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Decentralized Adaptive Control of Synchronization of Dynamic System Networks at Bounded Disturbances

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Abstract—The problem is considered for adaptive synchronization in the output of the network of interconnected nonlinear dynamic systems with bounded disturbances. The structure of a controller and the adaptation algorithm are found with the aid of the velocity gradient method and the passification method. The sufficient conditions of synchronization and the upper bound for the convergence set relative to the leading subsystem are given. The conditions of the accessibility of synchronization are also obtained for a certain class of monotone nonlipschitz nonlinearities. The results are illustrated by an example of synchronization of the network of interconnected Chua chains with disturbances.

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1. INTRODUCTION

In recent years in published works interest is observed to the control of networks of interconnected systems. This is attributable not only to a relative novelty of the theme, but also to the practical significance because the sets of physical objects can be considered as interconnected systems. To these systems we can assign telecommunicational networks, molecular groups, biological objects, trophic chains, spatially distributed built-in systems, associations of robots or transport facilities, etc. The elaboration of similar systems is associated with the rapid development of information-communication technologies, including the wireless communication and wireless sensors. Interest grows to the modeling and control of biological, biochemical and social networks. However, the synthesis of controllers that afford the coordinate synchronization becomes increasingly difficult in view of the complexity of spatially distributed systems. Difficulties arise on account of a limited exchange of the information between subsystems. The available methods of decentralized control [1–3] are inapplicable for the solution of new, more complex problems, for example, the problems of control through a communication channel with a limited carrying capacity [4]. In the works on synchronization of network systems, it is mainly assumed that the states of network subsystems (agents) are available for a change, while the number of controls in agents coincides with the number of variables of the state. In [5, 6] consideration was given to the problems of synchronization in the networks of identical agents, which are described by differential equations in the Lur'e form, i.e., by the systems with the right sides broken down into linear and nonlinear components. On the basis of the passification method the synchronization conditions are obtained at an arbitrary number of the inputs, outputs and variables of the state of agents. But the effect of noises was not taken into account.

It is assumed in this article that the equations of objects (agents) of the network contain bounded disturbances, while the connections between agents can be nonlinear. As in [5, 6], it is thought that only some linear functions of state variables are accessible for the measurement and that the control does not enter into all state equations. It is also thought that the parameters of objects and the connections between them depend on the vector of unknown parameters. The fundamental (leading) subsystem is selected, which is isolated, i.e., not connected with the remaining subsystems, and which has a known control function. The task is set up for finding the control functions of interconnected subsystems and the conditions affording the synchronization, i.e., the approach of the state of each subsystem to the state of the leading subsystem. The control goal must be achieved for all values of the unknown parameters from a certain class. The nonlinearity of an isolated subsystem and the connections between subsystems are taken to be Lipschitz ones. The stated problem is solved with the aid of the results set forth in [2, 3, 5–7]. The adaptation algorithm is synthesized by the velocity gradient method. It is proved that for the achievement of the control goal it is sufficient to obtain a hyperminimum-phasing of the function of some form and a small number of interconnections. The obtained results are illustrated by an example of the synchronization of a few interconnected Chua chains that display a random behavior. The computational modeling is performed to illustrate the theoretical results.

2. STATEMENT OF THE PROBLEM

We will consider the network S consisting of d interconnected subsystems (agents) S_i, $i = 1, \ldots, d$. Let S_i be described in the following way:

$$\begin{gathered} \dot{x}_i = Ax_i + Bu_i + \varphi_0(x_i) + \sum_{j=1}^{d} \alpha_{ij}\varphi_{ij}(x_i - x_j) + f_i(t), \\ y_i = C^{\mathrm{T}}x_i, \end{gathered} \tag{2.1}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^1$, $\alpha_{ij} \in \mathbb{R}^1$, $y_i \in \mathbb{R}^l$. The functions $\varphi_{ij}(\cdot), i = 1, \ldots, d,\ j = 1, \ldots, d$ describe the interconnections between the subsystems and the $f_i(t)$ describes a bounded disturbance in the system S_i:

$$||f_i(t)|| \leqslant d_{f_i}. \tag{2.2}$$

Let us assume that $\varphi_{ii}(0) = 0$, $\alpha_{ii} = 0$, $i = 1, \ldots, d$. We will consider that the matrices A, B, C and the functions $\varphi_0(\cdot)$ are known, while $\varphi_{ij}(\cdot)$, $i = 1, \ldots, d$, $j = 1, \ldots, d$ depend on the vector of the unknown parameters $\xi \in \Xi$, where Ξ is the known set. Let the leading subsystem (leader) be described in such a way:

$$\dot{\overline{x}} = A\overline{x} + B\overline{u} + \varphi_0(\overline{x}), \quad \overline{y} = C^{\mathrm{T}}\overline{x}, \tag{2.3}$$

where $\overline{u}$ is the prescribed known control and the matrix C is the same as that in (2.1). Let the control goal involve the convergence of trajectories of all subsystems to a certain neighborhood of the leader trajectory:

$$\varlimsup_{t\to\infty} \|x_i(t) - \overline{x}(t)\| \leqslant \Delta_i. \tag{2.4}$$

The adaptive synchronization problem consists in finding the function of the decentralized control $u_i = \mathfrak{U}_i(y_i, \overline{y}, \overline{u}, t)$ that ensures the achievement of the control goal (2.4) for all values of the vector of the unknown parameters $\xi \in \Xi$.

3. SYNTHESIS OF CONTROL

Let us put $z_i = x_i - \overline{x}$, $\tilde{u}_i = u_i - \overline{u}$, and for the dynamics of z_i we obtain the system

$$\dot{z}_i = Az_i + B\tilde{u}_i + \varphi_0(x_i) - \varphi_0(\overline{x}) + f_i(t) + \sum_{j=1}^{d} \alpha_{ij}\varphi_{ij}(x_i - x_j), \quad \tilde{y}_i = C^{\mathrm{T}} z_i, \quad i = 1, \ldots, d. \tag{3.1}$$

We will seek a linear (in output) controller of the auxiliary subsystem in the form

$$\tilde{u}_i = \theta_i^{\mathrm{T}}(t)\tilde{y}_i, \quad \theta_i(t) \in \mathbb{R}^l, \quad i = 1, \ldots, d, \tag{3.2}$$

where $\tilde{y}_i$ is defined in the (3.1) and $\theta_i(t)$ are adjustable parameters. To define the $\theta_i(t)$, we will use the velocity gradient method [2]. This choice is motivated by the fact that the approach to the solution of the problem is based on the use of the Lyapunov quadratic function, and in this case the linear controllers with adjustable coefficients lead to the effectively solvable matrix inequalities.

To define the $\theta_i(t)$ according to the velocity gradient technique, we will prescribe the goal function of the form

$$Q(z_i) = \frac{1}{2} z_i^{\mathrm{T}} H z_i, \quad H = H^{\mathrm{T}} > 0. \tag{3.3}$$

We will take the derivative of $Q(z_i)$ in view of the isolated system:

$$\omega_i(x_i, \overline{x}) = z^{\mathrm{T}} H(Az + B\theta_i^{\mathrm{T}}\tilde{y}_i + \varphi_0(x_i) - \varphi_0(\overline{x})), \quad i = 1, \ldots, d. \tag{3.4}$$

Taking the gradient in $\theta_i(t)$, we obtain the basic adaptation algorithm. It is well known [2] that in the action of nonvanishing disturbances the basic algorithm is inoperative even in the case of one subsystem because it leads to the unlimited growth of adjustable coefficients. In order to ensure the boundedness of trajectories of the system, we will introduce into the initial algorithm the inert zone [2]:

$$\dot{\theta}_i(t) = \begin{cases} -g^{\mathrm{T}}\tilde{y}_i(t)\Gamma_i\tilde{y}_i(t), & Q_i(x_i(t), t) > \Delta_i \\ 0, & Q_i(x_i(t), t) \leqslant \Delta_i, \end{cases} \tag{3.5}$$

where $\Gamma_i = \Gamma_i^{\mathrm{T}} > 0$ are the positive definite matrices of order $l \times l$.

