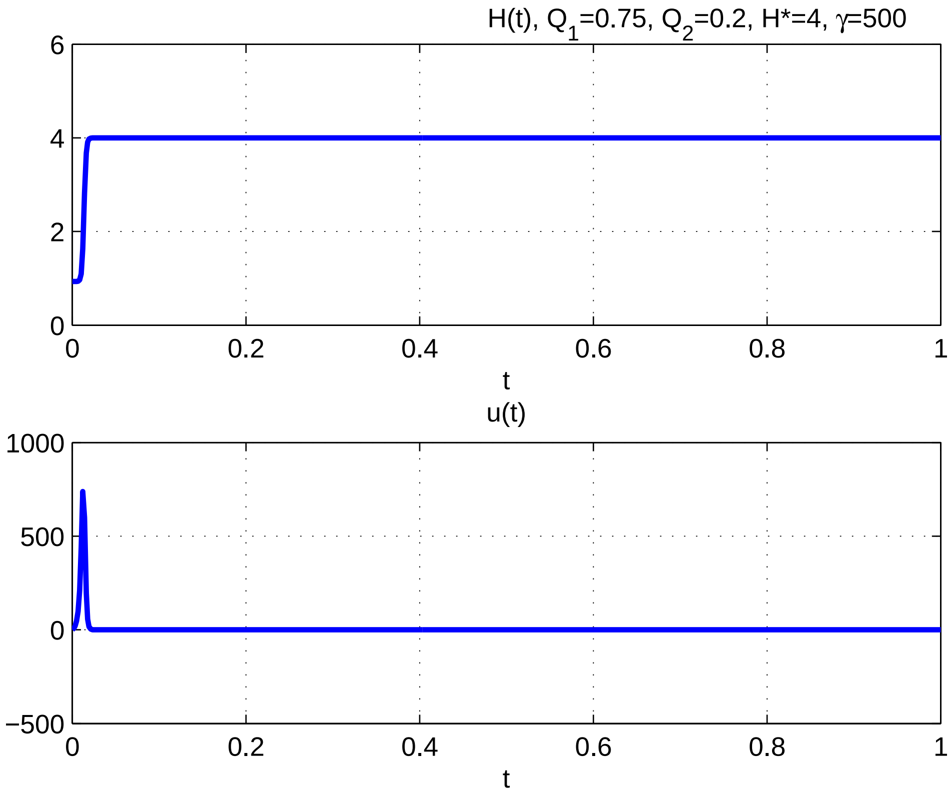

Рис. 3: Energy variation and switch off the control using the algorithm of control (5).

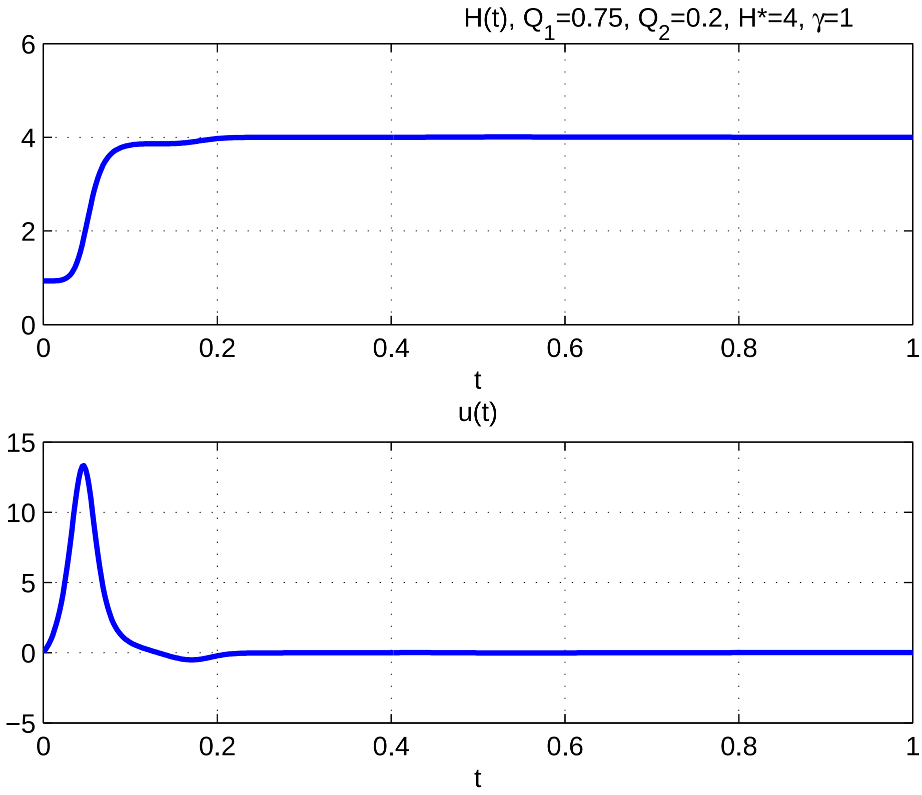

Рис. 4: Energy variation and switch off the control using the algorithm of control (8).