4. CONDITIONS OF SYNCHRONIZATION FOR LIPSCHITZ NONLINEARITIES

To state the synchronization conditions, we will need the definition of the hyperminimum-phasing [2, 7, 8].

Definition 1. Let $W(S) = \beta(s)/\alpha(s)$ be the correct rational function and $\beta(s)$, $\alpha(s)$ be real polynomials. The $W(s)$ function is termed minimum-phase if its numerator $\beta(s)$ is the Hurwitz polynomial. The $W(s)$ function is termed hyperminimum-phase if it is minimum-phase, while the number $\lim\limits_{s\to+\infty} sW(s)$ is positive.

Let us consider the real matrices $H = H^{\mathrm{T}} > 0$, g, θ_* of orders $n \times n$, $l \times 1$, $l \times 1$,respectively, and the number $\rho > 0$, such that

$$HA_* + A_*^{\mathrm{T}}H < -\rho H, \quad HB = Cg, \quad A_* = (A + LI_n) + B\theta_*^{\mathrm{T}}C^{\mathrm{T}}.$$

We will denote by $\lambda^-(H)$ and $\lambda^+(H)$ the minimum and the maximum eigenvalue of the matrix H and by $\lambda_* = \lambda^+(H)/\lambda^-(H)$, the condition number of the matrix H.

Theorem 1. *Let the functions* $\varphi_0(\cdot)$ *and* $\varphi_{ij}(\cdot)$, $i = 1,\dots,d$, $j = 1,\dots,d$ *be globally Lipchitz ones:*

$$||\varphi_0(x) - \varphi_0(x')|| \leqslant L||x - x'||, \quad L > 0,$$
$$||\varphi_{ij}(x) - \varphi_{ij}(x')|| \leqslant L_{ij}||x - x'||, \quad L_{ij} > 0.$$

Let for a certain number $g \in \mathbb{R}^l$ *the function* $g^{\mathrm{T}}\chi(s - L)$ *be hyperminimum-phase, where* $\chi(s) = C^{\mathrm{T}}(sI_n - A)^{-1}B$ *is the transfer function. Then there exist such matrices* $H = H^{\mathrm{T}}$, θ_* *of orders* $n \times n$, $l \times 1$, *respectively, and* $\rho > 0$ *that the following conditions are fulfilled:*

$$HA_* + A_*^{\mathrm{T}}H < -\rho H, \quad HB = Cg, \quad A_* = (A + LI_n) + B\theta_*^{\mathrm{T}}C^{\mathrm{T}}. \tag{4.1}$$

We will introduce the designation $\delta_i = \frac{\rho}{2}\frac{\lambda^-(H)}{\lambda^+(H)} - \sum_{j=1}^{d}|\alpha_{ij}L_{ij}|$. *If for all* $i = 1,\dots,d$ *the following conditions are fulfilled:*

$$\delta_i > 0, \tag{4.2}$$

then for all $i = 1,\dots,d$ *and all* $\xi \in \Xi$ *the adaptive control* (3.2), (3.5) *affords the achievement of the goal* (2.4) *at*

$$\Delta_i > \frac{d_{f_i}\lambda^+(H)}{2\rho\delta_i} \tag{4.3}$$

and the boundedness of all solutions of the closed system (2.1), (2.3), (3.2), (3.5).

The condition (4.3) is fairly natural, and it points to the connection of the level of disturbances with the dimensions of the region of attraction of subsystem trajectories to the trajectory of the leading subsystem: the higher the level of disturbances, the wider the boundaries of the region of attraction.

Remark. We will denote by ρ_* the stability degree of the numerator of the function $g^{\mathrm{T}}\chi(s-L)$. From the results of [8] we can conclude that if the function $g^{\mathrm{T}}\chi(s - L)$ is hyperminimum-phase, then as θ_* and ρ in (4.1) it is possible to take $\theta_* = \kappa g$ and any $\rho : 0 < \rho < \rho_*$, where the number $\kappa > 0$ is rather high. Thus the inequality (4.2) can be substituted for

$$\sum_{j=1}^{d}|\alpha_{ij}L_{ij}| < \gamma, \tag{4.4}$$

where $\gamma = \frac{\rho_*}{2\lambda_*}$.

5. THE CASE OF $\varphi_0(x_i) = B\psi_0(y_i)$

We will consider the case where nonlinearities in the models of agents depend only on the outputs and they can be represented in the form $\varphi_0(x_i) = B\psi_0(y_i)$, where $\psi_0 : \mathbb{R}^l \to \mathbb{R}^1$. Then the subsystem (2.1) can be rewritten in the following way:

$$\dot{x}_i = Ax_i + B(u_i + \psi_0(y_i)) + \sum_{j=1}^{d}\alpha_{ij}\varphi_{ij}(x_i - x_j) + f_i(t), \quad y_i = C^{\mathrm{T}}x_i. \tag{5.1}$$

We will rewrite the leader's equation in the form

$$\dot{\overline{x}} = A\overline{x} + B(\overline{u} + \psi_0(\overline{y})), \quad \overline{y} = C^{\mathrm{T}}\overline{x}, \tag{5.2}$$

where $\overline{u} \in \mathbb{R}^1$ is the preassigned control that is taken to be known. We will assume that the matrices A, B, C and $\psi_0(\cdot)$ are known and the functions $\varphi_{ij}(\cdot)$, $i = 1,\dots,d$, $j = 1,\dots,d$ depend on the vector of the unknown parameters $\xi \in \Xi$, where Ξ is the known set. For the further presentation we will need the definition of the g-monotone decreasing function [5].

Definition 2. Let $g \in \mathbb{R}^l$. The function $f : \mathbb{R}^l \to \mathbb{R}^1$ is said to be g-monotone decreasing if for any $x,\, y \in \mathbb{R}^l$ the following condition is fulfilled: $(x-y)^{\mathrm{T}} g(f(x) - f(y)) \leqslant 0$.

We will further assume that for all agents the following condition is valid:

(A1) *The functions φ_{ij} are globally Lipschitzian with the constant $L_{ij} > 0$, while the function $\psi_0(\cdot)$ is such that the existence and uniqueness of the solutions of the equations for each network object are ensured.*

As a control we can select the control from the Section 3: (3.2), (3.5).

We will consider the real matrices $H = H^{\mathrm{T}} > 0$, g, θ_* of orders $n \times n$, $l \times 1$, $l \times 1$, respectively, and the number $\rho > 0$, such that

$$HA_* + A_*^{\mathrm{T}} H < -\rho H, \quad HB = Cg, \quad A_* = A + B\theta_*^{\mathrm{T}} C^{\mathrm{T}}. \tag{5.3}$$

Let us note that the relations (5.3) can be obtained if we put $L = 0$ in (4.1).

Theorem 2. *Let for each $\xi \in \Xi$ the supposition* (A1) *be fulfilled and for some $g \in \mathbb{R}^l$ the function $g^{\mathrm{T}}\chi(s)$ be hyperminimum-phase, where $\chi(s) = C^{\mathrm{T}}(sI_n - A)^{-1}B$. Then there exist such matrices $H = H^{\mathrm{T}} > 0$ and θ_* of orders $n \times n$, $l \times 1$, respectively, and a positive number ρ that the relations* (5.3) *are fulfilled. Let, in addition, the function $\psi_0(\cdot)$ be g-monotone decreasing. We will denote by δ_i the expression $\frac{\rho}{2}\frac{\lambda^-(H)}{\lambda^+(H)} - \sum_{j=1}^d |\alpha_{ij} L_{ij}|$. Let for all $i = 1, \ldots, d$ the conditions given below be fulfilled:*

$$\begin{cases} \displaystyle\sum_{j=1}^{d} |\alpha_{ij} L_{ij}| > \gamma \\ \Delta_i > \dfrac{d_{f_i} \lambda^+(H)}{2\rho\delta_i}, \end{cases} \tag{5.4}$$

where $\gamma = \rho_*/(2\lambda_*)$, λ_* is the condition number of the matrix H, ρ_* is the stability degree of the numerator of the function $g^{\mathrm{T}}\chi(s)$. Then for all $i = 1, \ldots, d$ and for all $\xi \in \Xi$ the adaptive control (3.2), (3.5) *affords the achievement of the goal*

$$\overline{\lim_{t\to\infty}} \|x_i(t) - \overline{x}(t)\| \leqslant \Delta_i. \tag{5.5}$$

In this case all solutions of the closed system (3.2), (3.5), (5.1), (5.2) *remain bounded by* $[0, \infty)$.

6. SYNCHRONIZATION UNDER CONSISTENCY CONDITIONS

We will consider the case where the leader and the remaining subsystems of the network are described by various equations, but the so-called consistency conditions are fulfilled [2]. Let the leader be described by the equations

$$\dot{\overline{x}} = A_M \overline{x} + B_M(\overline{u} + \psi_0(\overline{y})), \quad \overline{y} = C^{\mathrm{T}} \overline{x}, \tag{6.1}$$

where $\overline{x} \in \mathbb{R}^n$, $\overline{u} \in \mathbb{R}^1$, $\overline{y} \in \mathbb{R}^l$, $\psi_0 : \mathbb{R}^l \to \mathbb{R}^1$. We consider, as before, that the $\overline{u}$ is the preset known control. We will assume that A_M, B_M, C and $\psi_0(\cdot)$ are known and do not depend on $\xi \in \Xi$, where Ξ is the known set. We will consider the network from d interconnected objects, each of which is described by the equation

$$\dot{x}_i = Ax_i + Bu_i + B_M \psi_0(y_i) + f_i(t) + \sum_{j=1}^{d} \alpha_{ij} \psi_{ij}(x_i - x_j), \quad y_i = C^{\mathrm{T}} x_i, \quad i = 1, \ldots, d, \tag{6.2}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^1$, $\alpha_{ij} \in \mathbb{R}^1$, $y_i \in \mathbb{R}^l$. We will consider that the matrices A, B and the functions $\varphi_{ij}(\cdot)$, $i = 1,\dots,d$, $j = 1,\dots,d$ depend on the vector of the unknown parameters $\xi \in \Xi$, while $f_i(t)$ is the bounded disturbance in the system S_i:

$$||f_i(t)|| \leqslant d_{f_i}. \tag{6.3}$$

We assume that the consistency conditions are fulfilled [9, §7.2, item 2] *of the structure of the leading subsystem* (6.1) *with the structure* (6.2) *of each network object.*

(A2) *For each $\xi \in \Xi$ there exists a vector $\nu_* = \nu_*(\xi) \in \mathbb{R}^l$ and a number $\theta_* = \theta_*(\xi) > 0$, such that the following equalities are valid*:

$$A_M = A + B\nu_*^{\mathrm{T}}C^{\mathrm{T}}, \quad B_M = \theta_* B.$$

Let us put $\sigma_i(t) = \mathrm{col}(y_i(t), \overline{u}(t))$. We will apply the adaptive controller in the same way as in [9]:

$$u_i(t) = \tau_i(t)^{\mathrm{T}}\sigma_i(t), \quad i = 1,\dots,d, \tag{6.4}$$

where $\tau_i(t) \in \mathbb{R}^{l+1}$ is the vector of adjustable parameters. We will consider the difference of solutions of the agent S_i and the leader for each $i = 1,\dots,d$:

$$\dot{x}_i - \dot{\overline{x}} = Ax_i + Bu_i + B_M\psi_0(y_i) + f_i(t) - A_M\overline{x} - B_M(\overline{u} + \varphi_0(\overline{y})). \tag{6.5}$$

We will perform the replacement of $z_i = x_i - \overline{x}$ and select the goal function

$$Q(z_i) = \frac{1}{2}z_i^{\mathrm{T}}Hz_i, \quad H = H^{\mathrm{T}} > 0. \tag{6.6}$$

Let us consider the derivative of the goal function in view of the isolated subsystem (6.5):

$$\omega(x_i, \overline{x}, \tau_i) = \dot{Q} = z_i^{\mathrm{T}}H\left(Ax_i + B\tau_i^{\mathrm{T}}(t)\sigma_i(t) + B_M\psi_0(y_i) + f_i(t) - A_M\overline{x} - B_M(\overline{u} + \psi_0(\overline{y}))\right).$$

Calculating the gradient in τ_i

$$\nabla_{\tau_i}\omega(x_i, \overline{x}, \tau_i) = z_i^{\mathrm{T}}HB\sigma_i(t),$$

we obtain the basic adaptation algorithm. Roughening it by introducing the inert zone, we obtain

$$\dot{\tau}_i = \begin{cases} -g^{\mathrm{T}}(y_i - \overline{y})\Gamma_i\sigma_i(t), & Q_i(x_i(t), t) > \Delta_i \\ 0, & Q_i(x_i(t), t) \leqslant \Delta_i, \end{cases} \tag{6.7}$$

where $\Gamma_i = \Gamma_i^{\mathrm{T}} > 0$ are the matrices of orders $(l+1)\times(l+1)$ and $g \in \mathbb{R}^l$.

We will consider the real matrices $H = H^{\mathrm{T}} > 0$, g of orders $n \times n$ and $l \times l$ respectively and the number $\rho > 0$, such that

$$HA_M + A_M^{\mathrm{T}}H < -\rho H, \quad HB_M = Cg. \tag{6.8}$$

Let us put $\chi(s) = C^{\mathrm{T}}(sI_n - A_M)^{-1}B_M$. In the following theorem we will present the sufficient conditions of adaptive synchronization.