The discretization of Eqs. (2), (6) is based on a uniform spatial splitting of the interval $[X_{\min}, X_{\max}]$ into $N = 2500$ points and on further integration of the obtained system of ordinary differential equations in MATLAB-7/Simulink using the method *ode45* (*Dormand-Prince*) with automatic step size and the relative accuracy 10^{-3}. The biggest step of integration is assumed to be equal to $2 \cdot 10^{-3}$.

Examination of both algorithms shows that the energy is achieved rather fast, see typical cases for both algorithms in Figs 3, 4. The algorithm (8) is faster than that of (8). It allows us to assign smaller value of γ for Eq. (8) than for Eq. (5). Achievement of the desired value of the energy, H^*, results in switching off the control u as shown in Figs 3, 4. After that equation under study corresponds to original sine- Gordon equation (1).

Further calculations were carried out at different values of Q_1 noted in Fig. 5 and for $Q_2 = 0.2$ noted in Fig. 6. Both algorithms of control assume their switching off when the desired value of energy is achieved. The values of γ are noted in the legend in the

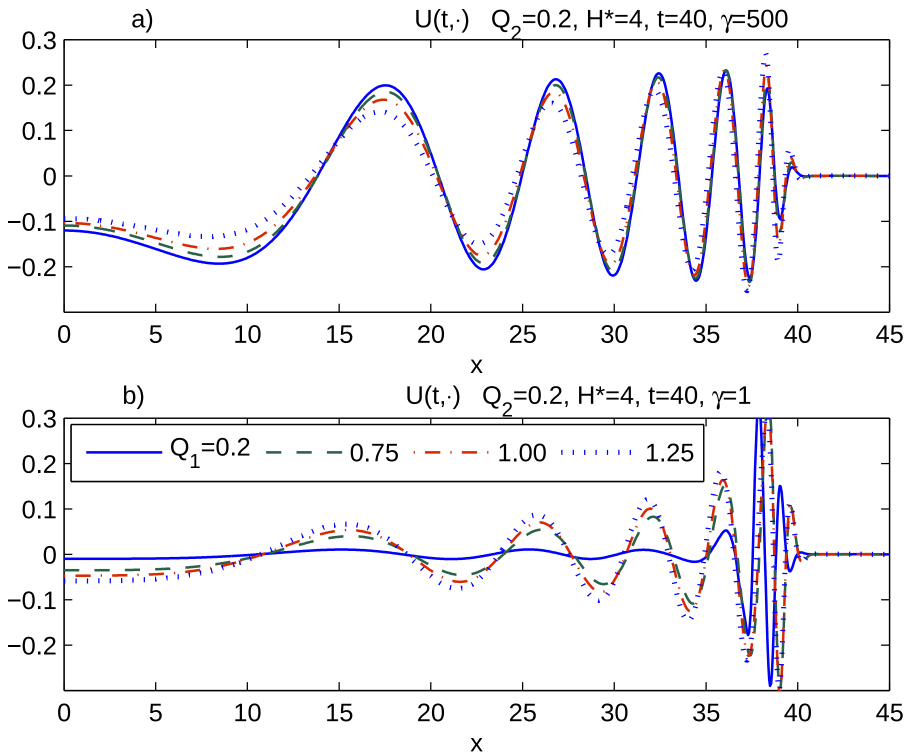

Рис. 5: Influence of the control with a feedback on the wave arising from the Gaussian input at $t = 40$ with the value of $Q_2 = 0.2$ and of Q_1 equal to $Q_1 = 0.2$; $Q_1 = 0.75$; $Q_1 = 1$. a) algorithm (5) b) algorithm (8)

figures. The result of the action of the control is shown in Figs.4, 5 for both developed algorithms (5), (8). One can note that the profiles in Figs. 5, 6 are shown at $t = 40$, much later the switching off the control. The inputs (9) have different energies not equal to the energy H^* prescribed by the control algorithms. Figure 5 demonstrates an influence of the control (5), Fig. 5(a), and the control (8), Fig. 5(b), on the wave profile. One can see that algorithm (5) provides better uniform profile for the different inputs than algorithm (8) in Fig. 5(b). However, the shape of the wave in Fig. 5(b) is closer to those shown in Fig. 1, and the wave turns out more localized than that of shown in Fig. 5(a).