Theorem 3. *Let* $\mathrm{rank}\, B_M = 1$, *the matrix* A_M *be the Hurwitz one and for some* $g \in \mathbb{R}^l$ *the frequency inequalities be valid:*

$$\mathrm{Re}\, g^{\mathrm{T}}\chi(i\omega) > 0, \quad \lim_{\omega\to\infty}\omega^2\mathrm{Re}\, g^{\mathrm{T}}\chi(i\omega) > 0 \tag{6.9}$$

for all $\omega \in \mathbb{R}^1$. *Then there exist such* $H = H^{\mathrm{T}} > 0$ *and* $\rho > 0$ *that* (6.8) *are fulfilled.*

Let for each $\xi \in \Xi$ *the suppositions* (A1) *and* (A2), *be fulfilled and the function* $\psi_0(\cdot)$ *be* g*-monotone decreasing. If for all* $i = 1, \ldots, d$ *the following conditions are fulfilled:*

$$\sum_{j=1}^{d} |\alpha_{ij} L_{ij}| < \gamma, \tag{6.10}$$

where $\gamma = \rho_*/(2\lambda_*)$, λ_* *is the condition number of the matrix* H, ρ_* *is the stability degree of the denominator of the function* $g^{\mathrm{T}}\chi(s)$, *then for all* $i = 1, \ldots, d$ *and for all* $\xi \in \Xi$ *the adaptive control* (6.4), (6.7) *ensures the achievement of the goal*

$$\overline{\lim_{t\to\infty}} \|x_i(t) - \overline{x}(t)\| < \Delta_i \tag{6.11}$$

for

$$\Delta_i > \frac{d_{f_i}\lambda^+(H)}{2\rho\delta_i}, \tag{6.12}$$

where $\delta_i = \frac{\rho}{2}\frac{\lambda^-(H)}{\lambda^+(H)} - \sum_{j=1}^{d} |\alpha_{ij} L_{ij}|$. *In this case vectors of the adjustable parameters* τ_i *remain bounded at* $[0, \infty)$ *for all solutions of the closed system* (6.1), (6.2), (6.4), (6.7).

7. EXAMPLE. SYNCHRONIZATION IN THE NETWORK OF CHUA CHAINS

7.1. System Description

As an example we will consider the network of six Chua chains [10]. The Chua chain is described in the following way:

$$\begin{cases} \dot{\overline{x}}_1 = p(\overline{x}_2 - \overline{x}_1 - \xi(\overline{x}_1)) + b\overline{u} \\ \dot{\overline{x}}_2 = \overline{x}_1 - \overline{x}_2 + \overline{x}_3 \\ \dot{\overline{x}}_3 = -q\overline{x}_2, \end{cases} \tag{7.1}$$
$$\overline{y}(t) = \overline{x}_1(t),$$

where $\overline{x} = \mathrm{col}(\overline{x}_1, \overline{x}_2, \overline{x}_3) \in \mathbb{R}^3$ is the state vector of the subsystem, $\overline{y}$ is the measurable output, $\overline{u}$ is the scalar control, $\xi(x) = m_1 x + \frac{m_0 - m_1}{2}(|x+1| - |x-1|)$ is the nonlinearity, and p, q, m_1, m_0 are the subsystem parameters.

This system can be easily brought to the form:

$$\dot{\overline{x}} = A\overline{x} + B\overline{u}, \tag{7.2}$$
$$\overline{y} = C^{\mathrm{T}}\overline{x}$$

with the matrices

$$A = \begin{pmatrix} -(1+m_0)p & p & 0 \\ 1 & -1 & 1 \\ 0 & -q & 0 \end{pmatrix}, \tag{7.3}$$

$B = \mathrm{col}(b, 0, 0)$, $C = \mathrm{col}(1, 0, 0)$ and with the nonlinearity of $\psi_0(\overline{y}) = p\nu(\overline{y})/b$, where $\nu(x) = -\frac{m_0 - m_1}{2}(|x+1| - |x-1| - 2x)$, $f = \mathrm{col}(f_1, f_2, f_3)$. It is seen from [5] that if $p/b > 0$, $g > 0$, $m_0 < m_1$, then $\psi_0(\cdot)$ will decrease g-monotonically, and for security of the hyperminimum-phasing of the transfer function

$$g^{\mathrm{T}}\chi(s) = gC^{\mathrm{T}}(sI - A)^{-1}B = gb\frac{s^2 + s + q}{s^3 + \beta_2 s^2 + \beta_1 s + \beta_0}, \tag{7.4}$$

where β_0, β_1, β_2 are some numbers, it is sufficient to select $g > 0$ and $b > 0$. To fulfill the Hurwitz condition of the polynomial $s^2 + s + q$ it is sufficient to take $q > 0$.

We put $x_i = \mathrm{col}(x_{i1}, x_{i2}, x_{i3})^{\mathrm{T}}$, $i = 1, \ldots, 6$, and consider six interconnected Chua chains:

$$\dot{x}_i = Ax_i + B(u_i + \psi_0(y_i)) + f_i + \sum_{j=1}^{6} \alpha_{ij}\varphi_{ij}(x_i - x_j), \tag{7.5}$$

$$y_i = C^{\mathrm{T}} x_i, \quad i = 1, \ldots, 6,$$

where u_i, $\alpha_{ij} \in \mathbb{R}^1$, f_i is the bounded disturbance, $\|f_i\| < d_{f_i}$. We will write $\varphi_{ij} = \varphi_{ij}(x_i - x_j)$, $i = 1, \ldots, 6$, $j = 1, \ldots, 6$. Let us assume that φ_{14}, φ_{25}, φ_{32}, φ_{42}, φ_{45}, φ_{52}, φ_{53} are equal to $(0, 0, 0)^{\mathrm{T}}$. Note that $\varphi_{ii} = (0, 0, 0)^{\mathrm{T}}$ for all $i = 1, \ldots, 6$. We set:

$$\begin{aligned}
&\varphi_{12} = (\sin(x_{11} - x_{21}), 0, 0)^{\mathrm{T}}, && \varphi_{13} = (0, x_{12} - x_{32}, 0)^{\mathrm{T}},\\
&\varphi_{15} = (0, 0, \sin(x_{13} - x_{53}))^{\mathrm{T}}, && \varphi_{21} = (x_{21} - x_{11}, 0, x_{23} - x_{13})^{\mathrm{T}},\\
&\varphi_{23} = (0, \sin(x_{22} - x_{32}), 0)^{\mathrm{T}}, && \varphi_{24} = (0, x_{22} - x_{42}, 0)^{\mathrm{T}},\\
&\varphi_{26} = (\sin(x_{21} - x_{61}), 0, 0)^{\mathrm{T}}, && \varphi_{31} = (\sin(x_{31} - x_{11}), 0, 0)^{\mathrm{T}},\\
&\varphi_{34} = (\sin(x_{31} - x_{41}), 0, 0)^{\mathrm{T}}, && \varphi_{35} = (x_{31} - x_{51}, x_{32} - x_{52}, x_{33} - x_{53})^{\mathrm{T}},\\
&\varphi_{41} = (0, \sin(x_{42} - x_{12}), 0)^{\mathrm{T}}, && \varphi_{43} = (\sin(x_{41} - x_{31}), 0, 0)^{\mathrm{T}},\\
&\varphi_{51} = (x_{51} - x_{11}, 0, x_{53} - x_{13})^{\mathrm{T}}, && \varphi_{54} = (0, x_{52} - x_{42}, 0)^{\mathrm{T}},\\
&\varphi_{56} = (0, x_{52} - x_{62}, x_{53} - x_{63})^{\mathrm{T}}, && \varphi_{61} = (x_{61} - x_{11}, 0, x_{63} - x_{13})^{\mathrm{T}},\\
&\varphi_{64} = (\sin(x_{61} - x_{41}), 0, 0)^{\mathrm{T}}.
\end{aligned}$$

Lipschitz constants of all nonzero values φ_{ij} are equal to unity.

According to the Theorem 2 for $\rho > 0$, $q > 0$, $b > 0$, $g > 0$, $m_0 < m_1$ and for sufficiently small values of $\sum_{j=1}^{6} |\alpha_{ij}|$, $i = 1, \ldots, 6$, the adaptive decentralized control

$$u_i = \theta_i[y_i - \overline{y}] + \overline{u},$$

$$\dot{\theta}_i(t) = \begin{cases} -g^{\mathrm{T}} \tilde{y}_i(t) \Gamma_i \tilde{y}_i(t), & Q_i(x_i(t), t) > \Delta_i \\ 0, & Q_i(x_i(t), t) \leqslant \Delta_i, \end{cases}$$

where $\Gamma_i = \Gamma_i^{\mathrm{T}} > 0$ are positive numbers, ensures synchronization in the network of Chua chains from the viewpoint of achievement of the goal (2.4), where $\overline{x}(t)$ is the solution of the system (7.1) with the input action $\overline{u}(t)$.

7.2. Modeling Results

Following [11] we put $m_0 = -8/7$, $m_1 = -5/7$, $p = 15.6$, $q = 25.58$, $b = g = 1$, $\Gamma_i = 1$, $d_{f_i} = 4$, $i = 1, \ldots, 6$, and

$$\begin{aligned}
&\overline{x}_1(0) = 0.5, && \overline{x}_2(0) = 0, && \overline{x}_3(0) = 0,\\
&x_1(0) = (7; 14; 0.4)^{\mathrm{T}}, && x_2(0) = (0; 4; 4)^{\mathrm{T}}, && x_3(0) = (1; -1; 4.5)^{\mathrm{T}},\\
&x_4(0) = (3; -4; 0.2)^{\mathrm{T}}, && x_5(0) = (2; 8; 15)^{\mathrm{T}}, && x_6(0) = (1; 1; 0.2)^{\mathrm{T}}.
\end{aligned}$$

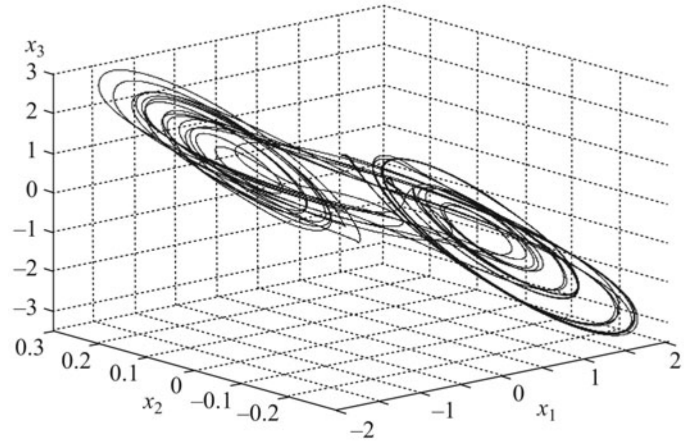

Fig. 1. The phase portrait of the leading subsystem.

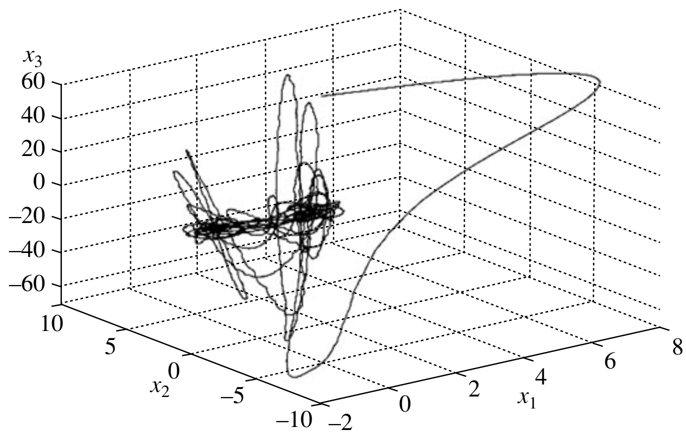

Fig. 2. The phase portrait of the first subsystem.

Thus the conditions $\rho > 0$, $q > 0$, $b > 0$, $g > 0$ and $m_0 > m_1$ are met. As a control for the leading subsystem we will take the impulse excitation with an amplitude of 1/2, zero phase and a period of $T = 5$ s. We will model the disturbance $f_i(t)$ as a uniformly distributed random quantity in the interval $[-d_{f_i}, d_{f_i}]$. We put

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{16} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{26} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{61} & \alpha_{62} & \dots & \alpha_{66} \end{pmatrix},$$

$$\hat{\alpha} = \begin{pmatrix} 0 & 0.0051 & 0.1395 & 0 & 0.1676 & 0 \\ 0.0662 & 0 & 0.0921 & 0.0065 & 0 & 0.0926 \\ 0.2013 & 0 & 0 & 0.2271 & 0.1430 & 0 \\ 0.0907 & 0 & 0.0675 & 0 & 0 & 0 \\ 0.0663 & 0 & 0 & 0.02773 & 0 & 0.1472 \\ 0.0662 & 0 & 0 & 0.0065 & 0 & 0 \end{pmatrix}.$$

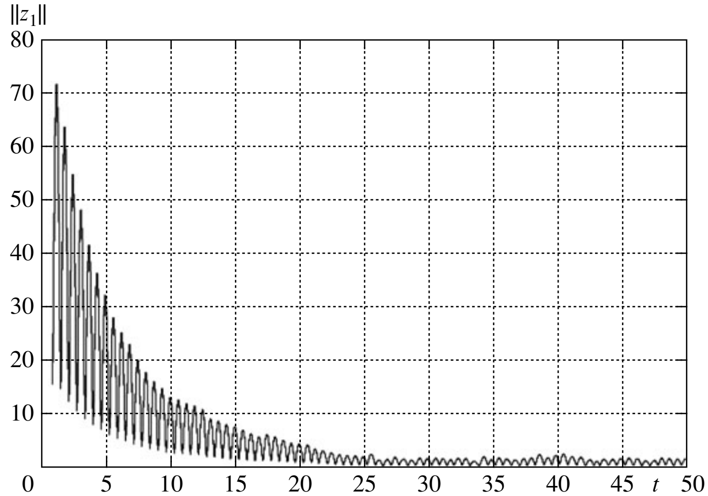

Fig. 3. The error norm for the first subsystem.

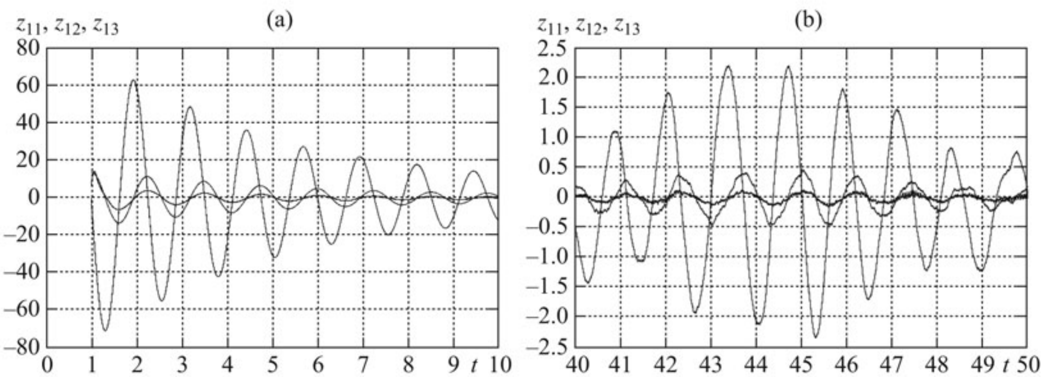

Fig. 4. The error in components of the first subsystem: (a) in the first 10 seconds; (b) in the last 10 seconds.