However, control algorithm (5) does not provide suitable merge of the profiles in Fig. 6(a) when Q_1 is fixed while Q_2 varies. Again a delocalization of the wave happens in Fig. 6(a) in comparison with the profiles shown in Fig. 2. The algorithm (8) makes profiles much closer in Fig. 6(b) than in Fig. 6(a). Comparison of the waves from Fig. 6(b) with those in Fig. 2 clearly demonstrates merge of the wave profiles but not so efficiently

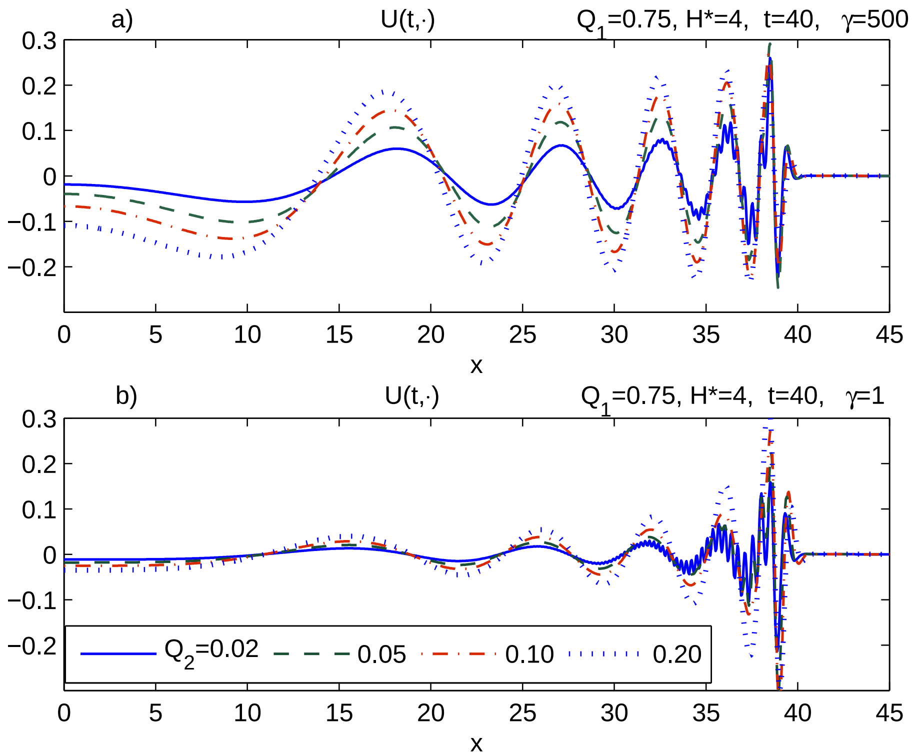

Рис. 6: Influence of the control with a feedback on the wave arising from the Gaussian input at $t = 40$ with the value of $Q_1 = 0.75$ and of Q_2 equal to $Q_2 = 0.02$; $Q_2 = 0.05$; $Q_2 = 0.1$. a) algorithm (5) b) algorithm (8)

as for the case of varying Q_1 shown in Fig. 4(b). Also the profiles of the resulting waves in Fig. 6(b) are localized to the same extent as those shown in Fig. 2.

4 Conclusions

Two speed gradient algorithms for state feedback control are developed for the sine- -Gordon equation. They are used for obtaining unified wave profile with prescribed energy arising from initial localized bell-shaped inputs with different energies. The results allow us to conclude that both algorithms allow one to reduce dependence of the wave shape on initial conditions. However, algorithm (8) turns out more efficient than algorithm (5). Also this algorithm is more physically reasonable as noted at the bottom of Sec. 1.

Despite algorithm (8) provides merge of the profiles at variations of different parameters of the input, this merge is not perfect. Further improvement of the algorithm is needed to achieve unified profile of the wave. Also, other initial conditions for the sine- Gordon equation will be studied to examine an efficiency of the control with a feedback for getting the unified wave.

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