The phase portraits of the leading and the first subsystem are displayed in Figs. 1 and 2 respectively. The modeling for 50 seconds at $\alpha = \hat{\alpha}$ shows that the goal is being achieved. In this case $||z_i|| < \Delta_i$, but it does not tend zero (see Figs. 3 and 4).

8. CONCLUSIONS

In contrast to the previous articles in this work we obtained the conditions of output feedback synchronization for nonlinear dynamic networks with bounded disturbances. The synthesis of the control algorithm affording the synchronization is based on the velocity gradient method [2, 7]. The synchronization conditions for nonlinear systems with disturbances are found with the aid of passification theorems [2, 8]. The qualitative result is obtained, which point to the possibility of adaptive synchronization for output feedback of the systems with bounded disturbances and also of the direct connection between the level of disturbances and the value of the limit deviation of subsystem trajectories from the trajectory of the leading subsystem. Let us note that a similar problem for systems with unlimited (stochastic) disturbances of the type of white noise is considered in [12].

Further investigations can be devoted to the study of the joint effect of disturbances and delay on the processes of synchronization [13, 14].

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APPENDIX

Proof of Theorem 1. For the proof it will be necessary to use two auxiliary lemmas. The first lemma is the changed lemma from [5].

Lemma 1. *For the existence of real matrices $H = H^* > 0$ and θ_*, such that $HA_* + A_*^{\mathrm{T}}H < 0$ and $HB = CG$, where $A_* = (A + LI_n) + B\theta_*^{\mathrm{T}}C^{\mathrm{T}}$, it is sufficient that the function $G^{\mathrm{T}}C^{\mathrm{T}}(sI_n - A - LI_n)^{-1}B$ should be hyperminimum-phase.*

The second lemma is the modified Theorem 2.19 from [3].

Lemma 2. *We will consider the system consisting of N interacting subsystems; the dynamics of each subsystem is described by the equation*

$$\dot{x}_i = F_i(x_i, \theta_i, t) + h_i(x, \theta, t), \quad i = 1, \ldots, K, \tag{A.1}$$

$$\dot{\theta}_i(t) = \begin{cases} -\Gamma_i \nabla_{\theta_i}\omega_i(x_i, \theta_i, t), & Q_i(x_i(t), t) > \Delta_i \\ 0, & Q_i(x_i(t), t) \leqslant \Delta_i, \end{cases} \tag{A.2}$$

where $x_i \in \mathbb{R}^{n_i}, \theta_i \in \mathbb{R}^{m_i}$,

$$\omega_i(x_i, \theta_i, t) = \frac{\partial Q_i}{\partial t} + \nabla Q_i(x_i, t)^{\mathrm{T}} F_i(x_i, \theta_i, t).$$

Here $Q(\cdot)$ is a certain goal function, $N = \sum n_i$, $m = \sum m_i$, $x = \mathrm{col}(x_1, \ldots, x_l) \in \mathbb{R}^N$. We assume that for (A.1) *the following groups of conditions are fulfilled:*

(1) *The functions $F_i(\cdot)$ are continuous at x_i and t_i, continuously differentiable with respect to θ_i and locally bounded at $t > 0$; the functions $\omega_i(x_i, \theta_i, t)$ are convex at θ_i; there exist vectors $\theta_i^* \in \mathbb{R}^{m_i}$ and scalar, continuously increasing functions $k_i(Q)$, $\rho_i(Q)$, such that $k_i(0) = \rho_i(0) = 0$, $k_i(Q) \to +\infty$ and $\rho_i(Q) \to \infty$ at $Q_i \to +\infty$,*

$$\omega_i(x_i, \theta_i^*, t) \leqslant -\rho_i(Q_i(x_i, t)) \tag{A.3}$$

and

$$Q_i(x_i, t) \geqslant k_i(||x_i - x_i^*(t)||),$$

where $x_i^* = \mathrm{argmin}_{x_i} Q_i(x_i, t)$ and $Q_i(x_i^*(t), t) \equiv 0$;

(2) *The functions $h_i(x, \theta, t)$ are continuous and satisfy the inequalities*

$$|\nabla_{x_i} Q_i(x_i, t)^{\mathrm{T}} h_i(x, \theta, t)| \leqslant \sum_{j=1}^{l} \mu_{ij}\rho_j(Q_j(x_j, t)) + d_i, \tag{A.4}$$

where the matrix $M - I$ is the Hurwitz one, $M = \{\mu_{ij}\}$, $\mu_{ij} \geqslant 0$, I is the identify matrix, $d_i > 0$; in this case Δ_i in (A.2) *are selected from the conditions*

$$\rho_i(\Delta_i) > r_i, \tag{A.5}$$

where $r = (I - M)^{-1}d$, $r = \mathrm{col}(r_1, \ldots, r_l)$, $d = \mathrm{col}(d_1, \ldots, d_l)$.

Then all trajectories of the system (A.1), (A.2) *are bounded and in this case*

$$\overline{\lim_{t\to\infty}} Q_i(x_i(t),t) \leqslant \Delta_i, \quad i = 1,\dots,l.$$

We will directly pass on to the proof of the Theorem 1. Let us consider the first group of conditions of the Lemma 2. The condition of the local boundedness on $t > 0$ is fulfilled because for each $i = 1,\dots,d$ the right side of the system (4) and the function $Q(z_i)$ are the smooth functions that do not depend on t. The convexity condition is afforded by the linearity in θ_i of the right side of the expression (3.4). As functions $\rho_i(\cdot)$, $i = 1,\dots,d$ that appear in Lemma 2 we will take the same linear function $Q \to \rho \times Q$. We will show that the existence condition of $\theta_* \in \mathbb{R}^l$ and ρ, such that $\omega_i(z_i,\theta_*) \leqslant -\rho Q(z_i)$ is afforded by the hyperminimum-phasing of the function $g^{\mathrm{T}}\chi(s)$. Indeed, according to Lemma 1, the hyperminimum-phasing of $g^{\mathrm{T}}\chi(s)$ ensures the existence of $H = H^{\mathrm{T}} > 0$ and θ_*, such that

$$HA_* + A_*^{\mathrm{T}}H < 0, \quad HB = Cg,$$

where $A_* = (A + LI_n) + B\theta_*^{\mathrm{T}}C^{\mathrm{T}}$,

$$\begin{aligned} F_i &= Az_i + B\tilde{u}_i + \varphi_0(x_i) - \varphi_0(\overline{x}),\\ \omega_i(z_i,\theta_*) = \dot{Q} &= z_i^{\mathrm{T}}H[Az_i + B\tilde{u}_i + \varphi_0(x_i) - \varphi_0(\overline{x})]\\ &= z_i^{\mathrm{T}}H(A + B\theta_i^{\mathrm{T}}C)z_i + z_i^{\mathrm{T}}H[\varphi_0(x_i) - \varphi_0(\overline{x})]\\ &\leqslant z_i^{\mathrm{T}}H(A + B\theta_i^{\mathrm{T}}C^{\mathrm{T}})z_i + ||z_i^{\mathrm{T}}|| \times ||H|| \times L \times ||z_i||\\ &\leqslant z_i^{\mathrm{T}}H(A + B\theta_i^{\mathrm{T}}C^{\mathrm{T}})z_i + L\lambda^+(H)||z_i||^2\\ &\leqslant z_i^{\mathrm{T}}H(A + B\theta_i^{\mathrm{T}}C^{\mathrm{T}})z_i + L\frac{\lambda^+(H)}{\lambda^-(H)}z_i^{\mathrm{T}}Hz_i = \frac{1}{2}z_i^{\mathrm{T}}[HA_* + A_*^{\mathrm{T}}H]z_i, \end{aligned} \tag{A.6}$$

where $A_* = (A + LI_n) + B\theta_*^{\mathrm{T}}C^{\mathrm{T}}$; on account of the relation $\frac{\lambda^+(H)}{\lambda^-(H)} \geqslant 1$, we have $L \leqslant L\frac{\lambda^+(H)}{\lambda^-(H)}$ and, consequently, with A_* the Lipschitz conditions will also be fulfilled.

Then, calculating $\dot{Q}_i$, in view of the equation of error for the ith node (3.1), we will obtain

$$\omega_i(z_i,\theta_*) \leqslant z_i^{\mathrm{T}}H[(A + LI_n) + B\theta_*^{\mathrm{T}}C^{\mathrm{T}}]z_i = \frac{1}{2}z_i^{\mathrm{T}}[HA_* + A_*^{\mathrm{T}}H]z_i, \quad i = 1,\dots,d.$$

As $HA_* + A_*^{\mathrm{T}}H$ is negatively definite, there exists such a $\rho > 0$ that $HA_* + A_*^{\mathrm{T}}H \leqslant -\rho H$, which thus ensures the condition

$$\omega_i(z_i,\theta_*) \leqslant -\rho Q(z_i), \quad i = 1,\dots,d.$$

We will pass on to the conditions of interconnection of subsystems (the second group of conditions of the Lemma 2). In the case of interest the conditions take the form

$$\left|\nabla_{z_i}Q(z_i)^{\mathrm{T}}\left[\sum_{j=1}^{d}\alpha_{ij}\varphi_{ij}(z_i - z_j) + f_i(t)\right]\right| \leqslant \sum_{j=1}^{d}\mu_{ij}\rho Q(z_j) + d_i, \quad i = 1,\dots,d,$$

where the matrix $M - I$ is the Hurwitz one, $M = \{\mu_{ij}\}$, $\mu_{ij} \geqslant 0$, I is the identity matrix. Let us rewrite the last inequality:

$$\left|z_i^{\mathrm{T}}H\left[\sum_{j=1}^{d}\alpha_{ij}\varphi_{ij}(z_i - z_j) + f_i(t)\right]\right| \leqslant \frac{\rho}{2}\sum_{j=1}^{d}\mu_{ij}z_j^{\mathrm{T}}Hz_j + d_i, \quad i = 1,\dots,d. \tag{A.7}$$

We will estimate the quantity in the left side of the (A.7):

$$\left|z_i^{\mathrm{T}} H\left[\sum_{j=1}^{d} \alpha_{ij}\varphi_{ij}(z_i-z_j)+f_i(t)\right]\right| \leqslant \sum_{j=1}^{d}|\alpha_{ij}L_{ij}|\lambda^{+}(H)(||z_i||^2+||z_i||\times||z_j||)+|z_i^{\mathrm{T}}Hf_i(t)|$$

$$\leqslant \sum_{j=1}^{d}|\alpha_{ij}L_{ij}|\lambda^{+}(H)(||z_i||^2+||z_i||\times||z_j||)+\frac{1}{2}\left|\delta_i z_i^{\mathrm{T}}Hz+\frac{1}{\delta_i}f_i(t)^{\mathrm{T}}Hf_i(t)\right|$$

$$\leqslant \sum_{j=1}^{d}|\alpha_{ij}L_{ij}|\lambda^{+}(H)(||z_i||^2+||z_i||\times||z_j||)+\frac{1}{2}\delta_i||z_i||^2\lambda^{+}(H)+\frac{d_{f_i}}{2\delta_i}\lambda^{+}(H), \quad i=1,\dots,d,$$

where $\delta_i > 0$, $i=1,\dots,d$, are some numbers.

We will define the lower estimate of the right side of the (A.7):

$$\frac{\rho}{2}\sum_{j=1}^{d}\mu_{ij}z_j^{\mathrm{T}}Hz_j+d_i \geqslant \frac{\rho}{2}\sum_{j=1}^{d}\mu_{ij}\lambda^{-}(H)||z_j||^2+d_i, \quad i=1,\dots,d.$$

Thus it is sufficient to fulfill the inequality

$$\begin{gathered}\sum_{j=1}^{d}|\alpha_{ij}L_{ij}|\lambda^{+}(H)(||z_i||^2+||z_i||\times||z_j||)\\ +\frac{1}{2}\delta_i||z_i||^2\lambda^{+}(H)+\frac{d_{f_i}}{2\delta_i}\lambda^{+}(H)\leqslant\frac{\rho}{2}\sum_{j=1}^{d}\mu_{ij}\lambda^{-}(H)||z_j||^2+d_i\end{gathered} \tag{A.8}$$

or

$$\begin{cases}\dfrac{d_{f_i}}{2\delta_i}\lambda^{+}(H)\leqslant d_i\\ \displaystyle\sum_{j=1}^{d}|\alpha_{ij}L_{ij}|(||z_i||^2+||z_i||\times||z_j||)+\frac{1}{2}\delta_i||z_i||^2\leqslant\frac{\rho}{2}\frac{\lambda^{-}(H)}{\lambda^{+}(H)}\sum_{j=1}^{d}\mu_{ij}||z_j||^2,\end{cases} \tag{A.9}$$

where $i=1,\dots,d$.

We put $\mathbf{z}$, $\nu_i^{(2)}$, η_i as in [5], while for $\nu_i^{(1)}$ we will take

$$\nu_i^{(1)}=\begin{pmatrix}0&\dots&0&\dots&0\\0&\ddots&0&\dots&0\\\vdots&&\vdots&&\vdots\\0&\dots&\sum_{j=1}^{d}|\alpha_{ij}L_{ij}|+\frac{1}{2}\delta_i&\dots&0\\\vdots&&\vdots&\ddots&\vdots\\0&\dots&0&\dots&0\end{pmatrix}. \tag{A.10}$$

Then for the fulfilment of the (A.7) it is sufficient that for any $i=1,\dots,d$ the following inequality should be valid:

$$\mathbf{z}^{\mathrm{T}}(\nu_i^{(1)}+\nu_i^{(2)})\mathbf{z}\leqslant\mathbf{z}^{\mathrm{T}}\eta_i\mathbf{z}, \tag{A.11}$$

i.e., the matrix $\eta_i-\nu_i^{(1)}-\nu_i^{(2)}$ for each $i=1,\dots,d$ must be positive semidefinite.

As a matrix $M = \{\mu_{ij}\}$ we will take the diagonal matrix in the following way:

$$0 < \mu_{ii} < 1, \quad \mu_{ij} = 0, \quad i \neq j, \quad i = 1, \ldots, d, \quad j = 1, \ldots, d.$$

Then the matrix $M - I$ will be the Hurwitz one. The positive semidefiniteness of the matrix $\eta_i - \nu_i^{(1)} - \nu_i^{(2)}$ affords the condition

$$\mu_{ii} \frac{\rho}{2} \frac{\lambda^-(H)}{\lambda^+(H)} - \sum_{j=1}^{d} |\alpha_{ij} L_{ij}| - \frac{1}{2}\delta_i \geqslant 0, \quad i = 1, \ldots, d,$$

or

$$\mu_{ii} \geqslant \left(\frac{\rho}{2} \frac{\lambda^-(H)}{\lambda^+(H)} \right)^{-1} \left(\sum_{j=1}^{d} |\alpha_{ij} L_{ij}| + \frac{1}{2}\delta_i \right), \quad i = 1, \ldots, d. \tag{A.12}$$

Noting that $\mu_{ii} < 1$ we come to the condition (4.2). We will now consider the condition (A.5) together with the condition (A.12). Let us assume that $\delta_i = \frac{\rho}{2}\frac{\lambda^-(H)}{\lambda^+(H)} - \sum_{j=1}^{d} |\alpha_{ij} L_{ij}|$.

Then (A.9), (A.12) can be rewritten in such a way:

$$\begin{cases} \dfrac{d_{f_i} \lambda^+(H)}{2\delta_i} \leqslant d_i < \rho \Delta_i \\ \delta_i > 0, \end{cases} \tag{A.13}$$

which is carried out in view of the condition of Theorem 1. Thus it is possible to apply Lemma 2, which affords the fulfilment of the assertions of Theorem 1.

Proof of Theorem 2. To prove the Theorem 2 it will again be necessary to use two auxiliary lemmas from the proof of the Theorem 1: Lemmas 1 and 2.

We will consider the first group of conditions of the Lemma 2. The condition of the local boundedness on $t > 0$ is fulfilled because for each $i = 1, \ldots, d$ the right side of the system (4) and the function $Q(z_i)$ are the smooth functions that do not depend on t. The convexity condition is ensured by the linearity in θ_i of the right side of the expression (3.4). As functions $\rho_i(\cdot)$, $i = 1, \ldots, d$ appearing in the Lemma 2 we will take the same linear function $Q \to \rho \times Q$. We will show that the existence condition of $\theta_* \in \mathbb{R}^l$ and ρ, such that $\omega_i(z_i, \theta_*) \leqslant -\rho Q(z_i)$ is ensured by the hyperminimum-phasing of the function $g^{\mathrm{T}} \chi(s)$. Indeed, according to the Lemma 1, the hyperminimum-phasing of $g^{\mathrm{T}} \chi(s)$ affords the existence of $H = H^{\mathrm{T}} > 0$ and θ_* such that $HA_* + A_*^{\mathrm{T}} H < 0$, $HB = Cg$, where $A_* = A + B\theta_*^{\mathrm{T}} C^{\mathrm{T}}$. Let us estimate the rate of change of the goal function in view of the Eqs. (3.2), (3.5), (5.1), (5.2):

$$\begin{aligned} \omega_i(z_i, \theta_*) = \dot{Q} &= z_i^{\mathrm{T}} H[Az_i + B(\tilde{u}_i + \psi_0(y_i) - \psi_0(\overline{y}))] \\ &= z_i^{\mathrm{T}} H(A + B\theta_i^{\mathrm{T}} C^{\mathrm{T}}) z_i + z_i^{\mathrm{T}} HB(\psi_0(y_i) - \psi_0(\overline{y})) \\ &= z_i^{\mathrm{T}} H(A + B\theta_i^{\mathrm{T}} C^{\mathrm{T}}) z_i + z_i^{\mathrm{T}} Cg(\psi_0(y_i) - \psi_0(\overline{y})) \\ &= z_i^{\mathrm{T}} H(A + B\theta_i^{\mathrm{T}} C^{\mathrm{T}}) z_i + (\psi_0(y_i) - \psi_0(\overline{y}))^{\mathrm{T}} g^{\mathrm{T}} (y_i - \overline{y}) \\ &= z_i^{\mathrm{T}} H(A + B\theta_i^{\mathrm{T}} C^{\mathrm{T}}) z_i + (y_i - \overline{y})^{\mathrm{T}} g(\psi_0(y_i) - \psi_0(\overline{y})) \leqslant z_i^{\mathrm{T}} H(A + B\theta_i^{\mathrm{T}} C^{\mathrm{T}}) z_i. \end{aligned} \tag{A.14}$$

The further proof repeats the proof of the Theorem 1.

Proof of Theorem 3. For the proof we will need the Yakubovich–Kalman lemma (frequency theorem) [15] in the following form.

Lemma 3. *Let* $u \in \mathbb{R}^m$, $\chi(s) = C^{\mathrm{T}}(sI_n - A)^{-1}B$, $\operatorname{rank} B = m$. *Then the following two conditions are equivalent:*

(1) *There exists a positive definite matrix* $H = H^{\mathrm{T}} > 0$, *such that*

$$HA + A^{\mathrm{T}}H < 0, \quad HB = C;$$

(2) *The determinant of the matrix* $sI_n - A$ *is the Hurwitz one, and for all* $\omega \in \mathbb{R}^1$, $u \in \mathbb{R}^m$, $u \neq 0$ *the frequency inequalities are fulfilled:*

$$\mathrm{Re}\, u^{\mathrm{T}}\chi(i\omega)u > 0, \quad \lim_{\omega\to\infty} \omega^2 \mathrm{Re}\, u^{\mathrm{T}}\chi(i\omega)u > 0.$$

Let us note that in the case considered $m = 1$, i.e., u is scalar, and instead of C we will take Cg. Then the conditions of the Lemma 3 and Theorem 3 afford the existence of the matrix $H = H^{\mathrm{T}} > 0$, such that

$$HA_M + A_M^{\mathrm{T}}H < 0, \quad HB = Cg.$$

We will consider

$$\omega(x_i, \overline{x}, \tau_i) = \dot{Q} = z_i^{\mathrm{T}} H(Ax_i + B\tau_i^{\mathrm{T}}(t)\sigma_i(t) + B_M\psi_0(y_i) - A_M\overline{x} - B_M(\overline{u} + \psi_0(\overline{y}))).$$

Let us put $\tau_* = \mathrm{col}(\nu_*, \theta_*)$. At $\tau = \tau_*$ we obtain

$$\begin{aligned}\omega_i(x_i, \overline{x}, \tau_*) = \dot{Q} &= z_i^{\mathrm{T}} H(Ax_i + B(\nu_* C^{\mathrm{T}} x_i + \theta_*\overline{u}) + B_M\psi_0(y_i) - A_M\overline{x} - B_M(\overline{u} + \psi_0(\overline{y}))) \\ &= z_i^{\mathrm{T}} H(A_M z_i + B_M\psi_0(\overline{y})) \leqslant z_i^{\mathrm{T}} HA_M z_i, \quad i = 1, \ldots, d.\end{aligned} \tag{A.15}$$

The last inequality is fulfilled because $\psi(\cdot)$ is the g-monotone decreasing function. Thus

$$\omega_i(z_i, \tau_*) \leqslant z_i^{\mathrm{T}} HA_M z_i = \frac{1}{2} z_i^{\mathrm{T}}[HA_M + A_M^{\mathrm{T}}H]z_i, \quad i = 1, \ldots, d, \tag{A.16}$$

i.e., there exists a number $\rho > 0$, such that $\omega_i(z_i, \tau_*) \leqslant -\rho Q(z_i)$. Repeating further the proof of Theorem 1 we will obtain the required proof.

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